HILBERT-SCHMIDT DOUBLE DIFFERENCES OF COMPOSITION OPERATORS AND NON-RIGID PHENOMENON

BOO RIM CHOE, XIN GUO, TAKUYA HOSOKAWA, HYUNGWOON KOO, SHÛICHI OHNO, AND MAOFA WANG

ABSTRACT. In the setting of the standard weighted Bergman spaces over the unit disk, compactness characterizations for linear combinations of composition operators have been known. One of those characterizations asserts that degenerate double differences, compared with each single difference, do not improve the compactness at all in the sense that a degenerate double difference is compact only when each difference is individually compact. Such a rigid phenomenon is actually known to hold for a certain broader class of linear combinations. In this paper we investigate into similar properties for Hilbert-Schmidtness with main focus on double differences.

We first obtain a complete characterization for Hilbert-Schmidt double differences of composition operators. We then observe that double differences, compared with each single difference, can improve the Hilbert-Schmidtness even in the degenerate case, by constructing concrete examples of Hilbert-Schmidt double differences with each difference not being Hilbert-Schmidt. We also include some remarks concerning connection between Hilbert-Schmidtness on the standard weighted Bergman spaces and weak-to-strong boundedness on certain vector-valued weighted Bergman spaces.

1. INTRODUCTION

Let $H(\mathbf{D})$ be the class of all holomorphic functions on the unit disk $\mathbf{D}$ of the complex plane $\mathbf{C}$. Denote by $\mathcal{S}(\mathbf{D})$ the set of all holomorphic self-maps of $\mathbf{D}$. Given $\varphi \in \mathcal{S}(\mathbf{D})$, the *composition* operator C_φ with *symbol* φ is defined by

$$C_\varphi f := f \circ \varphi$$

for $f \in H(\mathbf{D})$. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. We refer to standard monographs by Cowen-MacCluer [5] and Shapiro [12] for an overview of various aspects on the theory of composition operators acting on classical holomorphic function spaces.

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During the past three decades, initiated by the Shapiro-Sundberg Question raised in 1990 (see [13]), many significant results concerning topics related to the topological structure of composition operators have been achieved in the literature. For general historical remarks on the progress of the Shapiro-Sundberg Question, we refer to [3] and references therein. One of such topics is to characterize compactness for differences, or more generally, for linear combinations of composition operators acting on the standard weighted Bergman spaces. As the first result in that direction, Moorhouse [11] characterized compactness for differences of composition operators. Her characterization shows that differences of composition operators, compared with each operator, can improve the compactness by way of certain natural cancellations.

Inspired by the aforementioned result of Moorhouse, Koo and Wang [9] characterized compactness for the *degenerate* double differences, i.e., operators of the form $2C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$. Quite unexpectedly, their characterization reveals a rigid phenomenon asserting that such an operator is compact only when both $C_{\varphi_1} - C_{\varphi_2}$ and $C_{\varphi_1} - C_{\varphi_3}$ are individually compact. Thus, intuitively speaking, degenerate double differences, compared with each single difference, do not improve the compactness anymore. This rigid phenomenon was later extended to general linear combinations under the so-called Coefficient Non-cancellation Condition(CNC), which means that the whole sum of the coefficients vanishes, but any proper sum does not; see [4, Theorem 1.2]. More recently, Choe, Koo and Wang characterized (see [3, Theorem 1.1]) compactness for double differences, i.e., operators of the form $(C_{\varphi_1} - C_{\varphi_2}) - (C_{\varphi_3} - C_{\varphi_4})$, and exhibited an explicit example demonstrating that non-compact differences can form a compact double difference; see [3, Example 6.4]. Note that CNC does not hold in general for double differences.

The purpose of the current paper is to study Hilbert-Schmidt analogues of what have been mentioned in the preceding paragraph; see Section 2.3 for the notion of Hilbert-Schmidt operators. We obtain a complete characterization for Hilbert-Schmidt double differences. Using our result, we also exhibit concrete examples demonstrating that the rigid phenomenon for compactness mentioned above does not extend to Hilbert-Schmidtness.

Before proceeding, we first recall our function spaces to work on. For $\alpha > -1$, we denote by dA_α the normalized weighted measure defined by

$$dA_\alpha(z) := (\alpha + 1)(1 - |z|^2)^\alpha dA(z), \qquad z \in \mathbf{D}$$

where dA denotes the area measure on $\mathbf{D}$ normalized to have the total mass 1. Now, the α-weighted Bergman space $A^2_\alpha(\mathbf{D})$ is the space consisting of all $f \in H(\mathbf{D})$ for which

$$\|f\|_{A^2_\alpha} := \left\{ \int_{\mathbf{D}} |f|^2 \, dA_\alpha \right\}^{1/2} < \infty.$$

As is well known, the space $A^2_\alpha(\mathbf{D})$ is a closed subspace of the Lebesgue space $L^2_\alpha(\mathbf{D}) := L^2(\mathbf{D}, dA_\alpha)$ and thus is a Hilbert space. Also is well known that every composition operator is bounded on $A^2_\alpha(\mathbf{D})$ thanks to the Littlewood Subordination Principle. In case $\alpha = 0$, we write $A^2(\mathbf{D}) := A^2_0(\mathbf{D})$ and $L^2(\mathbf{D}) := L^2_0(\mathbf{D})$.

We also introduce some notation to be used throughout the paper. Given φ_1, φ_2, φ_3, $\varphi_4 \in \mathcal{S}(\mathbf{D})$, not necessarily distinct, we will save notation by setting

$$\rho_{ij} := \rho(\varphi_i, \varphi_j)$$

and

$$M_{ij} := \rho_{ij}\left(\|K_{\varphi_i}\|_{A^2_\alpha} + \|K_{\varphi_j}\|_{A^2_\alpha}\right)$$

for $i, j = 1, 2, 3, 4$ where $\|K_{\varphi_i}\|_{A^2_\alpha}$ denotes the function $z \mapsto \|K_{\varphi_i(z)}\|_{A^2_\alpha}$. Here, ρ denotes the pseudohyperbolic distance on $\mathbf{D}$ and $K_{(\cdot)}$ denotes the reproducing kernel for $A^2_\alpha(\mathbf{D})$; see Sections 2.1 and 2.2, respectively.

In addition, we use the abbreviated notation

$$R_1 := M_{12} + M_{34} \quad \text{and} \quad R_2 := M_{13} + M_{24}. \tag{1.1}$$

Using the function σ to be specified in Section 2.1, we further set

$$\sigma_{ij} := \sigma(\varphi_i, \varphi_j)$$

for $i, j = 1, 2, 3, 4$ and put

$$R_3 := \left[\frac{|\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4|}{\sum_{j=1}^4 (1 - |\varphi_j|^2)} + (\rho_{12} + \rho_{34})(\rho_{13} + \rho_{24})\right] \sum_{j=1}^4 \|K_{\varphi_j}\|_{A^2_\alpha},$$

$$R_4 := \left(|\sigma_{12} + \sigma_{13}| + \rho_{12}^2 + \rho_{13}^2\right) \sum_{j=1}^3 \|K_{\varphi_j}\|_{A^2_\alpha}.$$

Finally, we put

$$\Omega_s := \left\{ z \in \mathbf{D} : \max_{1 \le i < j \le 4} \rho_{ij}(z) < s \right\}$$

for $0 < s < 1$. For all the abbreviated notation specified above, the dependency on the self-maps φ_j's (and α) should be clear from the context.

We obtain the next characterization for Hilbert-Schmidt double differences.

Theorem 1.1. *For φ_1, φ_2, φ_3, $\varphi_4 \in \mathcal{S}(\mathbf{D})$, put*

$$T := C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3} + C_{\varphi_4}.$$

Given $\alpha > -1$, there exists a positive number $s_0 \in (0, 1)$, depending only on α, with the following property: T is Hilbert-Schmidt on $A^2_\alpha(\mathbf{D})$ iff

$$\int_{\mathbf{D}\setminus\Omega_s} (\min\{R_1, R_2\})^2 \, dA_\alpha + \int_{\Omega_s} R_3^2 \, dA_\alpha < \infty \tag{1.2}$$

for some/all $s \in (s_0, 1)$. Moreover, the above condition reduces to

$$\int_{\mathbf{D}} R_4^2 \, dA_\alpha < \infty \tag{1.3}$$

in the degenerate case $\varphi_1 = \varphi_4$.

Note. Condition (1.2) can be replaced by a bit simpler one which is not symmetric; see Remark 3.6. Meanwhile, we have $R_4 \approx M_{12}$ when both $\varphi_1 = \varphi_4$ and $\varphi_2 = \varphi_3$ hold. Thus, the known characterization for Hilbert-Schmidt differences, which can be derived from (2.9) and (2.8) below, is recovered by the degenerate case of Theorem 1.1.

Applying Theorem 1.1, we also construct examples $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{S}(\mathbf{D})$ demonstrating the following on $A^2(\mathbf{D})$:

- ♦ $C_{\varphi_1} - C_{\varphi_j}$ *is not Hilbert-Schmidt for* $j = 2, 3$, *but* $2C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3}$ *is*;

see Example 5.2. See also Example 5.4 for a more general example. So, in sharp contrast to the rigid phenomenon for compactness mentioned above, we see that double differences, compared with each single difference, can improve the Hilbert-Schmidtness even in the degenerate case.

Our proof of Theorem 1.1 is supported by many technical lemmas and thus is quite long. So, in order to help readers to keep up with the direction of the details on the whole, it seems worth mentioning the overall scheme regarding the motivation, the key steps, the subtlety, and the strategies.

Overall Scheme. The Hilbert-Schmidt norm of T is precisely the same (see (2.9)) as the square root of the integral

$$\int_{\mathbf{D}} \left\| K_{\varphi_1(z)} - K_{\varphi_2(z)} - K_{\varphi_3(z)} + K_{\varphi_4(z)} \right\|^2_{A^2_\alpha} dA_\alpha(z). \tag{1.4}$$

The main difficulty is caused by the fact that the integrand behaves quite differently, according as all the points $\varphi_j(z)$'s are hyperbolically close to one another or some of them are apart.

To show the subtlety of our estimations, we briefly compare with the compactness case where one needs to estimate (see [3,4,9]) $(1-|a|)^{\alpha+2}\|TK_a\|^2_{A^2_\alpha}$, or more explicitly,

$$(1-|a|)^{\alpha+2} \int_{\mathbf{D}} \left| K_{\varphi_1(z)}(a) - K_{\varphi_2(z)}(a) - K_{\varphi_3(z)}(a) + K_{\varphi_4(z)}(a) \right|^2 dA_\alpha(z).$$

For the sake of simplicity, consider the degenerate case $\varphi_1 = \varphi_4$. In [9] it is proved that this quantity vanishes as $|a| \to 1$ only when each of the integrals

$$(1-|a|)^{\alpha+2} \int_{\mathbf{D}} \left| K_{\varphi_1(z)}(a) - K_{\varphi_j(z)}(a) \right|^2 dA_\alpha(z), \qquad j = 2, 3$$

does, which turns out to be equivalent to the compactness of corresponding differences of composition operators. This means that compactness cannot be achieved through double cancellation in general.

On the contrary in the case of Hilbert-Schmidtness, it is possibly the case that the integral in (1.4) is finite, but at the same time

$$\int_{\mathbf{D}} \left\| K_{\varphi_i(z)} - K_{\varphi_j(z)} \right\|^2_{A^2_\alpha} dA_\alpha(z) = \infty$$

for $(i, j) \in \{(1, 2), (3, 4)\}$. This demonstrates that double cancellation in a nontrivial way may end up with Hilbert-Schmidtness. It is possibility of such double cancellation which causes the subtlety in studying Hilbert-Schmidtness of double differences.

Eventually, the hardest and most crucial part in our proof of Theorem 1.1 is to establish a sharp estimate of the integrand in (1.4). So, we are forced to establish a sharp estimate of the ratio

$$\frac{\|K_{z_1} - K_{z_2} - K_{z_3} + K_{z_4}\|_{A^2_\alpha}}{\sum_{j=1}^{4} \|K_{z_j}\|_{A^2_\alpha}} \tag{1.5}$$

for general $z_1, z_2, z_3, z_4 \in \mathbf{D}$ in terms of the hyperbolic distances between the points. Our approaches are quite different in two cases: (i) when all the points are close to one another and (ii) when they are not. Both cases are quite delicate as follows:

- Case (i), where cancellation occurs, is a bit more delicate to handle. Using the binomial expansions, we express $|K_{z_1}(w) - K_{z_2}(w) - K_{z_3}(w) + K_{z_4}(w)|$ as the sum of a major term and an error term, which might be *probably* the only way in general one may think of. We estimate the integral of the major term and then use it to control the error term. Section 3 is entirely devoted to Case (i).
- Case (ii), where no cancellation is expected, is not easy at all, either. The difficulty is not only the presence of too many subcases to consider, but also the need to devise right approaches depending on the subcases. Section 4 is entirely devoted to Case (ii).

We hope these preliminary information to help readers in understanding the long and technical steps towards Theorem 1.1.

In Section 2 we collect basic known results to be used in later sections.

In Sections 3 and 4 we establish the aforementioned ratio estimate. For the case when the four points are close to one another, see Theorems 3.5 and 3.7. For the remaining case, see Theorem 4.6.

In Section 5, relying on the optimal ratio estimates, we prove Theorem 1.1. Our proof actually produces optimal Hilbert-Schmidt norm estimates; see Remark 5.1. Applying our result, we exhibit an explicit example demonstrating that the rigid phenomenon, analogous to the one for compactness mentioned before, fails to hold Hilbert-Schmidtness; see Example 5.2. We also construct a more general example demonstrating the same pathology; see Example 5.4.

Finally, in Section 6, for linear combinations of composition operators and related operators, we notice some remarks revealing the connection between Hilbert-Schmidtness on $A^2_\alpha(\mathbf{D})$ and boundedness from certain weak to strong vector-valued weighted Bergman spaces.

Constants. Throughout the paper various constants C are used with no attempt to calculate their exact values, which may change from one occurrence to the next. Variables indicating the dependency of constants C will be often specified in the parenthesis. Given two non-negative quantities A and B, we write $A \lesssim B$ to indicate that there exists some inessential constant $C > 0$ so that $A \leq CB$. The converse relation $A \gtrsim B$ is defined in an analogous manner, and if $A \lesssim B$ and $B \lesssim A$ both hold, we write $A \approx B$.

2. Preliminaries

In this section, we collect some basic facts and preliminary results to be used in later sections.

2.1. **Pseudohyperbolic Distance.** Put

$$\sigma(a,b) := \frac{a-b}{1-\overline{a}b}$$

for $a, b \in \mathbf{D}$. The hyperbolic distance between a and b is given by

$$\tanh^{-1}\rho(a,b) \quad \text{where} \quad \rho(a,b) := |\sigma(a,b)|.$$

It is well known that ρ itself is a distance, called the *pseudohyperbolic* distance. In most cases it is more convenient to work with this pseudohyperbolic distance rather than the original hyperbolic distance.

The well-known identity

$$1-\rho^2(a,b) = \frac{(1-|a|^2)(1-|b|^2)}{|1-\overline{a}b|^2}$$

is straightforward. This yields an inequality

$$\frac{1-|a|^2}{2\sqrt{1-\rho(a,b)}} \leq |1-\overline{a}b| \quad \text{for} \ \ |b| \leq |a|, \tag{2.1}$$

which is useful for our purpose. We also recall the following inequalities:

$$\frac{\rho(a,b)}{1+\rho(a,b)} \leq \frac{|a-b|}{1-|a|^2} \leq \frac{\rho(a,b)}{1-\rho(a,b)}; \tag{2.2}$$

$$\frac{1-\rho(a,b)}{1+\rho(a,b)} \leq \frac{1-|a|^2}{1-|b|^2} \leq \frac{1+\rho(a,b)}{1-\rho(a,b)}; \tag{2.3}$$

$$1-\rho(a,b) \leq \frac{1-|a|^2}{|1-\overline{a}b|} \leq 1+\rho(a,b). \tag{2.4}$$

These inequalities, which are well-known and elementary to prove, will be quite frequently used later in our proofs.

2.2. **Reproducing Kernel.** Given $\alpha > -1$, subharmonicity yields

$$|g(0)|^2 \le \int_{\mathbf{D}} |g(w)|^2 \, dA_\alpha(w)$$

for $g \in H(\mathbf{D})$. Given $z \in \mathbf{D}$, applying this inequality to the function

$$g(w) := (f \circ \sigma_z)(w) \left(\frac{\sqrt{1-|z|^2}}{1 - w\bar{z}} \right)^{\alpha+2} \quad \text{where } \sigma_z := \sigma(z, \cdot)$$

and then making a change-of-variable, one obtains

$$(1-|z|^2)^{\alpha+2} |f(z)|^2 \le \|f\|^2_{A^2_\alpha} \tag{2.5}$$

for $f \in H(\mathbf{D})$. In particular, this shows that each point evaluation is a continuous linear functional on $A^2_\alpha(\mathbf{D})$. Thus, to each $z \in \mathbf{D}$ corresponds a unique reproducing kernel $K_z^{(\alpha)}$ whose explicit formula is well known as

$$K_z^{(\alpha)}(w) := \frac{1}{(1-w\bar{z})^{\alpha+2}}, \qquad w \in \mathbf{D}. \tag{2.6}$$

We will use the abbreviated notation

$$K_z := K_z^{(\alpha)};$$

this will cause no confusion, because α is the only weight parameter we consider throughout the paper.

Using the reproducing property, one may explicitly compute the norm of reproducing kernels as

$$\|K_z\|^2_{A^2_\alpha} = K_z(z) = \frac{1}{(1-|z|^2)^{\alpha+2}} \tag{2.7}$$

for $z \in \mathbf{D}$. As for the differences of reproducing kernels, we have the norm estimates (see [2, Proposition 3.5])

$$\|K_z - K_w\|_{A^2_\alpha} \approx \rho(z,w) \left(\|K_z\|_{A^2_\alpha} + \|K_w\|_{A^2_\alpha} \right) \tag{2.8}$$

for all $z, w \in \mathbf{D}$; the constants suppressed above depend only on α.

2.3. **Hilbert-Schmidt Operator.** Let X be a separable Hilbert space with orthonormal basis $\{e_n\}$. A linear operator $S : X \to X$ is called *Hilbert-Schmidt* if

$$\|S\|_{HS(X)} := \left\{ \sum_n \|Se_n\|^2_X \right\}^{1/2} < \infty.$$

The above *Hilbert-Schmidt norm* of S is known to be independent of the choice of orthonormal basis $\{e_n\}$. It is well known that every Hilbert-Schmidt operator is compact and therefore bounded.

Given $\alpha > -1$, the Hilbert-Schmidt norm of a linear combination of composition operators on $A^2_\alpha(\mathbf{D})$ is known to be represented as an elegant integral. Namely, given a positive integer n, we have

$$\left\|\sum_{j=1}^{n} c_j C_{\varphi_j}\right\|^2_{HS(A^2_\alpha)} = \int_{\mathbf{D}} \left\|\sum_{j=1}^{n} \overline{c_j} K_{\varphi_j(z)}\right\|^2_{A^2_\alpha} dA_\alpha(z) \tag{2.9}$$

for all $c_1, \dots, c_n \in \mathbf{C}$ and for all $\varphi_1, \dots, \varphi_n \in \mathcal{S}(\mathbf{D})$; see [2, Proposition 3.1]. So, for the operators

$$T := C_{\varphi_1} - C_{\varphi_2} - C_{\varphi_3} + C_{\varphi_4}$$

being considered throughout the paper, we have

$$\|T\|^2_{HS(A^2_\alpha)} = \int_{\mathbf{D}} \|K\|^2_{A^2_\alpha}\, dA_\alpha \tag{2.10}$$

where $K := K_{\varphi_1} - K_{\varphi_2} - K_{\varphi_3} + K_{\varphi_4}$.

3. The ratio estimate: Part 1

In this section we establish optimal norm estimates for the ratio (1.5) when the points are close to one another. For that purpose we need several technical lemmas. In what follows we denote by $D_r(a)$ the Euclidean disk with center a and radius r, i.e.,

$$D_r(a) := \{z \in \mathbf{C} : |z - a| < r\}$$

for $a \in \mathbf{C}$ and $r > 0$.

Lemma 3.1. *The inequality*

$$|a_1p_1 + a_2p_2| + \left|a_1p_1\lambda + a_2p_2\lambda^2\right| \geq \frac{\lambda(1-\lambda)}{2}(|a_1p_1| + |a_2p_2|) \tag{3.1}$$

holds for all $\lambda \in (0,1)$ and $a_1, a_2, p_1, p_2 \in \mathbf{C}$.

Proof. For arbitrary $\lambda \in (0,1)$ and $a_1, a_2, p_1, p_2 \in \mathbf{C}$, we note

$$\begin{pmatrix} a_1p_1 \\ a_2p_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda & \lambda^2 \end{pmatrix}^{-1} \begin{pmatrix} a_1p_1 + a_2p_2 \\ a_1p_1\lambda + a_2p_2\lambda^2 \end{pmatrix},$$

or said differently,

$$\lambda(\lambda - 1)\begin{pmatrix} a_1p_1 \\ a_2p_2 \end{pmatrix} = \begin{pmatrix} \lambda^2 & -1 \\ -\lambda & 1 \end{pmatrix} \begin{pmatrix} a_1p_1 + a_2p_2 \\ a_1p_1\lambda + a_2p_2\lambda^2 \end{pmatrix}.$$

This implies the asserted inequality. The proof is complete. □

Lemma 3.2. *For $r > 0$ the following assertions hold:*

(a) *The estimate*

$$|z + w - \xi| + |z^2 + w^2 - \xi^2| \approx |z + w - \xi| + |zw|$$

holds for $z, w, \xi \in D_r(0)$; the constants suppressed in this estimate depends only on r.

(b) *The inequality*

$$|z^3 + w^3 - \xi^3| \leq 5r^2|z + w - \xi| + 4r|zw|$$

holds for $z, w, \xi \in D_r(0)$.

Proof. Note

$$|(z^2+w^2-\xi^2)+2zw| = |(z+w+\xi)(z+w-\xi)| \le 3r|z+w-\xi|$$

and thus

$$\Big|2|zw| - |z^2+w^2-\xi^2|\Big| \le 3r|z+w-\xi|$$

for $z, w, \xi \in D_r(0)$. This implies (a). Meanwhile, (b) is an easy consequence of the identity

$$\begin{aligned} z^3+w^3-\xi^3 &= -zw(z+w+2\xi) \\ &\qquad + (z+w-\xi)(w^2+z^2+\xi^2+z\xi+w\xi). \end{aligned}$$

The proof is complete. □

Lemma 3.3. *Let $\eta > 0$. Then there is a constant $C = C(\eta) > 0$ such that*

$$\left|\frac{z^3}{(1-z)^\eta} + \frac{w^3}{(1-w)^\eta} - \frac{\xi^3}{(1-\xi)^\eta}\right| \le C\left(r^2|z+w-\xi| + r|zw|\right)$$

for all $0 < r < \frac{1}{3}$ and $z, w, \xi \in D_r(0)$.

Proof. Let $0 < r < \frac{1}{3}$ and consider arbitrary $z, w, \xi \in D_r(0)$. Setting

$$\begin{aligned} A_0 &:= z^3+w^3-\xi^3, \\ A_1 &:= \frac{1-(1-z)^\eta}{(1-w)^\eta} w^3, \\ A_2 &:= \left[1 - \frac{1}{(1-w)^\eta}\right](w^3-\xi^3), \\ A_3 &:= \left[\frac{(1-z)^\eta}{(1-\xi)^\eta} - \frac{1}{(1-w)^\eta}\right]\xi^3, \end{aligned}$$

we note

$$\frac{z^3}{(1-z)^\eta} + \frac{w^3}{(1-w)^\eta} - \frac{\xi^3}{(1-\xi)^\eta} = \frac{A_0 - A_1 - A_2 - A_3}{(1-z)^\eta}$$

and hence

$$\left|\frac{z^3}{(1-z)^\eta} + \frac{w^3}{(1-w)^\eta} - \frac{\xi^3}{(1-\xi)^\eta}\right| \lesssim |A_0| + |A_1| + |A_2| + |A_3|. \tag{3.2}$$

For A_0, we have by Lemma 3.2(b)

$$|A_0| \lesssim r^2|z+w-\xi| + r|zw|. \tag{3.3}$$

For A_1, since $|1-(1-z)^\eta| \lesssim |z|$, we have

$$|A_1| \lesssim |zw^3| \le r^2|zw|. \tag{3.4}$$

For A_2, we have

$$\begin{aligned}|A_2| &= \frac{|(1-w)^\eta - 1||w^3-\xi^3|}{|1-w|^\eta}\\ &\lesssim r^2|w||w-\xi|\\ &\le r^2(|z+w-\xi|+|zw|). \qquad (3.5)\end{aligned}$$

For A_3, since $|z+w-zw| \le 2r+r^2 < \frac{7}{9}$, we have

$$\begin{aligned}|(1-z)^\eta(1-w)^\eta - (1-\xi)^\eta| &= |(1-z-w+zw)^\eta - (1-\xi)^\eta|\\ &\lesssim |z+w-zw-\xi|\end{aligned}$$

and thus

$$\begin{aligned}|A_3| &= |\xi|^3\left|\frac{(1-z)^\eta(1-w)^\eta-(1-\xi)^\eta}{(1-\xi)^\eta(1-w)^\eta}\right|\\ &\lesssim r^3|z+w-zw-\xi|\\ &\le r^3(|z+w-\xi|+|zw|). \qquad (3.6)\end{aligned}$$

One may check that all the constants suppressed so far depend only on η. So, inserting the estimates (3.3), (3.4), (3.5) and (3.6) into (3.2), we conclude the lemma. ☐

Lemma 3.4. *Let $\eta > 0$. Given $z, w, \xi \in \mathbf{D}$, put*

$$\begin{aligned}A_0 &:= 1 - \frac{1}{(1-z)^\eta} - \frac{1}{(1-w)^\eta} + \frac{1}{(1-\xi)^\eta},\\ A_1 &:= z+w-\xi,\\ A_2 &:= z^2+w^2-\xi^2.\end{aligned}$$

Then there is a constant $C = C(\eta) > 0$ such that

$$\left|A_0 + \eta A_1 + \frac{\eta(\eta+1)}{2}A_2\right| \le C\left(r^2|z+w-\xi| + r|zw|\right)$$

for all $0 < r \le \frac{1}{3}$ and $z, w, \xi \in D_r(0)$.

Proof. Let $0 < r \le \frac{1}{3}$ and consider arbitrary $z, w, \xi \in D_r(0)$. Put

$$g(\lambda) := 1 - \frac{1}{(1-\lambda z)^\eta} - \frac{1}{(1-\lambda w)^\eta} + \frac{1}{(1-\lambda\xi)^\eta}$$

for $\lambda \in \overline{\mathbf{D}}$. Clearly, g is holomorphic in an open set containing $\overline{\mathbf{D}}$. Applying integration by parts successively, we note

$$g(1) = g(0) + g'(0) + \frac{g''(0)}{2} + \frac{1}{2}\int_0^1 (1-\lambda)^2 g'''(\lambda)\,d\lambda.$$

Noting that $g(1) = A_0$, $g(0) = 0$, $g'(0) = -\eta A_1$ and $g''(0) = -\eta(\eta+1)A_2$, we obtain

$$\left|A_0 + \eta A_1 + \frac{\eta(\eta+1)}{2}A_2\right| \le \frac{1}{2}\max_{0\le\lambda\le1}|g'''(\lambda)|. \qquad (3.7)$$

In conjunction with this, we note

$$g'''(\lambda) = -\eta(\eta+1)(\eta+2)\left[\frac{z^3}{(1-\lambda z)^{\eta+3}} + \frac{w^3}{(1-\lambda w)^{\eta+3}} - \frac{\xi^3}{(1-\lambda\xi)^{\eta+3}}\right].$$

Thus, for $0 \le \lambda \le 1$, we have by Lemma 3.3

$$|\lambda^3 g'''(\lambda)| \le C[(r\lambda)^2|\lambda z + \lambda w - \lambda\xi| + r\lambda|(\lambda z)(\lambda w)|]$$

for some constant $C = C(\eta) > 0$, i.e.,

$$|g'''(\lambda)| \le C(r^2|z + w - \xi| + r|zw|).$$

This, together with (3.7), yields the asserted inequality. The proof is complete. □

We are now ready to prove the following optimal norm estimate for double differences of kernel functions when the points are close to one another.

Theorem 3.5. *Let $\alpha > -1$ and $0 < s < 1$. Given $z_1, z_2, z_3, z_4 \in \mathbf{D}$, put*

$$A := \frac{|z_1 - z_2 - z_3 + z_4|}{\sum_{j=1}^4 (1 - |z_j|^2)}$$

and

$$B := [\rho(z_1, z_2) + \rho(z_3, z_4)][\rho(z_1, z_3) + \rho(z_2, z_4)].$$

Then there exists $r = r(\alpha, s) \in (0, 1)$ such that

$$\|K_{z_1} - K_{z_2} - K_{z_3} + K_{z_4}\|_{A^2_\alpha} \approx (A + B)\sum_{j=1}^4 \|K_{z_j}\|_{A^2_\alpha}$$

whenever $1 - |z_j| \le r$ and $\rho(z_i, z_j) \le s$ for $i, j = 1, 2, 3, 4$; the constants suppressed in this estimate depend only on α and s.

Proof. Throughout the proof we use the notation

$$c_1 = c_4 := 1 \text{ and } c_2 = c_3 := -1$$

for simplicity. For a large number $N = N(\alpha, s) > 4$ to be specified later, put

$$r := \frac{1}{4N\sqrt{N}}. \tag{3.8}$$

With such r, note $|z_j| > \frac{1}{2}$.

Consider arbitrary $z_1, z_2, z_3, z_4 \in \mathbf{D}$ such that $\rho(z_i, z_j) \le s$ for all i, j with $i \ne j$. We also assume $1 - |z_j| \le r$ for all j. By symmetry and (2.3) it suffices to establish

$$\frac{\|\sum_{j=1}^4 c_j K_{z_j}\|_{A^2_\alpha}}{\sum_{j=1}^4 \|K_{z_j}\|_{A^2_\alpha}} \approx \frac{|\sum_{j=1}^4 c_j z_j|}{1 - |z_1|^2} + \rho(z_1, z_2)\rho(z_1, z_3) \tag{3.9}$$

where the hidden constants depend only on α and s.

First, we proceed to establish the lower estimate. Put

$$f := \sum_{j=1}^{4} c_j K_{z_j} \quad \text{and} \quad f_j(z) := \frac{z(\overline{z_j} - \overline{z_1})}{1 - z\overline{z_1}}$$

for $j = 1, \dots, 4$. Note $f_1 \equiv 0$ and

$$f(z) = K_{z_1}(z) \sum_{j=1}^{4} \frac{c_j}{(1 - f_j(z))^{\alpha+2}} \tag{3.10}$$

for $z \in \mathbf{D}$.

Put

$$b_i := \left(1 - M_i N(1 - |z_1|^2)\right) z_1 \qquad (i = 1, 2) \tag{3.11}$$

where $M_1 := 1$ and $M_2 := \sqrt{N}$. Note

$$1 - |b_i| = (1 - |z_1|)\left[1 + M_i N|z_1|(1 + |z_1|)\right]$$

and

$$1 - b_i\overline{z_1} = (1 - |z_1|^2)(1 + M_i N|z_1|^2) \tag{3.12}$$

for each i. This yields $b_i \in \mathbf{D}$ and

$$1 - |b_i| \approx M_i N(1 - |z_1|^2) \approx 1 - b_i\overline{z_1} \tag{3.13}$$

for $i = 1, 2$; the constants suppressed here are absolute. Combining this with (2.5), we obtain

$$\begin{aligned}\frac{\|f\|_{A^2_\alpha}}{\|K_{z_1}\|_{A^2_\alpha}} &\gtrsim \left[(1 - |b_i|^2)(1 - |z_1|^2)\right])^{\alpha/2+1}|f(b_i)| \\ &\approx \frac{1}{(M_i N)^{\alpha/2+1}} \frac{|f(b_i)|}{|K_{z_1}(b_i)|} \\ &\ge \frac{1}{(N\sqrt{N})^{\alpha/2+1}} \frac{|f(b_i)|}{|K_{z_1}(b_i)|}\end{aligned}$$

for $i = 1, 2$. Since

$$\|K_{z_1}\|_{A^2_\alpha} \approx \sum_{j=1}^{4} \|K_{z_j}\|_{A^2_\alpha} \text{ by (2.3),} \tag{3.14}$$

we obtain

$$\|f\|_{A^2_\alpha} \left(\sum_{j=1}^{4} \|K_{z_j}\|_{A^2_\alpha} \right)^{-1} \gtrsim \frac{1}{(N\sqrt{N})^{\alpha/2+1}} \sum_{i=1}^{2} \frac{|f(b_i)|}{|K_{z_1}(b_i)|}; \tag{3.15}$$

the constants suppressed above depend only on α and s.

In order to estimate $\frac{|f(b_i)|}{|K_{z_1}(b_i)|}$ in (3.15) we first note

$$f_j(b_1) = \frac{b_1(\overline{z_j} - \overline{z_1})}{1 - b_1\overline{z_1}} = \frac{b_1(\overline{z_j} - \overline{z_1})}{(1 - |z_1|^2)(1 + N|z_1|^2)}$$

and thus

$$|f_j(b_1)| = \frac{|b_1|\rho(z_1,z_j)|1-\overline{z_1}z_j|}{(1+N|z_1|^2)(1-|z_1|^2)}. \tag{3.16}$$

We also have

$$\frac{f_j(b_2)}{f_j(b_1)} = \frac{1+N|z_1|^2}{1+N\sqrt{N}|z_1|^2}\cdot\frac{1-N\sqrt{N}(1-|z_1|^2)}{1-N(1-|z_1|^2)} < 1, \tag{3.17}$$

So, since $|z_1| \geq \frac{1}{2}$, we have by (3.16) and (2.4)

$$|f_j(b_2)| < |f_j(b_1)| \leq \frac{4}{N(1-s)}\rho(z_1,z_j) \tag{3.18}$$

for all j. In conjunction with this we take N so large that

$$N \geq \frac{12}{1-s} \tag{3.19}$$

so that $|f_j(b_i)| \leq \frac{1}{3}$ by (3.18) for all i and j. Accordingly, setting $p_1 := \alpha+2$ and $p_2 := (\alpha+2)(\alpha+3)/2$, we note by Lemma 3.4

$$\begin{aligned}\left|\frac{f(b_i)}{K_{z_1}(b_i)}\right| &= \left|\sum_{j=1}^{4}\frac{c_j}{\big(1-f_j(b_i)\big)^{\alpha+2}}\right| \\ &\geq \left|\sum_{k=1}^{2}p_k\sum_{j=2}^{4}c_j[f_j(b_i)]^k\right| \\ &\qquad - \frac{C}{N(1-s)}\left(\left|\sum_{j=2}^{4}c_jf_j(b_i)\right| + \Big|f_2(b_i)f_3(b_i)\Big|\right)\end{aligned} \tag{3.20}$$

for $i = 1,2$ and for some constant $C = C(\alpha) > 0$.

For the last term in (3.20), we note

$$\Big|f_2(b_2)f_3(b_2)\Big| \leq \Big|f_2(b_1)f_3(b_1)\Big|.$$

by (3.17). We now consider the remaining terms in(3.20). To begin with, we note $1-|z_1|^2 \leq 2r = \frac{1}{2N\sqrt{N}}$. So, in conjunction with (3.17), put

$$\lambda(t) := \frac{1+Nt}{1+N\sqrt{N}t}\cdot\frac{1-N\sqrt{N}(1-t)}{1-N(1-t)}, \qquad 1-\frac{1}{2N\sqrt{N}} \leq t \leq 1.$$

For t as above, we have

$$\frac{1}{2} \leq 1-N\sqrt{N}(1-t) \leq 1-N(1-t) \leq 1$$

and thus

$$\begin{aligned}\lambda(t)[1-\lambda(t)] &\geq \frac{1+Nt}{2(1+N\sqrt{N}t)}\cdot\frac{N(\sqrt{N}-1)}{(1+N\sqrt{N}t)(1-N(1-t))}\\ &\geq \frac{1+N}{2(1+N\sqrt{N})}\cdot\frac{N(\sqrt{N}-1)}{1+N\sqrt{N}}\\ &\geq \frac{N^2(\sqrt{N}-1)}{8N^3}\\ &> \frac{1}{16\sqrt{N}}. \end{aligned}\tag{3.21}$$

Accordingly, setting $\lambda_1(t)\equiv 1$ and $\lambda_2(t)=\lambda(t)$, we note from the equality in (3.17)

$$\begin{aligned}&\sum_{i=1}^{2}\left|\sum_{k=1}^{2}p_k\sum_{j=2}^{4}c_j[f_j(b_i)]^k\right|\\ &\quad=\sum_{i=1}^{2}\left|\sum_{k=1}^{2}p_k\left(\sum_{j=2}^{4}c_j[f_j(b_1)]^k\right)\lambda_i^k(|z_1|^2)\right|\\ &\quad\geq\frac{1}{32\sqrt{N}}\sum_{k=1}^{2}\left|\sum_{j=2}^{4}c_j[f_j(b_1)]^k\right|\qquad\text{by Lemma 3.1 and (3.21)}\\ &\quad\approx\frac{1}{\sqrt{N}}\left(\left|\sum_{j=2}^{4}c_jf_j(b_1)\right|+\left|f_2(b_1)f_3(b_1)\right|\right)\qquad\text{by Lemma 3.2(a);}\end{aligned}$$

the constants hidden in the last estimate are absolute (recall $|f_j(b_1)|\leq\frac{1}{3}$). Meanwhile, we have

$$\left|\sum_{j=2}^{4}c_jf_j(b_i)\right|=\frac{|z_1-z_2-z_3+z_4|}{1-|z_1|^2}\cdot\frac{|b_i|}{1+M_iN|z_1|^2}\leq\left|\sum_{j=2}^{4}c_jf_j(b_1)\right|\tag{3.22}$$

for $i=1,2$; note $|b_i|\leq|b_1|$ for the last inequality.

Combining the estimates observed in the preceding paragraph, we obtain

$$\sum_{i=1}^{2}\left|\frac{f(b_i)}{K_{z_1}(b_i)}\right|\gtrsim\frac{1}{\sqrt{N}}\left(1-\frac{C_1}{\sqrt{N}}\right)\left(\left|\sum_{j=2}^{4}c_jf_j(b_1)\right|+\left|f_2(b_1)f_3(b_1)\right|\right)\tag{3.23}$$

for some constant C_1 depending only on α and s; the constant hidden in this estimate is absolute. Note from (3.22)

$$\left|\sum_{j=2}^{4}c_jf_j(b_1)\right|\approx\frac{|z_1-z_2-z_3+z_4|}{1-|z_1|^2};\tag{3.24}$$

the constants suppressed above depend only on N. Finally, in addition to (3.19), requiring N to be so large that

$$\frac{C_1}{\sqrt{N}} \leq \frac{1}{2},$$

we conclude the lower estimate in (3.9) by (3.24), (3.23), (3.18) and (3.15), as required.

We now consider the upper estimate in (3.9). By symmetry we may assume

$$\rho(z_1, z_2) \leq \rho(z_1, z_3).$$

Put

$$\tau := \frac{(1-s)^2}{8} \quad \text{and} \quad \eta := \frac{\tau^2}{1+\tau}.$$

Consider the following subcases:

(i) $\rho(z_1, z_2) \geq \eta$;
(ii) $\tau\rho(z_1, z_2) \geq \rho(z_3, z_4)$ or $\tau\rho(z_3, z_4) \geq \rho(z_1, z_2)$;
(iii) $\tau\rho(z_1, z_2) \leq \rho(z_3, z_4) \leq \frac{1}{\tau}\rho(z_1, z_2)$ and $\rho(z_1, z_2) \leq \eta \leq \rho(z_1, z_3)$;
(iv) $\tau\rho(z_1, z_2) \leq \rho(z_3, z_4) \leq \frac{1}{\tau}\rho(z_1, z_2)$ and $\rho(z_1, z_3) \leq \eta$.

Now, we will save notation by setting

$$G := K_{z_1} - K_{z_2} - K_{z_3} + K_{z_4} \tag{3.25}$$

for the rest of the proof.

In Case (i), we have $\rho(z_1, z_2)\rho(z_1, z_3) \geq \eta^2$ and thus

$$\|G\|_{A^2_\alpha} \leq \frac{\rho(z_1, z_2)\rho(z_1, z_3)}{\eta^2} \sum_{j=1}^{4} \|K_{z_j}\|_{A^2_\alpha},$$

as desired.

Consider Case (ii). First, assume $\tau\rho(z_1, z_2) \geq \rho(z_3, z_4)$. We then have

$$\begin{aligned} \frac{|z_1 - z_2 - z_3 + z_4|}{1 - |z_1|^2} &\geq \frac{|z_1 - z_2|}{1 - |z_1|^2} - \frac{|z_3 - z_4|}{1 - |z_3|^2} \cdot \frac{1 - |z_3|^2}{1 - |z_1|^2} \\ &\geq \frac{\rho(z_1, z_2)}{1+s} - \frac{\rho(z_3, z_4)}{1-s} \cdot \frac{2}{1-s} \quad \text{by (2.2) and (2.3)} \\ &\geq \frac{\rho(z_1, z_2)}{2} \left[1 - \frac{4\tau}{(1-s)^2}\right] \\ &= \frac{\rho(z_1, z_2)}{4}. \end{aligned}$$

It follows that

$$\begin{aligned}\|G\|_{A^2_\alpha} &\le \|K_{z_1} - K_{z_2}\|_{A^2_\alpha} + \|K_{z_3} - K_{z_4}\|_{A^2_\alpha}\\ &\lesssim (1+\tau)\rho(z_1,z_2)\sum_{j=1}^4 \|K_{z_j}\|_{A^2_\alpha} \quad \text{by (2.8)}\\ &\le 4(1+\tau)\frac{|z_1 - z_2 - z_3 + z_4|}{1-|z_1|^2}\sum_{j=1}^4 \|K_{z_j}\|_{A^2_\alpha},\end{aligned}$$

as desired. Since $1-|z_1| \approx 1-|z_4|$ by (2.3), the case $\tau\rho(z_3,z_4) \ge \rho(z_1,z_2)$ can be treated similarly.

In Case(iii), we have

$$\begin{aligned}\rho(z_1,z_2)+\rho(z_3,z_4) &\le \left(1+\frac{1}{\tau}\right)\rho(z_1,z_2)\\ &\le \frac{1}{\eta}\left(1+\frac{1}{\tau}\right)\rho(z_1,z_2)\rho(z_1,z_3)\end{aligned}$$

and thus

$$\begin{aligned}\|G\|_{A^2_\alpha} &\le \|K_{z_1} - K_{z_2}\|_{A^2_\alpha} + \|K_{z_3} - K_{z_4}\|_{A^2_\alpha}\\ &\lesssim [\rho(z_1,z_2)+\rho(z_3,z_4)]\sum_{j=1}^4 \|K_{z_j}\|_{A^2_\alpha} \quad \text{by (2.8)}\\ &\le \frac{1}{\eta}\left(1+\frac{1}{\tau}\right)\rho(z_1,z_2)\rho(z_1,z_3)\sum_{j=1}^4 \|K_{z_j}\|_{A^2_\alpha},\end{aligned}$$

as desired.

Finally, consider Case (iv). Note from (2.4)

$$|f_j(z)| \le \rho(z_1,z_j)\frac{|z||1-z_1\overline{z_j}|}{1-|z_1|} \le \frac{2}{1-s}\rho(z_1,z_j) \tag{3.26}$$

for $z \in \mathbf{D}$. So, since $\eta < \tau$ and

$$\rho(z_1,z_4) \le \rho(z_1,z_3)+\rho(z_3,z_4) \le \eta + \frac{\eta}{\tau} = \tau,$$

we obtain

$$|f_j(z)| \le \frac{2\tau}{1-s} \le \frac{1}{4}$$

for all $z \in \mathbf{D}$ and j. Thus, applying Lemma 3.4, we obtain

$$\begin{aligned}\left|\frac{f(z)}{K_{z_1}(z)}\right| &= \left|\sum_{j=1}^{4} \frac{c_j}{\big(1-f_j(z)\big)^{\alpha+2}}\right| \\ &\lesssim \sum_{k=1}^{2}\left|\sum_{j=2}^{4} c_j[f_j(z)]^k\right| + \left(\left|\sum_{j=2}^{4} c_j f_j(z)\right| + \Big|f_2(z)f_3(z)\Big|\right) \\ &\lesssim \left|\sum_{j=2}^{4} c_j f_j(z)\right| + \Big|f_2(z)f_3(z)\Big|\end{aligned}$$

for all $z \in \mathbf{D}$; the last inequality holds by Lemma 3.2(a). Moreover, we have

$$f_j(z) = \frac{z}{b_1}\cdot\frac{1-b_1\overline{z_1}}{1-z\overline{z_1}} f_j(b_1)$$

and

$$\left|\frac{z}{b_1}\cdot\frac{1-b_1\overline{z_1}}{1-z\overline{z_1}}\right| \le \frac{|1-b_1\overline{z_1}|}{|b_1|(1-|z_1|)} \approx 1;$$

the last estimate holds by (3.12). Combining these observations, we obtain

$$\left|\frac{f(z)}{K_{z_1}(z)}\right| \lesssim \left|\sum_{j=2}^{4} c_j f_j(b_1)\right| + \Big|f_2(b_1)f_3(b_1)\Big|$$

for all $z \in \mathbf{D}$. This, together with (3.24) and (3.18), implies the upper estimate in (3.9). The proof is complete. □

Remark 3.6. As a byproduct of the proof of Theorem 3.5, we see that the second integral in (1.2) can be replaced by a bit simpler one. Namely, by (3.9) and (2.3), we can replace R_3 in (1.2) by the quantity

$$\frac{|\varphi_1-\varphi_2-\varphi_3+\varphi_4|}{1-|\varphi_1|^2} + \rho_{12}\rho_{13}.$$

While this quantity is not symmetric, it might be more useful in applications because of its simplicity.

When $z_1 = z_4$, we have two additional optimal estimates as in the next theorem.

Theorem 3.7. *Let $\alpha > -1$ and $0 < s < 1$. Given $z_1, z_2, z_3 \in \mathbf{D}$, put*

$$A_1 := \frac{|2z_1-z_2-z_3|}{1-|z_1|^2} \quad \textit{and} \quad A_2 := |\sigma(z_1,z_2)+\sigma(z_1,z_3)|\,.$$

Let $A = A_1$ or A_2. Then there exists $r = r(s) > 0$ such that

$$\|2K_{z_1}-K_{z_2}-K_{z_3}\|_{A^2_\alpha} \approx \Big[A+\rho^2(z_1,z_2)+\rho^2(z_1,z_3)\Big]\sum_{j=1}^{3}\|K_{z_j}\|_{A^2_\alpha}$$

whenever $1-|z_j| \le r$ and $\rho(z_i,z_j) \le s$ for $i,j = 1,2,3$ with $i \ne j$; the constants suppressed above depend only on α and s.

Proof. Consider arbitrary $z_1, z_2, z_3 \in \mathbf{D}$ such that $\rho(z_i, z_j) \le s$ for all $i \ne j$. The estimate for $A = A_1$ is immediate from Theorem 3.5 (with $z_1 = z_4$) and (2.3).

For the case $A = A_2$, we note

$$\frac{2z_1 - z_2 - z_3}{1 - |z_1|^2} - [\sigma(z_1, z_2) + \sigma(z_1, z_3)] = \sum_{j=2}^{3} \sigma(z_1, z_j) \frac{\overline{z_1}(z_1 - z_j)}{1 - |z_1|^2} \tag{3.27}$$

and thus by (2.2)

$$|A_1 - A_2| \le \frac{1}{1-s} \sum_{j=2}^{3} \rho^2(z_1, z_j),$$

which yields

$$A_1 + \sum_{j=2}^{3} \rho^2(z_1, z_j) \approx A_2 + \sum_{j=2}^{3} \rho^2(z_1, z_j).$$

So, we also conclude the estimate for $A = A_2$. The proof is complete. □

4. The ratio estimate: Part II

In Theorem 3.5 we established optimal estimates for the ratio (1.5) when the points are close to one another. In this section we complete such estimates for the remaining case, which are not covered by Theorem 3.5. We will split the remaining case into four subcases and treat them one by one in separate lemmas.

Before proceeding, we first fix some notation. Given a positive number $N \ge 2$, we put

$$a_N := [1 - N(1 - |a|^2)]a$$

for $a \in \mathbf{D}$ with $2N(1 - |a|^2) < 1$. For easier reference later, we note

$$1 - a_N \overline{a} = (1 - |a|^2)(1 + N|a|^2). \tag{4.1}$$

Meanwhile, since

$$1 - |a_N| = (1 - |a|)\big[1 + N|a|(1 + |a|)\big] \ge 1 - |a|,$$

we note $|a_N| \le |a|$ and thus

$$1 - |a_N|^2 \ge 1 - |a|^2. \tag{4.2}$$

Also, since $N \ge 2$, we note $|a| > \frac{1}{2}$ and hence $|a_N| > \frac{1}{4}$.

The following technical lemma will play essential roles in the proofs of subsequent four lemmas.

Lemma 4.1. *Let $\alpha > -1$, $N \ge 2$ and $0 < \epsilon < \delta < 1$. Then the following assertions hold for $a, z, w \in \mathbf{D}$ with $2N(1 - |a|^2) < 1$ and for some constant $c_\alpha > 0$ depending only on α.*

(a) *If $N(1 - \epsilon) \ge 6$ and $\rho(a, z) \le \epsilon$, then*

$$\frac{1}{c_\alpha} \cdot \frac{\rho(a, z)}{N} \le \left| 1 - \frac{K_z(a_N)}{K_a(a_N)} \right| \le c_\alpha \cdot \frac{\rho(a, z)}{N(1 - \epsilon)}.$$

If, in addition, $N(1-\epsilon) \geq 2c_\alpha$, *then*

$$\left|1 + \frac{K_z(a_N)}{K_a(a_N)}\right| \geq \frac{3}{2}.$$

(b) *If* $|z| \leq |a|$ *and*

$$\frac{N\sqrt{1-\rho(a,z)}}{|a_N|\rho(a,z)} \leq \frac{1}{8}, \tag{4.3}$$

then

$$\left|\frac{K_z(a_N)}{K_a(a_N)}\right| \leq c_\alpha \left(\frac{N\sqrt{1-\rho(a,z)}}{|a_N|\rho(a,z)}\right)^{\alpha+2}.$$

(c) *If* $|w| \leq |a|$, $\rho(z,w) \leq \epsilon$, $\rho(a,w) \geq \delta$, $M \geq 5$ *and*

$$\frac{1-|w|^2}{1-|a|^2} \leq M \leq \left(\frac{1-\epsilon}{\sqrt{1-\delta}}\right)^{1/2}, \tag{4.4}$$

then

$$\left|1 - \frac{K_w(a_N)}{K_z(a_N)}\right| \leq c_\alpha \rho(z,w) \left(\frac{\sqrt{1-\delta}}{1-\epsilon}\right)^{1/2}.$$

Proof. Consider $a \in \mathbf{D}$ with $2N(1-|a|^2) < 1$ and let $z, w \in \mathbf{D}$. For simplicity we put $b := a_N$ throughout the proof. Recall $|a| > \frac{1}{2}$ and $|a_N| > \frac{1}{4}$.

First, we prove (a). To begin with, we note

$$\frac{K_b(z)}{K_b(a)} = \left[1 - \frac{\overline{b}(a-z)}{1-z\overline{b}}\right]^{\alpha+2} = \left[1 - \overline{b}\sigma(a,z)\cdot\frac{1-z\overline{a}}{1-z\overline{b}}\right]^{\alpha+2}. \tag{4.5}$$

In order to see whether the binomial expansion is legitimate for the last expression, we need to estimate the size of $|\frac{1-z\overline{a}}{1-z\overline{b}}|$. Assuming $\rho(a,z) \leq \epsilon$, we have by (2.2)

$$|z| \geq |a| - |z-a| \geq |a|^2 - \frac{1-|a|^2}{1-\epsilon} = 1 - (1-|a|^2)\left(1 + \frac{1}{1-\epsilon}\right)$$

and thus

$$|z| \geq 1 - \frac{1}{2N}\left(1 + \frac{1}{1-\epsilon}\right) \geq 1 - \frac{1}{N(1-\epsilon)}.$$

It follows that

$$N \geq \frac{|z||a-b|}{1-|a|^2} = |z||a|N \geq \frac{1}{2}\left(N - \frac{1}{1-\epsilon}\right).$$

This, together with (2.4), yields

$$\frac{|1-z\overline{b}|}{1-|a|^2} = \frac{|1-z\overline{a}+z(\overline{a}-\overline{b})|}{1-|a|^2} \leq \frac{|z||a-b|}{1-|a|^2} + \frac{|1-z\overline{a}|}{1-|a|^2} \leq N + \frac{1}{1-\epsilon}$$

and

$$\frac{|1-z\overline{b}|}{1-|a|^2} \geq \frac{|z||a-b|}{1-|a|^2} - \frac{|1-z\overline{a}|}{1-|a|^2} \geq \frac{1}{2}\left(N - \frac{3}{1-\epsilon}\right).$$

Now, further assuming $N(1-\epsilon) \ge 6$ and applying (2.4) once more, we obtain

$$\left|\frac{1-z\overline{a}}{1-z\overline{b}}\right| \ge \frac{1}{1+\epsilon} \cdot \frac{1-|a|^2}{|1-z\overline{b}|} \ge \frac{1}{2N} \cdot \frac{N(1-\epsilon)}{N(1-\epsilon)+1} \ge \frac{3}{7N} \tag{4.6}$$

and

$$\left|\frac{1-z\overline{a}}{1-z\overline{b}}\right| \le \frac{1}{1-\epsilon} \cdot \frac{1-|a|^2}{|1-z\overline{b}|} \le \frac{2}{N(1-\epsilon)-3} \le \frac{4}{N(1-\epsilon)}. \tag{4.7}$$

Note

$$\left|\overline{b}\sigma(a,z) \cdot \frac{1-z\overline{a}}{1-z\overline{b}}\right| = |b|\rho(a,z)\left|\frac{1-z\overline{a}}{1-z\overline{b}}\right| \le \frac{2}{3}$$

by (4.7). Accordingly, we may consider the binomial expansion of (4.5) to see that

$$\left|1-\frac{K_z(b)}{K_a(b)}\right| = \left|1-\frac{K_b(z)}{K_b(a)}\right| \approx \left|\sigma(a,z)\frac{\overline{b}(1-z\overline{a})}{1-z\overline{b}}\right| \approx \rho(a,z)\left|\frac{1-z\overline{a}}{1-z\overline{b}}\right|;$$

the constants suppressed in these estimates depend only on α. So, combining this with (4.6) and (4.7), we conclude the first part of (a). Since

$$\left|1+\frac{K_z(b)}{K_a(b)}\right| = \left|2+\left(\frac{K_z(b)}{K_a(b)}-1\right)\right| \ge 2-\left|1-\frac{K_z(b)}{K_a(b)}\right|,$$

one may see that the second part is immediate from the first part.

Next, we prove (b). Note

$$\frac{K_z(b)}{K_a(b)} = \left(\frac{1-b\overline{a}}{1-b\overline{z}}\right)^{\alpha+2}. \tag{4.8}$$

To estimate the size of the right-hand side of the above, we note

$$\frac{1-b\overline{z}}{1-b\overline{a}} = 1+\frac{b(\overline{a}-\overline{z})}{1-b\overline{a}} = 1+\frac{b}{1+|a|^2N} \cdot \frac{\overline{a}-\overline{z}}{1-a\overline{z}} \cdot \frac{1-a\overline{z}}{1-|a|^2}.$$

So, assuming $|z| \le |a|$ and (4.3), we obtain

$$\begin{aligned}\left|\frac{1-b\overline{z}}{1-b\overline{a}}\right| &\ge \frac{|b|\rho(a,z)}{2N} \cdot \frac{|1-a\overline{z}|}{1-|a|^2} - 1\\ &\ge \frac{|b|\rho(a,z)}{4N\sqrt{1-\rho(a,z)}} - 1 \qquad \text{by (2.1)}\\ &\ge \frac{|b|\rho(a,z)}{8N\sqrt{1-\rho(a,z)}} \qquad \text{by (4.3).}\end{aligned}$$

This, together with (4.8), yields the asserted inequality.

Finally, we prove (c). To begin with, we note

$$\frac{K_b(w)}{K_b(z)} = \left[1-\frac{\overline{b}(z-w)}{1-w\overline{b}}\right]^{\alpha+2} \tag{4.9}$$

as in the proof of (a). In conjunction with this representation, we note

$$\frac{z-w}{1-w\overline{b}} = \sigma(z,w)\left(\frac{1-w\overline{z}}{1-w\overline{a}}\right)\left(\frac{1-w\overline{a}}{1-w\overline{b}}\right).$$

Now, assume $|w| \le |a|$, $\rho(z,w) \le \epsilon$, $\rho(a,w) \ge \delta$ and (4.4). Since $|w| \le |a|$, we have

$$|1 - w\overline{z}| \cdot \frac{1}{|1 - w\overline{a}|} \le \frac{1 - |w|^2}{1-\epsilon} \cdot \frac{2\sqrt{1-\delta}}{1-|a|^2} \le 2\left(\frac{\sqrt{1-\delta}}{1-\epsilon}\right)^{1/2};$$

the first inequality holds by (2.4) and (2.1), and the second one by (4.4). We also have

$$\left|\frac{1-w\overline{a}}{1-w\overline{b}}\right| = \left|1 - \frac{w(\overline{a}-\overline{b})}{1-w\overline{b}}\right| \le 1 + \frac{|a-b|}{1-|b|} = 1 + \frac{|a|N(1+|a|)}{1+N(1+|a|)} < 2.$$

Combining these observations, we obtain by the second inequality of (4.4)

$$\left|\frac{\overline{b}(z-w)}{1-w\overline{b}}\right| \le \left|\frac{z-w}{1-w\overline{b}}\right| \le 4\left(\frac{\sqrt{1-\delta}}{1-\epsilon}\right)^{1/2} \le \frac{4}{M} \le \frac{4}{5}$$

for $M \ge 5$. So, considering the binomial expansion of (4.9), we conclude the asserted inequality as in the proof of (a). The proof is complete. □

We now introduce several auxiliary notation and auxiliary numbers, which will be used repeatedly in the rest of the paper.

Auxiliary notation. Let $\alpha > -1$. Given $z_1, z_2, z_3, z_4 \in \mathbf{D}$, we continue using the notation G specified in (3.25). We also put

$$Q_1 := \|K_{z_1} - K_{z_2}\|_{A^2_\alpha} + \|K_{z_3} - K_{z_4}\|_{A^2_\alpha},$$
$$Q_2 := \|K_{z_1} - K_{z_3}\|_{A^2_\alpha} + \|K_{z_2} - K_{z_4}\|_{A^2_\alpha}.$$

For these abbreviated notation, the dependency on the points z_j's (and α) should be clear from the context. Clearly, we have

$$\|G\|_{A^2_\alpha} \le \min\{Q_1, Q_2\}. \tag{4.10}$$

We will see that this estimate can be reversed, except when the underlying four points are all close enough to one another. To this end we introduce below certain positive numbers, depending only on α. For the rest of the section, these notation G, Q_1 and Q_2 will be used without any further reference.

Auxiliary numbers. Given $\alpha > -1$, let $c_\alpha > 0$ be the constant provided by Lemma 4.1 and pick $\beta_1 = \beta_1(\alpha) \in (0,1)$ such that

$$\left(2\sqrt{1-\beta_1}\right)^{\alpha+2} < \frac{1}{8}. \tag{4.11}$$

Take $N_1 = N_1(\alpha) = N_1(c_\alpha, \beta_1) \ge 2$ such that

$$N_1(1-\beta_1) \ge \max\{6, 4c_\alpha\}. \tag{4.12}$$

This N_1 will be required to have additional properties specified later in (4.29), (4.31), (4.34) and (4.35), which are conditions depending only on α (but independent of the choice of β_2 below). For such a choice of N_1, choose $\beta_2 = \beta_2(N_1) \in (\beta_1, 1)$ with the following three properties:

$$\frac{\sqrt{1-\beta_2}}{\beta_2} \le \frac{1}{4N_1^7}; \tag{4.13}$$

$$\max\left\{N_1^{2(\alpha+4)}, N_1^{12}\right\} \le \frac{1-\beta_1}{\sqrt{1-\beta_2}}; \tag{4.14}$$

$$c_\alpha \left(\frac{4N_1\sqrt{1-\beta_2}}{\beta_2}\right)^{\alpha+2} \le \frac{1}{4N_1}. \tag{4.15}$$

Once $\beta_1 \in (0,1)$ satisfying (4.11) is given, one may find $N_1 \ge 2$ and $\beta_2 \in (\beta_1, 1)$ satisfying (4.12)-(4.15). Noting $\beta_1 < \beta_2 < 1$, one may start with β_2 in place of β_1 to find $N_2 \ge N_1$ and $\beta_3 \in (\beta_2, 1)$ such that Lemmas 4.2-4.5 hold with (β_2, N_2, β_3) in place of (β_1, N_1, β_2). Repeating the same procedure, one can find (β_3, N_3, β_4). Note that each β_j can be chosen as close to 1 as we want. Fix those two triples (β_2, N_2, β_3) and (β_3, N_3, β_4) satisfying

$$2\tanh^{-1}\beta_3 \le \tanh^{-1}\beta_4; \tag{4.16}$$

this will be used in the proof of Theorem 4.6 later. For the rest of the section, these numbers c_α, β_j and N_j will be used without any further reference.

Lemma 4.2. *Let $\alpha > -1$. For $z_1, z_2, z_3, z_4 \in \mathbf{D}$, assume the following:*

(i) $|z_j| \le |z_1|$ *for* $j = 2, 3, 4$*;*
(ii) $\beta_1 \le \rho(z_1, z_j)$ *for* $j = 2, 3, 4$.

Then the estimate

$$\|G\|_{A^2_\alpha} \approx Q_1 \approx Q_2 \approx \|K_{z_1}\|_{A^2_\alpha}$$

holds; the constants suppressed above are absolute.

Proof. Using (i) and (ii), we obtain by (2.7) and (2.1)

$$\frac{|G(z_1)|}{\|K_{z_1}\|^2_{A^2_\alpha}} = \frac{|G(z_1)|}{K_{z_1}(z_1)} \ge 1 - \sum_{j=2}^{4}\left(2\sqrt{1-\beta_1}\right)^{\alpha+2} > \frac{1}{2};$$

the last inequality holds by (4.11). So, we obtain by (2.5)

$$\|G\|_{A^2_\alpha} \ge \frac{|G(z_1)|}{\|K_{z_1}\|_{A^2_\alpha}} \ge \frac{1}{2}\|K_{z_1}\|_{A^2_\alpha}.$$

Meanwhile, it is clear by (i) and (2.7) that

$$\|G\|_{A^2_\alpha} \le Q_j \le 4\|K_{z_1}\|_{A^2_\alpha} \tag{4.17}$$

for $j = 1, 2$. So, the proof is complete. □

Lemma 4.3. *Let $\alpha > -1$. For $z_1, z_2, z_3, z_4 \in \mathbf{D}$, assume the following:*

(i) $2N_1(1-|z_1|^2) < 1$ *and* $|z_j| \le |z_1|$ *for* $j = 2, 3, 4$*;*
(ii) $\rho(z_1, z_4) \le \beta_1$*;*
(iii) *either one of the following holds;*
  (iii-a) $\beta_2 \le \rho(z_1, z_2) \le \rho(z_1, z_3)$.
  (iii-b) $\rho(z_1, z_2) \le \beta_1$ *and* $\beta_2 \le \rho(z_1, z_3)$.

Then the estimate

$$\|G\|_{A^2_\alpha} \approx Q_1 \approx Q_2 \approx \|K_{z_1}\|_{A^2_\alpha}$$

holds; the constants suppressed above depend only on α.

Proof. Throughout the proof, we use the notation

$$b := (z_1)_{N_1} = \left[1 - N_1(1 - |z_1|^2)\right] z_1 \tag{4.18}$$

for simplicity. As in the proof of Lemma 4.2, it suffices to establish

$$\|G\|_{A^2_\alpha} \geq C\|K_{z_1}\|_{A^2_\alpha} \tag{4.19}$$

for some constant $C = C(\alpha) > 0$.

To begin with, we note

$$\begin{aligned}
\|G\|_{A^2_\alpha} &\geq \frac{(1-|b|^2)^{1+\alpha/2}}{|1-b\overline{z_1}|^{\alpha+2}} \cdot \frac{|G(b)|}{|K_{z_1}(b)|} && \text{by (2.5)} \\
&\geq \frac{\|K_{z_1}\|_{A^2_\alpha}}{(1+N_1|z_1|^2)^{\alpha+2}} \cdot \frac{|G(b)|}{|K_{z_1}(b)|} && \text{by (4.1) and (4.2)} \\
&\geq \frac{\|K_{z_1}\|_{A^2_\alpha}}{(2N_1)^{\alpha+2}} \cdot \frac{|G(b)|}{|K_{z_1}(b)|}.
\end{aligned} \tag{4.20}$$

In addition, we note

$$\begin{aligned}
\frac{|G(b)|}{|K_{z_1}(b)|} &\geq \left|1 + \frac{K_{z_4}(b)}{K_{z_1}(b)}\right| - \left|\frac{K_{z_2}(b)}{K_{z_1}(b)} + \frac{K_{z_3}(b)}{K_{z_1}(b)}\right| \\
&\geq \frac{3}{2} - \left|\frac{K_{z_2}(b)}{K_{z_1}(b)} + \frac{K_{z_3}(b)}{K_{z_1}(b)}\right|;
\end{aligned} \tag{4.21}$$

the second inequality holds by (4.12) and Lemma 4.1(a).

To estimate the quantity in (4.21), we first consider the case (iii-a). Note from (4.13)

$$\frac{\sqrt{1-\rho(z_1,z_j)}}{\rho(z_1,z_j)} \leq \frac{\sqrt{1-\beta_2}}{\beta_2} \leq \frac{1}{4N_1^7} < \frac{|b|}{N_1^7} \tag{4.22}$$

for $j = 2, 3$; recall $|b| > \frac{1}{4}$ for the last inequality. Thus we have by Lemma 4.1(b)

$$\begin{aligned}
\left|\frac{K_{z_2}(b)}{K_{z_1}(b)} + \frac{K_{z_3}(b)}{K_{z_1}(b)}\right| &\leq \left|\frac{K_{z_2}(b)}{K_{z_1}(b)}\right| + \left|\frac{K_{z_3}(b)}{K_{z_1}(b)}\right| \\
&\leq c_\alpha \sum_{j=2}^{3} \left(\frac{N_1\sqrt{1-\rho(z_1,z_j)}}{|b|\rho(z_1,z_j)}\right)^{2+\alpha} \\
&\leq 2c_\alpha \left(\frac{4N_1\sqrt{1-\beta_2}}{\beta_2}\right)^{2+\alpha} \\
&\leq \frac{1}{4};
\end{aligned} \tag{4.23}$$

the last inequality holds by (4.15). Combining this with (4.20) and (4.21), we obtain

$$\|G\|_{A^2_\alpha} \geq \frac{5}{4} \cdot \frac{\|K_{z_1}\|_{A^2_\alpha}}{(2N_1)^{\alpha+2}}$$

and thus conclude (4.19) for the case (iii-a).

We now consider the case (iii-b). Note that (4.22) is still valid for $j = 3$. Thus

$$\left|\frac{K_{z_3}(b)}{K_{z_1}(b)}\right| \le \frac{1}{8}.$$

We also have by Lemma 4.1(a)

$$\left|\frac{K_{z_2}(b)}{K_{z_1}(b)} - 1\right| \le c_\alpha \frac{\rho(z_1, z_2)}{N_1(1-\beta_1)} \le \frac{1}{4};$$

the last inequality holds by (4.12). It follows that

$$\left|\frac{K_{z_2}(b)}{K_{z_1}(b)} + \frac{K_{z_3}(b)}{K_{z_1}(b)}\right| \le 1 + \left|\frac{K_{z_2}(b)}{K_{z_1}(b)} - 1\right| + \left|\frac{K_{z_3}(b)}{K_{z_1}(b)}\right| \le \frac{11}{8}.$$

Combining this with (4.20) and (4.21), we obtain

$$\|G\|_{A^2_\alpha} \ge \frac{1}{8} \cdot \frac{\|K_{z_1}\|_{A^2_\alpha}}{(2N_1)^{\alpha+2}}$$

and thus conclude (4.19) for the case (iii-b). The proof is complete. □

Lemma 4.4. *Let $\alpha > -1$. For $z_1, z_2, z_3, z_4 \in \mathbf{D}$, assume the following:*

(i) $2N_1(1 - |z_1|^2) < 1$ *and* $|z_j| \le |z_1|$ *for* $j = 2, 3, 4$*;*
(ii) $\rho(z_1, z_2) \le \beta_1$*;*
(iii) $\beta_2 \le \rho(z_1, z_j)$ *for* $j = 3, 4$.

Then the estimate

$$\|G\|_{A^2_\alpha} \approx \min\{Q_1, Q_2\}$$

holds; the constants suppressed above depend only on α.

Proof. As in the proof of Lemma 4.2, it suffices to establish

$$\|G\|_{A^2_\alpha} \ge C \min\{Q_1, Q_2\} \tag{4.24}$$

for some constant $C = C(\alpha, N) = C(\alpha) > 0$.

If

$$\frac{\|K_{z_1} - K_{z_2}\|_{A^2_\alpha}}{\|K_{z_3} - K_{z_4}\|_{A^2_\alpha}} \le \frac{1}{2} \quad \text{or} \quad \frac{\|K_{z_1} - K_{z_2}\|_{A^2_\alpha}}{\|K_{z_3} - K_{z_4}\|_{A^2_\alpha}} \ge 2,$$

then we have

$$\frac{Q_1}{3} \le \left| \|K_{z_1} - K_{z_2}\|_{A^2_\alpha} - \|K_{z_3} - K_{z_4}\|_{A^2_\alpha} \right| \le \|G\|_{A^2_\alpha} \le Q_2$$

and thus (4.24) holds with $C = \frac{1}{3}$.

We now assume

$$\frac{1}{2} < \frac{\|K_{z_1} - K_{z_2}\|_{A^2_\alpha}}{\|K_{z_3} - K_{z_4}\|_{A^2_\alpha}} < 2 \tag{4.25}$$

for the rest of the proof. By symmetry we may further assume

$$|z_4| \le |z_3| \tag{4.26}$$

for the rest of the proof. Setting

$$M := \max\{N_1^{\alpha+4}, N_1^6\}$$

for short, we consider the following three subcases:

(a) $\rho(z_3, z_4) \le \beta_1$ and $\dfrac{1-|z_3|^2}{1-|z_1|^2} \le M$;

(b) $\rho(z_3, z_4) \le \beta_1$ and $\dfrac{1-|z_3|^2}{1-|z_1|^2} > M$;

(c) $\rho(z_3, z_4) > \beta_1$.

For the rest of the proof, we keep using the notation specified in (4.18).

Case (a): Since

$$|K_{z_4}(b)| = K_{z_4}(z_4)\left(\frac{1-|z_4|^2}{|1-z_4\bar{b}|}\right)^{2+\alpha} \le 2^{\alpha+2}\|K_{z_4}\|^2_{A^2_\alpha}$$

and

$$M^2 \le \frac{1-\beta_1}{\sqrt{1-\beta_2}},$$

we obtain by Lemma 4.1(c)

$$\begin{aligned}|K_{z_3}(b) - K_{z_4}(b)| &= \left|1 - \frac{K_{z_3}(b)}{K_{z_4}(b)}\right| |K_{z_4}(b)| \\ &\lesssim \rho(z_3, z_4)\left(\frac{\sqrt{1-\beta_2}}{1-\beta_1}\right)^{1/2}\|K_{z_4}\|^2_{A^2_\alpha} \\ &\le \frac{\rho(z_3, z_4)}{M}\|K_{z_1}\|_{A^2_\alpha}\|K_{z_4}\|_{A^2_\alpha};\end{aligned}$$

recall $|z_4| \le |z_1|$ for the last inequality. Meanwhile, since

$$|K_{z_1}(b)| \ge \frac{\|K_{z_1}\|^2_{A^2_\alpha}}{(2N_1)^{\alpha+2}}$$

by (4.1), we note from (4.12) and Lemma 4.1(a) that

$$\begin{aligned}|K_{z_1}(b) - K_{z_2}(b)| &= |K_{z_1}(b)|\left|1 - \frac{K_{z_2}(b)}{K_{z_1}(b)}\right| \\ &\gtrsim \frac{\rho(z_1, z_2)}{N_1^{\alpha+3}}\|K_{z_1}\|^2_{A^2_\alpha}.\end{aligned} \tag{4.27}$$

Combining these observations and using $M \ge N_1^{\alpha+4}$, we obtain

$$\begin{aligned}|G(b)| &\ge |K_{z_1}(b) - K_{z_2}(b)| - |K_{z_3}(b) - K_{z_4}(b)| \\ &\gtrsim \frac{\|K_{z_1}\|_{A^2_\alpha}}{N_1^{\alpha+3}}\left[\rho(z_1, z_2)\|K_{z_1}\|_{A^2_\alpha} - \frac{C}{N_1}\rho(z_3, z_4)\|K_{z_4}\|_{A^2_\alpha}\right]\end{aligned} \tag{4.28}$$

where $C > 0$ is a constant depending only on α. In addition, we have by (i) and (2.8)

$$\rho(z_1, z_2)\|K_{z_1}\|_{A^2_\alpha} \approx \|K_{z_1} - K_{z_2}\|_{A^2_\alpha}.$$

Similarly, by (4.26) and (2.3), we have

$$\rho(z_3, z_4)\|K_{z_4}\|_{A^2_\alpha} \approx \|K_{z_3} - K_{z_4}\|_{A^2_\alpha}.$$

The constants suppressed so far depend only on α (and β_1). Inserting these estimates into (4.28), we obtain by (2.5) and (4.2)

$$\begin{aligned}\|G\|_{A^2_\alpha} &\ge (1-|z_1|^2)^{\alpha/2+1}|G(b)|\\ &\ge \frac{C_1}{N_1^{\alpha+3}}\left(\|K_{z_1}-K_{z_2}\|_{A^2_\alpha}-\frac{C_2}{N_1}\|K_{z_3}-K_{z_4}\|_{A^2_\alpha}\right)\\ &\ge \frac{C_1}{N_1^{\alpha+3}}\left(1-\frac{2C_2}{N_1}\right)\|K_{z_1}-K_{z_2}\|_{A^2_\alpha}\end{aligned}$$

where C_1 and C_2 are positive constants depending only on α. Accordingly, choosing N_1 with

$$N_1 \ge 4C_2, \tag{4.29}$$

we obtain

$$Q_2 \ge \|G\|_{A^2_\alpha} \ge \frac{C_1}{2N_1^{\alpha+3}}\|K_{z_1}-K_{z_2}\|_{A^2_\alpha} \ge \frac{C_1}{6N_1^{\alpha+3}}Q_1;$$

the last inequality holds by (4.25). So, we conclude (4.24) for Case (a).

Case (b): Using the elementary inequality (see, for example, [2, Lemma 3.2])

$$\left|\frac{1}{(1-\xi)^t}-\frac{1}{(1-\zeta)^t}\right| \le t|\xi-\zeta|\left[\frac{1}{(1-|\xi|)^{t+1}}+\frac{1}{(1-|\zeta|)^{t+1}}\right]$$

valid for all $\xi,\zeta\in\mathbf{D}$ and $t>0$, we obtain

$$\begin{aligned}|K_{z_3}(b)-K_{z_4}(b)| &\lesssim \frac{|z_3-z_4|}{(1-|z_3|)^{\alpha+3}} \qquad \text{(recall } |z_4|\le|z_3|)\\ &= \frac{\rho(z_3,z_4)}{(1-|z_3|)^{\alpha+3}}\cdot|1-\overline{z_3}z_4|\\ &\le \frac{\rho(z_3,z_4)}{(1-|z_3|)^{\alpha+3}}\cdot\frac{1-|z_3|^2}{1-\beta_1} \qquad \text{by (2.4)}\\ &\approx \frac{1}{1-\beta_1}\cdot\frac{\rho(z_3,z_4)}{(1-|z_3|^2)^{\alpha+2}}.\end{aligned} \tag{4.30}$$

In addition, we have by (2.7) and the second condition in (b)

$$\begin{aligned}\frac{1}{(1-|z_3|^2)^{\alpha+2}} &= \left(\frac{1-|z_1|^2}{1-|z_3|^2}\right)^{\alpha/2+1}\|K_{z_1}\|_{A^2_\alpha}\|K_{z_3}\|_{A^2_\alpha}\\ &\le \frac{\|K_{z_1}\|_{A^2_\alpha}\|K_{z_3}\|_{A^2_\alpha}}{N_1^{3(\alpha+2)}}\\ &< \frac{\|K_{z_1}\|_{A^2_\alpha}\|K_{z_3}\|_{A^2_\alpha}}{N_1^{\alpha+4}};\end{aligned}$$

we have used $\alpha>-1$ for the last inequality. Inserting this into (4.30), we obtain

$$|K_{z_3}(b)-K_{z_4}(b)| \lesssim \frac{\rho(z_3,z_4)\|K_{z_1}\|_{A^2_\alpha}\|K_{z_3}\|_{A^2_\alpha}}{N_1^{\alpha+4}(1-\beta_1)};$$

the constant suppressed here depends only on α. Note that (4.27) is still valid. So, combining the above with (4.27), we obtain

$$\begin{aligned}|G(b)| &\geq |K_{z_1}(b) - K_{z_2}(b)| - |K_{z_3}(b) - K_{z_4}(b)| \\ &\gtrsim \frac{\|K_{z_1}\|_{A^2_\alpha}}{N_1^{\alpha+3}} \left[\rho(z_1, z_2)\|K_{z_1}\|_{A^2_\alpha} - \frac{C}{N_1(1-\beta_1)}\rho(z_3, z_4)\|K_{z_3}\|_{A^2_\alpha}\right]\end{aligned}$$

for some constant $C > 0$ depending only on α. Thus, proceeding as in the proof of Case (a), we obtain

$$\|G\|_{A^2_\alpha} \geq \frac{C_3}{N_1^{\alpha+3}} \left(1 - \frac{C_4}{N_1(1-\beta_1)}\right) \|K_{z_1} - K_{z_2}\|_{A^2_\alpha}$$

where C_3 and C_4 are positive constants depending only on α. Accordingly, choosing N_1 with

$$N_1 \geq \frac{2C_4}{1-\beta_1}, \tag{4.31}$$

we obtain

$$Q_2 \geq \|G\|_{A^2_\alpha} \geq \frac{C_3}{6N_1^{\alpha+3}} Q_1$$

as in the proof of Case (a). So, we conclude (4.24) for Case (b).

Case (c): Taking $j_0 \in \{3, 4\}$ such that

$$\max\{|K_{z_3}(b)|, |K_{z_4}(b)|\} = |K_{z_{j_0}}(b)|,$$

we claim

$$(1-|b|^2)^{\alpha/2+1}|K_{z_{j_0}}(b)| \leq \frac{C}{N_1^{\alpha+4}}\|K_{z_3}\|_{A^2_\alpha} \tag{4.32}$$

for some constant $C = C(\alpha) > 0$. To see this we consider two subcases

$$(\text{c1}) : \frac{1-|b|}{1-|z_{j_0}|} \leq \frac{1}{N_1^6} \quad \text{and} \quad (\text{c2}) : \frac{1-|z_{j_0}|}{1-|b|} \leq N_1^6.$$

In case of (c1), we have

$$\begin{aligned}(1-|b|)^{\alpha/2+1}|K_{z_{j_0}}(b)| &\leq \left(\frac{1-|b|}{1-|z_{j_0}|}\right)^{\alpha/2+1} \frac{1}{(1-|z_{j_0}|)^{\alpha/2+1}} \\ &\lesssim \frac{1}{N_1^{3(\alpha+2)}}\|K_{z_3}\|_{A^2_\alpha};\end{aligned}$$

recall $|z_4| \leq |z_3|$ for the second inequality. Since $\alpha > -1$, this yields (4.32). In case of (c2), we note from (iii) and (4.13)

$$\frac{N_1\sqrt{1-\rho(z_1, z_{j_0})}}{|b|\rho(z_1, z_{j_0})} \leq \frac{4N_1\sqrt{1-\beta_2}}{\beta_2} \leq \frac{1}{N_1^6};$$

recall $|b| > \frac{1}{4}$ for the second inequality. It follows from Lemma 4.1(b) that

$$|K_{z_{j_0}}(b)| \lesssim \frac{|K_{z_1}(b)|}{N_1^{6(\alpha+2)}} \le \frac{1}{N_1^{6(\alpha+2)}} \left(\frac{1}{1-|b|}\right)^{\alpha+2}.$$

We thus obtain

$$\begin{aligned}(1-|b|)^{\alpha/2+1}|K_{z_{j_0}}(b)| &\le \frac{1}{N_1^{6(\alpha+2)}} \left(\frac{1-|z_{j_0}|}{1-|b|}\right)^{\alpha/2+1} \frac{1}{(1-|z_{j_0}|)^{\alpha/2+1}} \\ &\lesssim \frac{\|K_{z_3}\|_{A^2_\alpha}}{N_1^{3(\alpha+2)}}\end{aligned}$$

and thus conclude (4.32) as above.

Now, noting that (4.27) is still valid, we have by (2.5), (4.2) and (4.32)

$$\begin{aligned}\|G\|_{A^2_\alpha} &\ge (1-|b|^2)^{\alpha/2+1}|G(b)| \\ &\ge (1-|z_1|^2)^{\alpha/2+1}|K_{z_1}(b) - K_{z_2}(b)| - 2(1-|b|^2)^{\alpha/2+1}|K_{j_0}(b)| \\ &\gtrsim \frac{1}{N_1^{\alpha+3}} \left[\rho(z_1,z_2)\|K_{z_1}\|_{A^2_\alpha} - C\frac{\|K_{z_3}\|_{A^2_\alpha}}{N_1}\right] \qquad (4.33)\end{aligned}$$

for some constant $C = C(\alpha) > 0$. In connection with this, we recall from (2.8) and (2.3) (recall $|z_2| \le |z_1|$ and $|z_4| \le |z_3|$)

$$\rho(z_1,z_2)\|K_{z_1}\|_{A^2_\alpha} \approx \|K_{z_1} - K_{z_2}\|_{A^2_\alpha} \approx \|K_{z_3} - K_{z_4}\|_{A^2_\alpha} \approx \|K_{z_3}\|_{A^2_\alpha},$$

which, in turn, yields

$$\rho(z_1,z_2)\|K_{z_1}\|_{A^2_\alpha} \approx Q_1 \approx \|K_{z_3}\|_{A^2_\alpha}.$$

The constants suppressed so far depend only on α. Thus, combining the above with (4.33), we obtain

$$\|G\|_{A^2_\alpha} \ge \frac{C_5}{N_1^{\alpha+3}} \left(1 - \frac{C_6}{N_1}\right) Q_1$$

where C_5 and C_6 are positive constants depending only on α. Accordingly, choosing N_1 with

$$N_1 \ge 2C_6, \qquad (4.34)$$

we obtain

$$Q_2 \ge \|G\|_{A^2_\alpha} \ge \frac{C_5}{2N_1^{\alpha+3}} Q_1.$$

So, we conclude (4.24), as required. This completes the proof for Case (c) and thus the proof of the lemma. □

Lemma 4.5. *Let $\alpha > -1$. For $z_1, z_2, z_3, z_4 \in \mathbf{D}$, assume the following:*

(i) $2N_1(1-|z_1|^2) < 1$ *and* $|z_j| \le |z_1|$ *for* $j = 2, 3, 4$*;*
(ii) $\rho(z_1,z_2) \le \beta_1$*;*
(iii) $\rho(z_1,z_3) \le \beta_1$ *and* $\beta_2 \le \rho(z_1,z_4)$.

Then the estimate

$$\|G\|_{A^2_\alpha} \approx Q_1 \approx Q_2 \approx \|K_{z_1}\|_{A^2_\alpha}$$

holds; the constants suppressed above depend only on α*.*

Proof. We keep using the notation (4.18). Using (4.12), we obtain by Lemma 4.1(a)

$$\left|1 - \frac{K_{z_j}(b)}{K_{z_1}(b)}\right| \le \frac{c_\alpha}{N_1(1-\beta_1)}$$

for $j = 2, 3$. Meanwhile, using (4.13) and (4.15), we obtain by Lemma 4.1(b)

$$\left|\frac{K_{z_4}(b)}{K_{z_1}(b)}\right| \le \frac{1}{4N_1}.$$

It follows that

$$\begin{aligned} |G(b)| &\ge |K_{z_1}(b)| \left(\left|1 - \frac{K_{z_2}(b)}{K_{z_1}(b)} - \frac{K_{z_3}(b)}{K_{z_1}(b)}\right| - \frac{|K_{z_4}(b)|}{|K_{z_1}(b)|} \right) \\ &\ge |K_{z_1}(b)| \left(1 - \left|1 - \frac{K_{z_2}(b)}{K_{z_1}(b)}\right| - \left|1 - \frac{K_{z_3}(b)}{K_{z_1}(b)}\right| - \frac{|K_{z_4}(b)|}{|K_{z_1}(b)|} \right) \\ &\ge \frac{\|K_{z_1}\|^2_{A^2_\alpha}}{(N_1+1)^{\alpha+2}} \left[1 - \frac{2c_\alpha}{N_1(1-\beta_1)} - \frac{1}{4N_1} \right]. \end{aligned}$$

Hence, taking N_1 with

$$\frac{2c_\alpha}{N_1(1-\beta_1)} + \frac{1}{4N_1} \le \frac{1}{2}, \tag{4.35}$$

we obtain by (2.5) and (4.2)

$$\|G\|_{A^2_\alpha} \ge (1-|b|^2)^{1+\alpha/2} |G(b)| \ge \frac{\|K_{z_1}\|_{A^2_\alpha}}{2(N_1+1)^{\alpha+2}}.$$

By this and (4.17) we conclude the lemma. The proof is complete. □

Having established Lemmas 4.2-4.5, we now proceed to obtain optimal ratio estimates for the remaining case, which are not covered by Theorem 3.5.

Theorem 4.6. *Let* $\alpha > -1$*. For* $z_1, z_2, z_3, z_4 \in \mathbf{D}$*, assume*

$$\beta_4 \le \max_{1 \le i < j \le 4} \rho(z_i, z_j) \tag{4.36}$$

and

$$\min_{1 \le j \le 4} (1 - |z_j|^2) \le \frac{1}{2N_2}. \tag{4.37}$$

Then the inequalities

$$C \min\{Q_1, Q_2\} \le \|G\|_{A^2_\alpha} \le \min\{Q_1, Q_2\}$$

hold for some constant $C = C(\alpha) > 0$*.*

Proof. The roles of z_1 and z_4 are the same and the roles of z_2 and z_3 are the same for the estimate of $\|G\|_{A^2_\alpha}$. Moreover, the roles of $\{z_1, z_4\}$ and $\{z_2, z_3\}$ can be interchanged by taking $-G$. Therefore, we may assume

$$\max_{2\le j\le 4} |z_j| \le |z_1| \quad \text{and} \quad \rho(z_1, z_2) \le \rho(z_1, z_3).$$

Note $2N_2(1 - |z_1|^2) < 1$ by (4.37). We have

$$C \min\{Q_1, Q_2\} \le \|G\|_{A^2_\alpha} \le \min\{Q_1, Q_2\}; \tag{4.38}$$

for some constant $C = C(\beta_1, N_1, \beta_2) = C(\alpha) > 0$ in the following four cases:

(a) $\min_{2\le j\le 4} \rho(z_1, z_j) \ge \beta_1$ (by Lemma 4.2);
(b) $\rho(z_1, z_4) \le \beta_1$, $\rho(z_1, z_2) \notin [\beta_1, \beta_2]$ and $\beta_2 \le \rho(z_1, z_3)$ (by Lemma 4.3);
(c) $\rho(z_1, z_2) \le \beta_1$ and $\beta_2 \le \min_{j=3,4} \rho(z_1, z_j)$ (by Lemma 4.4);
(d) $\max_{j=2,3} \rho(z_1, z_j) \le \beta_1$ and $\beta_2 \le \rho(z_1, z_4)$ (by Lemma 4.5).

Recalling that $\tanh^{-1} \rho$ is the well-known hyperbolic distance on $\mathbf{D}$, we have by the triangle inequality

$$\max_{1\le i<j\le 4} \tanh^{-1} \rho(z_i, z_j) \le 2 \max_{2\le j\le 4} \tanh^{-1} \rho(z_1, z_j).$$

Thus, we see from (4.16) and (4.36) that

$$\beta_3 \le \max_{2\le j\le 4} \rho(z_1, z_j).$$

So, one may check that the following three cases are missing in the four cases listed above:

(b1) $\rho(z_1, z_4) \le \beta_1 \le \rho(z_1, z_2) \le \beta_2$ and $\beta_3 \le \rho(z_1, z_3)$;
(c1) $\rho(z_1, z_2) \le \beta_1 \le \rho(z_1, z_4) \le \beta_2$ and $\beta_3 \le \rho(z_1, z_3)$;
(c2) $\rho(z_1, z_2) \le \beta_1 \le \rho(z_1, z_3) \le \beta_2$ and $\beta_3 \le \rho(z_1, z_4)$.

Note that Cases (b1) and (c1) reduce to Case (b) with (β_2, β_3) in place of (β_1, β_2). Case (c2) reduces Case (d) in the similar way. The proof is complete. □

5. HILBERT-SCHMIDT DOUBLE DIFFERENCES

In this section, applying the optimal ratio estimates established in the previous two sections, we prove Theorem 1.1. We also provide explicit examples demonstrating that the rigid phenomenon for compactness mentioned in the Introduction is no longer available for Hilbert-Schmidtness.

In the proof below we will use the auxiliary sets Γ_t given by

$$\Gamma_t := \left\{ z \in \mathbf{D} : \min_{1\le j\le 4} (1 - |\varphi_j(z)|) < t \right\}$$

for $0 < t < 1$. Also, we continue using auxiliary numbers specified in Section 4.

Proof of Theorem 1.1. Put $s_0 = s_0(\alpha) := \beta_4$. For $s \in (s_0, 1)$, let $r = r(s) > 0$ be a number provided by Theorem 3.5. Pick a sufficiently small $t = t(\alpha, s) > 0$ with the following two properties:

(i) $t \le \min\{\frac{1}{4N_2}, \frac{r}{2}\}$;

(ii) $\rho(a,b) \geq s$ whenever $1-|a| \leq t$ and $1-|b| > r$.

Note from (2.10)

$$\|T\|^2_{HS(A^2_\alpha)} = \int_{\mathbf{D}\setminus\Gamma_t} + \int_{\Gamma_t\setminus\Omega_s} + \int_{\Gamma_t\cap\Omega_s} \|K\|^2_{A^2_\alpha}\, dA_\alpha$$

where $K := K_{\varphi_1} - K_{\varphi_2} - K_{\varphi_3} + K_{\varphi_4}$. First, note from (i) that Theorem 4.6 holds with $2t$ in place of $\frac{1}{2N_2}$. Thus we see from Theorem 4.6 and (2.8)

$$\|K\|_{A^2_\alpha} \approx \min\{R_1, R_2\} \quad \text{on } \Gamma_t \setminus \Omega_s.$$

Next, note from (ii) that $1 - |\varphi_j(z)| \leq r$ for all j and $z \in \Gamma_t \cap \Omega_s$. Thus we see from Theorem 3.5

$$\|K\|_{A^2_\alpha} \approx R_3 \quad \text{on } \Gamma_t \cap \Omega_s.$$

The constants suppressed so far depend only on α, s and t. Finally, since $\varphi_j(\mathbf{D}\setminus\Gamma_t)$ is contained in the closed disk $\overline{D_{1-t}(0)}$ for each j, it is clear that the integrals over $\mathbf{D}\setminus\Gamma_t$ of $\|K\|^2_{A^2_\alpha}$, $(\min\{R_1,R_2\})^2$ and R_3^2 are all finite. So, we conclude the first part of the theorem.

Now assume $\varphi_1 = \varphi_4$ so that $R_1 = R_2 =: R$. Note

$$R = (\rho_{12} + \rho_{13})\|K_{\varphi_1}\|_{A^2_\alpha} + \rho_{12}\|K_{\varphi_2}\|_{A^2_\alpha} + \rho_{13}\|K_{\varphi_3}\|_{A^2_\alpha}. \tag{5.1}$$

Also, note by Theorem 3.7

$$R_4 \approx R_3 \quad \text{on } \Gamma_t \cap \Omega_s;$$

the constants suppressed in this estimate depend only α, s and t. Thus, as in the proof of the first part, it suffices to show that

$$R_4 \approx \sum_{j=1}^{3} \|K_{\varphi_j}\|_{A^2_\alpha} \approx R \quad \text{on } \mathbf{D}\setminus\Omega_s; \tag{5.2}$$

the constants suppressed in this estimate depend only α and s. Since

$$\rho_{12} + \rho_{13} \geq \max\{\rho_{12}, \rho_{13}\} \geq \frac{s}{2} \quad \text{on } \mathbf{D}\setminus\Omega_s, \tag{5.3}$$

the first estimate in (5.2) is clear. To see the second estimate, consider arbitrary $z \in \mathbf{D}\setminus\Omega_s$. Using the inequality in (5.3), we may assume $\rho_{12}(z) \leq \rho_{13}(z)$ by symmetry so that $\rho_{13}(z) \geq \frac{s}{2}$. If $\rho_{12}(z) \geq \frac{s}{2}$, then it is clear from (5.1) that

$$R(z) \approx \sum_{j=1}^{3} \|K_{\varphi_j(z)}\|_{A^2_\alpha}.$$

If $\rho_{12}(z) < \frac{s}{2}$, then we also have the same estimate by (5.1), (2.3) and (2.7). Accordingly, we conclude the second estimate in (5.2). The proof is complete. □

Remark 5.1. When (1.2) holds, closely looking at the proof of Theorem 1.1, one may obtain the norm estimate

$$\|T\|_{HS(A^2_\alpha)}^2 \approx \int_{\mathbf{D}\setminus\Gamma_t} \|K\|_{A^2_\alpha}^2 \, dA_\alpha + \int_{\Gamma_t\setminus\Omega_s} (\min\{R_1, R_2\})^2 \, dA_\alpha + \int_{\Gamma_t\cap\Omega_s} R_3^2 \, dA_\alpha.$$

Here, we use the same notation introduced in the proof of Theorem 1.1. Similarly, when (1.3) holds, one may obtain the norm estimate

$$\|T\|_{HS(A^2_\alpha)}^2 \approx \int_{\mathbf{D}\setminus\Gamma_t} \|K\|_{A^2_\alpha}^2 \, dA_\alpha + \int_{\Gamma_t} R_4^2 \, dA_\alpha.$$

The constants suppressed in these norm estimates depend only on α, s and t.

As is mentioned in the Introduction, non-compact differences of composition operators cannot form a compact linear combination, when coefficients in that combination satisfy CNC. In the next two examples we will exhibit explicit examples demonstrating that such a rigid phenomenon for compact combinations does not extend to Hilbert-Schmidt combinations. We take $\alpha = 0$ for simplicity. Recall $A^2(\mathbf{D}) = A^2_0(\mathbf{D})$.

Example 5.2. *Let* $\frac{1}{4} < \delta \le \frac{1}{2}$ *and* $0 < \epsilon < \frac{1}{2}$. *For* $\varphi_1(z) := \frac{z+1}{2}$, *put*

$$\varphi_2 := \varphi_1 + \epsilon(1-\varphi_1)^{2+\delta} \quad \text{and} \quad \varphi_3 := \varphi_1 - \epsilon(1-\varphi_1)^{2+\delta}.$$

Put $T := 2T_1 - T_2 - T_3$ *where* $T_j := C_{\varphi_j}$. *Then the following assertions hold on* $A^2(\mathbf{D})$*:*

(a) $T_1 - T_2$ *and* $T_1 - T_3$ *are not Hilbert-Schmidt;*
(b) T *is Hilbert-Schmidt.*

Proof. To begin with, we note

$$1 - |\varphi_1(z)|^2 = |1 - \varphi_1(z)|^2 + \frac{1-|z|^2}{2}$$

and thus

$$|1-\varphi_1|^{2+\delta} \le |1-\varphi_1|^2 < 1 - |\varphi_1|^2 \le 2(1-|\varphi_1|) \tag{5.4}$$

for any $\delta > 0$. So, we obtain

$$|(1-|\varphi_1|) - (1-|\varphi_2|)| \le |\varphi_1 - \varphi_2| \le 2\epsilon(1-|\varphi_1|)$$

and thus (recall $2\epsilon < 1$)

$$1 - |\varphi_1| \approx 1 - |\varphi_2|.$$

In particular, $\varphi_2 \in \mathcal{S}(\mathbf{D})$. We also note

$$\rho_{12} = \frac{\epsilon|1-\varphi_1|^{2+\delta}}{(1-|\varphi_1|^2)\left|1 - \epsilon\frac{\varphi_1(1-\overline{\varphi_1})^{2+\delta}}{1-|\varphi_1|^2}\right|} \approx \frac{|1-\varphi_1|^{2+\delta}}{1-|\varphi_1|^2};$$

the last estimate holds by (5.4). Similarly, we have the same estimates with φ_3 in place of φ_2.

In summary, we have $\varphi_2, \varphi_3 \in \mathcal{S}(\mathbf{D})$ and, in addition,

$$1 - |\varphi_j| \approx 1 - |\varphi_1| \quad \text{and} \quad \rho_{1j} \approx \frac{|1-\varphi_1|^{2+\delta}}{1-|\varphi_1|} \quad \text{on } \mathbf{D} \tag{5.5}$$

for each $j = 2, 3$. This, together with (2.8), yields

$$\begin{aligned} \|K_{\varphi_1(z)} - K_{\varphi_j(z)}\|_{A^2} &\approx \rho_{1j}(z) \left(\|K_{\varphi_1(z)}\|_{A^2} + \|K_{\varphi_j(z)}\|_{A^2} \right) \\ &\approx \frac{|1-\varphi_1(z)|^{2+\delta}}{(1-|\varphi_1(z)|)^2} \\ &\approx (1-x)^\delta + \frac{y^{2+\delta}}{(1-x)^2} \end{aligned}$$

for $z \in \mathbf{D}$ and $j = 2, 3$. Here, $x := \operatorname{Re} z$ and $y := \operatorname{Im} z$. For the last estimate above, we used the inequality $\frac{1-x}{2} \le 1 - |\varphi_1(z)|^2 \le 1 - x$. We note by elementary calculus

$$\int_{\mathbf{D}} \frac{y^{4+2\delta}}{(1-x)^4}\, dA(z) \approx \int_0^1 \frac{dx}{(1-x)^{\frac{3}{2}-\delta}} = \infty;$$

recall $\delta \le 1/2$. Thus (a) holds by (2.9).

We now prove (b). Note

$$|1 - \overline{\varphi}_1 \varphi_j| \le 1 - |\varphi_1|^2 + \epsilon |1-\varphi_1|^{2+\delta}$$

for $j = 2, 3$. So, since $2\varphi_1 = \varphi_2 + \varphi_3$, we have

$$\begin{aligned} \left| \sum_{j=2}^{3} \sigma(\varphi_1, \varphi_j) \right| &= \left| \sum_{j=2}^{3} \sigma^2(\varphi_1, \varphi_j) \frac{\overline{\varphi}_1 (1 - \overline{\varphi}_1 \varphi_j)}{1 - |\varphi_1|^2} \right| \quad \text{by (3.27)} \\ &\le \sum_{j=2}^{3} \rho_{1j}^2 \left(1 + \frac{\epsilon |1-\varphi_1|^{2+\delta}}{1-|\varphi_1|^2} \right) \\ &\approx \rho_{12}^2(z) + \rho_{13}^2 \quad \text{by (5.5)} \end{aligned}$$

and thus

$$R_4 \approx \left(\rho_{12}^2 + \rho_{13}^2\right) \sum_{j=1}^{3} \|K_{\varphi_j(z)}\|_{A^2} \approx \frac{|1-\varphi_1|^{2(2+\delta)}}{(1-|\varphi_1|)^3} \quad \text{by (5.5).}$$

Now, since

$$R_4^2(z) \approx \frac{|1-\varphi_1(z)|^{4(2+\delta)}}{(1-|\varphi_1(z)|)^6} \approx (1-x)^{2+4\delta} + \frac{y^{8+4\delta}}{(1-x)^6}$$

for $z \in \mathbf{D}$ and

$$\int_{\mathbf{D}} \frac{y^{8+2\delta}}{(1-x)^6}\, dA(z) \approx \int_0^1 \frac{dx}{(1-x)^{\frac{3}{2}-2\delta}} < \infty \quad \text{(recall } \delta > 1/4\text{)},$$

we conclude $\int_{\mathbf{D}} R_4^2\, dA < \infty$. Thus (b) holds by Theorem 1.1. The proof is complete. □

In fact Example 5.2 can be generalized to general linear combinations. To this end we need the following lemma.

Lemma 5.3. *Let $\alpha > -1$ and $0 < s < 1$. Let $n \geq 2$ be a positive integer. Given $z_1, \dots, z_n \in \mathbf{D}$ and $c_1, \dots, c_n \in \mathbf{C}$ with $\sum_{j=1}^n c_j = 0$ and $\max_{1\leq j\leq n} |c_j| = 1$, put*

$$A := \sum_{k=1}^{2} \left| \sum_{j=2}^{n} \overline{c_j} \left(\frac{z_j - z_1}{1 - |z_1|^2} \right)^k \right| \text{ and } B := \sum_{j=2}^{n} \rho^3(z_1, z_j).$$

Then there is a constant $C = C(\alpha, s, n) > 0$ such that

$$\left\| \sum_{j=1}^{n} c_j K_{z_j} \right\|_{A^2_\alpha} \leq C(A + B) \sum_{j=1}^{n} \|K_{z_j}\|_{A^2_\alpha}$$

whenever $\rho(z_1, z_j) \leq s$ for all j.

Proof. Let $z_1, \dots, z_n \in \mathbf{D}$ and consider $c_1, \dots, c_n \in \mathbf{C}$ with $\sum_{j=1}^n c_j = 0$ and $\max_{1\leq j\leq n} |c_j| = 1$. Assume $\rho(z_1, z_j) \leq s$ for all j in the rest of the proof.

As in the proof of Lemma 3.5, we set

$$f := \sum_{j=1}^{n} c_j K_{z_j} \text{ and } f_j(z) := \frac{z(\overline{z_j} - \overline{z_1})}{1 - z\overline{z_1}}$$

for $j = 1, \dots, n$. We note

$$f(z) = K_{z_1}(z) \sum_{j=1}^{n} \frac{c_j}{(1 - f_j(z))^{\alpha+2}} \tag{5.6}$$

for $z \in \mathbf{D}$. Also, we note from (2.4)

$$|f_j(z)| \leq \rho(z_1, z_j) \frac{|z||1 - z_1\overline{z_j}|}{1 - |z_1|} \leq \frac{2}{1 - s} \rho(z_1, z_j), \qquad z \in \mathbf{D} \tag{5.7}$$

for all j. In case $\rho(z_1, z_j) \geq \frac{1-s}{4}$ for some j, the desired estimate is trivial by the triangle inequality. So, we may assume $\rho(z_1, z_j) < \frac{1-s}{4}$ for all j so that $|f_j(z)| \leq \frac{1}{2}$ for all j and $z \in \mathbf{D}$. Thus, setting

$$p_k := \frac{(\alpha + 2)(\alpha + 3) \cdots (\alpha + 1 + k)}{k!}$$

for $k = 1, 2, \dots$ and

$$h(z) := \sum_{k=3}^{\infty} p_k z^k,$$

we may represent the right hand side of (5.6) in the binomial series to obtain

$$\frac{f(z)}{K_{z_1}(z)} = \sum_{k=1}^{2} p_k \sum_{j=2}^{n} c_j \big[f_j(z)\big]^k + \sum_{j=1}^{n} c_j h\big(f_j(z)\big) \tag{5.8}$$

for all $z \in \mathbf{D}$; recall $f_1 \equiv 0$ and $\sum_{j=1}^{n} c_j = 0$. It follows that

$$\frac{|f(z)|}{|K_{z_1}(z)|} \leq \sum_{k=1}^{2} \left| \sum_{j=2}^{n} c_j (f_j(z))^k \right| + \sum_{j=1}^{n} \left| h\big(f_j(z)\big) \right| =: I + II;$$

recall $|c_j| \leq 1$. It is clear that

$$I \leq (1 + |z_1|)^2 \sum_{k=1}^{2} \left| \sum_{j=2}^{n} \overline{c_j} \left(\frac{z_j - z_1}{1 - |z_1|^2} \right)^k \right| \leq 4A.$$

Since $|f_j(z)| \leq \frac{1}{2}$, we also have by (5.7)

$$II \lesssim \sum_{j=2}^{n} |f_j(z)|^3 \lesssim B;$$

the constants suppressed above depend only on α and s. Combining these observations, we conclude the lemma. The proof is complete. □

The following example generalizes Example 5.2 to general linear combinations.

Example 5.4. *Let $\frac{1}{4} < \delta \leq \frac{1}{2}$ and J be a set of finitely many real numbers containing 0. For $\varphi_0(z) := \frac{z+1}{2}$, put*

$$\varphi_\beta = \varphi_0 + \beta\epsilon(1 - \varphi_0)^{2+\delta}, \qquad \beta \in J$$

where $\epsilon > 0$ is chosen so that $\varphi_\beta \in \mathcal{S}(\mathbf{D})$ for all $\beta \in J$. Put $L_\beta := C_{\varphi_\beta}$ and

$$L := \sum_{\beta \in J} c_\beta L_\beta$$

where c_β's are coefficients satisfying

$$\sum_{\beta \in J} c_\beta = \sum_{\beta \in J} \beta c_\beta = 0 \quad \textit{but} \quad \sum_{\beta \in J} \beta^2 c_\beta \neq 0. \tag{5.9}$$

Then the following assertions hold on $A^2(\mathbf{D})$:

(a) *$L_\beta - L_\gamma$ is not Hilbert-Schmidt for all distinct $\beta, \gamma \in J$;*
(b) *L is Hilbert-Schmidt.*

Proof. As in the proof of Example 5.2, we have

$$1 - |\varphi_\beta| \approx 1 - |\varphi_0| \quad \text{and} \quad \rho_{\beta\gamma} \approx \rho_{0\beta} \approx \frac{|1 - \varphi_0|^{2+\delta}}{1 - |\varphi_0|} \quad \text{on } \mathbf{D} \tag{5.10}$$

for all $\beta, \gamma \in J$ with $\beta \neq 0$ and $\beta \neq \gamma$. Here, $\rho_{\beta\gamma} := \rho(\varphi_\beta, \varphi_\gamma)$. Thus (a) holds by the same proof of Example 5.2(a).

Pick $s \in (0, 1)$ and choose $t = t(s) \in (0, 1)$ as in the proof of Theorem 1.1. Also, choose $r = r(s)$ provided by Theorem 1.1. Shrinking r if necessary, we may further assume that r plays the role of ϵ in Lemma 5.3. Note that $\varphi_\beta(z)$ tends to 1 as $z \to 1$ for each β. Also, note from (5.10) that $\rho_{\beta\gamma}(z)$ and $\rho_{0\beta}(z)$ tend to 0 as $z \to 1$ for all β and γ under consideration. Thus, there is an open disk U centered at 1 such that $U \cap \mathbf{D} \subset \Gamma_t \cap \Omega_s$. Accordingly, in order to prove (b), it suffices to show

$$\int_{U\cap\mathbf{D}} \left\| \sum_{\beta\in J} \overline{c_\beta} K_{\varphi_\beta} \right\|_{A^2}^2 dA < \infty \tag{5.11}$$

by (2.9).

In order to see (5.11), put

$$\eta := \left| \sum_{\beta\in J} \beta^2 c_\beta \right| > 0 \quad \text{and} \quad B := \sum_{\beta<\gamma} \rho_{\beta\gamma}^2$$

for short. Since $\sum_{\beta\in J} \beta c_\beta = 0$ by assumption, we note

$$\begin{aligned} \sum_{k=1}^{2} \left| \sum_{\beta\in J} c_\beta \frac{(\varphi_0-\varphi_\beta)^k}{(1-|\varphi_0|)^k} \right| &= \left| \sum_{\beta\in J} c_\beta \frac{(\varphi_0-\varphi_\beta)^2}{(1-|\varphi_0|)^2} \right| \\ &= \frac{\eta\epsilon^2|1-\varphi_0|^{2(2+\delta)}}{(1-|\varphi_0|)^2} \\ &\approx \eta B \quad \text{by (5.10)} \end{aligned} \tag{5.12}$$

on $\mathbf{D}$. Now, assuming $\max_\beta |c_\beta| = 1$ for simplicity and applying Lemma 5.3, we obtain

$$\left\| \sum_{\beta\in J} \overline{c_\beta} K_{\varphi_\beta} \right\|_{A^2} \lesssim \eta B \sum_{\beta\in J} \|K_{\varphi_\beta}\|_{A^2} \tag{5.13}$$

on $\Gamma_t \cap \Omega_s$; the constant suppressed above depends only on s and the number of elements of J. This implies (5.11) as in the proof of Example 5.2(b). Thus (b) holds. The proof is complete. □

Remark 5.5. (1) In the notation of Example 5.4, we note that the operator considered in Example 5.2 is precisely $2L_0 - L_1 - L_{-1}$. More generally, double differences of the form

$$L_0 - L_\beta - L_\gamma + L_{\beta+\gamma} \quad \text{with } \beta\gamma \neq 0 \tag{5.14}$$

are covered by Example 5.4. Of course, one may find various examples of different type. For example:

$$\begin{aligned} &3L_0 - L_\beta - L_{-\gamma} - L_{\gamma-\beta}; \\ &5L_0 - 2L_\beta - L_{-2\gamma} - 2L_{\gamma-\beta}; \\ &5L_0 - L_1 - L_2 - L_\beta - L_{-\gamma} - L_{\gamma-\beta-3}. \end{aligned}$$

Except for a few exceptional choices of β and γ in each operator, one may check that these operators are covered by Example 5.4. One may also check that these operators satisfy CNC mentioned in the Introduction.

(2) Note from (2.8), (2.10) and (4.10) that if

$$\min\{R_1, R_2\} \in L^2_\alpha(\mathbf{D}), \tag{5.15}$$

then the corresponding double difference is Hilbert-Schmidt on $A^2_\alpha(\mathbf{D})$. So, one may ask whether the converse also holds, or said differently, whether Condition (1.2) can be reduced to Condition (5.15). The answer is *no*. To see it, closely look at the proofs of Example 5.2(a) and Example 5.4(a) (for the operators as in (5.14)).

6. REMARKS

Recently, study of composition operators on vector-valued holomorphic functions has been of growing interest. In this section we consider general linear combinations of composition operators, as well as related operators, and notice some remarks on the Hilbert-Schmidtness of such operators. Our results involve composition operators between certain vector-valued weighted Bergman spaces which are described below.

Given a (complex) Banach space X, let $H(\mathbf{D}, X)$ be the class of all X-valued holomorphic functions on $\mathbf{D}$. For $\alpha > -1$, we denote by $A^2_\alpha(\mathbf{D}, X)$ the *strong* X-valued α-weighted Bergman space consisting of all functions $g \in H(\mathbf{D}, X)$ for which

$$\|g\|_{A^2_\alpha(\mathbf{D},X)} := \left\{ \int_{\mathbf{D}} \|g(z)\|^2_X \, dA_\alpha(z) \right\}^{1/2} < \infty$$

where $\|\cdot\|_X$ denotes the norm on X. We also denote by $wA^2_\alpha(\mathbf{D}, X)$ the *weak* X-valued α-weighted Bergman space consisting of all functions $g \in H(\mathbf{D}, X)$ for which

$$\|g\|_{wA^2_\alpha(\mathbf{D},X)} := \sup_{x^* \in B_{X^*}} \|x^* \circ g\|_{A^2_\alpha} < \infty.$$

Here, X^* is the dual space of X and B_{X^*} is the closed unit ball of X^*.

The notion of composition operators naturally extends to the vector-valued setting. To be more precise, we note $g \circ \varphi \in H(\mathbf{D}, X)$ for all $\varphi \in \mathcal{S}(\mathbf{D})$ and $g \in H(\mathbf{D}, X)$. We will use the same notation C_φ to denote the composition operator $g \mapsto g \circ \varphi$ for $g \in H(\mathbf{D}, X)$. With this convention we have

$$x^*(C_\varphi g) = x^* \circ g \circ \varphi = C_\varphi(x^* \circ g) \tag{6.1}$$

for $x^* \in X^*$ and $g \in H(\mathbf{D}, X)$.

We now recall the well-known notion of order-boundedness which is closely related to the Hilbert-Schmidtness. Let X be a Banach space and μ be a positive finite Borel measure on $\mathbf{D}$. A linear operator $S : X \to L^2(\mu)$ is called *order-bounded* if there exists a nonnegative $h \in L^2(\mu)$ such that $|Sg| \leq h$ $[\mu]$-a.e. for each g in the closed unit ball of X; we refer to [6, Chapters 4-5] for more general approach to order-boundedness and related facts. We recall the following well-known result for a linear operator $S : L^2(\mu) \to L^2(\mu)$:

$$S \textit{ is Hilbert-Schmidt iff it is order-bounded}; \tag{6.2}$$

see [8, Page 226].

In what follows, we say that a linear operator $S : A^2_\alpha(\mathbf{D}) \to A^2_\alpha(\mathbf{D})$ is order-bounded if $S : A^2_\alpha(\mathbf{D}) \to L^2_\alpha(\mathbf{D})$ is order-bounded. Recall $L^2_\alpha(\mathbf{D}) := L^2(\mathbf{D}, A_\alpha)$. The following is an easy consequence of (6.2).

Corollary 6.1. *Let $\alpha > -1$ and S be a linear operator on $A^2_\alpha(\mathbf{D})$. Then S is Hilbert-Schmidt iff it is order-bounded.*

Proof. Let $P : L^2_\alpha(\mathbf{D}) \to A^2_\alpha(\mathbf{D})$ be the Hilbert-space orthogonal projection. Regard the operator SP acting from $L^2_\alpha(\mathbf{D})$ into itself. Then it is not hard to verify that SP is Hilbert-Schmidt(order-bounded, resp) on $L^2_\alpha(\mathbf{D})$ iff S is Hilbert-Schmidt(order-bounded, resp) on $A^2_\alpha(\mathbf{D})$. Thus the corollary holds by (6.2). $\square$

Laitila, Tylli and Wang [10] first investigated the boundedness of composition operators in the setting of the vector-valued Bergman spaces, revealing that the spaces $A^2_\alpha(\mathbf{D}, X)$ and $wA^2_\alpha(\mathbf{D}, X)$ are quite different for any *infinite*-dimensional Banach space X. They noticed an interesting result (see [10, Theorem 3.2]): The boundedness of composition operators acting from weak to strong vector-valued weighted Bergman spaces is equivalent to their Hilbert-Schmidtness (or equivalently, to the order-boundedness by Corollary 6.1) on the corresponding scalar-valued weighted Bergman spaces.

Inspired by the ideas of [10], Guo and Wang extended the aforementioned result of Laitila-Tylli-Wang to the difference of composition operators; see [7, Corollary 3.5]. Such a result actually extends to general linear combinations of composition operators as in the next theorem.

Theorem 6.2. *Let $\alpha > -1$ and X be an infinite-dimensional Banach space. Let T be a linear combination of composition operators. Then the following assertions are equivalent:*

(a) $T : A^2_\alpha(\mathbf{D}) \to A^2_\alpha(\mathbf{D})$ *is Hilbert-Schmidt/order-bounded;*
(b) $T : wA^2_\alpha(\mathbf{D}, X) \to A^2_\alpha(\mathbf{D}, X)$ *is bounded.*

Proof. First, we prove that (a) implies (b). So, assume (a). By Corollary 6.1 we may assume that $T : A^2_\alpha(\mathbf{D}) \to A^2_\alpha(\mathbf{D})$ is order-bounded. Pick a nonnegative $h \in L^2_\alpha(\mathbf{D})$ such that $|Tf| \le h\|f\|_{A^2_\alpha}$ almost everywhere on $\mathbf{D}$ for all $f \in A^2_\alpha(\mathbf{D})$. We will complete the proof by showing

$$\|T\|_{wA^2_\alpha(\mathbf{D},X)\to A^2_\alpha(\mathbf{D},X)} \le \|h\|_{L^2_\alpha}; \tag{6.3}$$

the left-hand side denotes the operator norm of $T : wA^2_\alpha(\mathbf{D}, X) \to A^2_\alpha(\mathbf{D}, X)$.

Consider arbitrary $g \in wA^2_\alpha(\mathbf{D}, X)$. Let $x^* \in B_{X^*}$. Note $x^* \circ g \in A^2_\alpha(\mathbf{D})$ by definition of the space $wA^2_\alpha(\mathbf{D}, X)$. Using the aforementioned property of the function h, we also note

$$\begin{aligned}
\|Tg(z)\|_X &= \sup_{x^*\in B_{X^*}} |x^*[Tg(z)]| \\
&= \sup_{x^*\in B_{X^*}} |T(x^* \circ g)(z)| \quad \text{by (6.1)} \\
&\le h(z) \sup_{x^*\in B_{X^*}} \|x^* \circ g\|_{A^2_\alpha} \\
&\le h(z)\|g\|_{wA^2_\alpha(\mathbf{D},X)}
\end{aligned}$$

for almost every $z \in \mathbf{D}$. We thus obtain

$$\|Tg\|^2_{A^2_\alpha(\mathbf{D},X)} = \int_{\mathbf{D}} \|Tg(z)\|^2_X \, dA_\alpha(z) \le \|h\|^2_{L^2_\alpha} \|g\|^2_{wA^2_\alpha(\mathbf{D},X)}.$$

Since this holds for arbitrary $g \in wA^2_\alpha(\mathbf{D}, X)$, we conclude (6.3), as desired.

Now, we prove that (b) implies (a). So, assume (b). By Corollary 6.1, it suffices to show

$$\|T\|^2_{HS(A^2_\alpha(\mathbf{D}))} \le \|T\|^2_{wA^2_\alpha(\mathbf{D},X)\to A^2_\alpha(\mathbf{D},X)}. \tag{6.4}$$

In order to prove this, one may modify in a straightforward way the proof of the implication (a) $\Longrightarrow$ (h) in [7, Theorem 4.3] based on Dvoretzky's Theorem and obtain

$$\begin{aligned} \|T\|^2_{wA^2_\alpha(\mathbf{D},X)\to A^2_\alpha(\mathbf{D},X)} &\ge \frac{1}{(1+\epsilon)^2} \sum_{k=0}^\infty \int_{\mathbf{D}} |T\omega_k|^2 \, dA_\alpha \\ &= \frac{1}{(1+\epsilon)^2} \|T\|^2_{HS(A^2_\alpha)} \end{aligned}$$

for arbitrary $\epsilon > 0$ where ω_k is the A^2_α-normalized monomial of degree k. Thus we conclude (6.4), as desired. The proof is complete. □

Let $u \in H(\mathbf{D})$ and $\varphi \in \mathcal{S}(\mathbf{D})$. For a non-negative integer n, the *weighted differentiation composition operator* $D^n_{u,\varphi}$ (with weight u and symbol φ) is defined as

$$D^n_{u,\varphi} f := u \cdot \left(f^{(n)} \circ \varphi\right), \quad f \in H(\mathbf{D}).$$

Given $u_j \in H(\mathbf{D})$ and $\varphi_j \in \mathcal{S}(\mathbf{D})$ for $j = 0, 1, \ldots, n$, put

$$T^n_{\mathbf{u},\mathbf{\Phi}} := \sum_{j=0}^n D^j_{u_j,\varphi_j}. \tag{6.5}$$

In the setting of the weighted Bergman spaces, the order-boundedness of these operators has been recently characterized by Acharyya and Ferguson [1]. We may extend their result to the vector-valued setting as in the next theorem.

Theorem 6.3. *Let $\alpha > -1$ and X be an infinite-dimensional Banach space. Given a positive integer n, let $T^n_{\mathbf{u},\mathbf{\Phi}}$ be the operator given in* (6.5)*. Then the following assertions are equivalent:*

(a) $T^n_{\mathbf{u},\mathbf{\Phi}} : A^2_\alpha(\mathbf{D}) \to A^2_\alpha(\mathbf{D})$ *is Hilbert-Schmidt/order-bounded;*
(b) $T^n_{\mathbf{u},\mathbf{\Phi}} : wA^2_\alpha(\mathbf{D}, X) \to A^2_\alpha(\mathbf{D}, X)$ *is bounded;*
(c) $D^j_{u_j,\varphi_j} : A^2_\alpha(\mathbf{D}) \to A^2_\alpha(\mathbf{D})$ *is Hilbert-Schmidt/order-bounded for each j;*
(d) $D^j_{u_j,\varphi_j} : wA^2_\alpha(\mathbf{D}, X) \to A^2_\alpha(\mathbf{D}, X)$ *is bounded for each j;*
(e) u_j *and* φ_j *satisfy*

$$\int_{\mathbf{D}} \frac{|u_j|^2}{(1 - |\varphi_j|^2)^{\alpha+2+2j}} \, dA_\alpha < \infty$$

for each j.

Proof. For the Hilbert-Schmidtness, the equivalence of (a), (c) and (e) is due to Acharyya and Ferguson [1, Theorem 1]. So, we deduce from Corollary 6.1 the equivalences (a) $\iff$ (c) $\iff$ (e). The implication (d) $\implies$ (b) is trivial. Following the argument in the second part of the proof of Theorem 6.2, one may derive the implication (b) $\implies$ (a). Now, in order to complete the proof, we show below the implication (e) $\implies$ (d).

Fix j and consider an arbitrary $g \in wA^2_\alpha(\mathbf{D}, X)$. For $z \in \mathbf{D}$, note

$$\left\|g^{(j)}(z)\right\|_X = \sup_{x^* \in B_{X^*}} \left|x^*\left[g^{(j)}(z)\right]\right| = \sup_{x^* \in B_{X^*}} \left|(x^* \circ g)^{(j)}(z)\right|.$$

In addition, by the Cauchy Estimates based on (2.5), we have

$$\left|(x^* \circ g)^{(j)}(z)\right|^2 \lesssim \frac{\|x^* \circ g\|^2_{A^2_\alpha}}{(1-|z|^2)^{\alpha+2+2j}} \le \frac{\|g\|^2_{wA^2_\alpha(\mathbf{D},X)}}{(1-|z|^2)^{\alpha+2+2j}}$$

for $x^* \in B_{X^*}$. It follows that

$$\begin{aligned}\left\|D^j_{u_j,\varphi_j} g\right\|^2_{A^2_\alpha(\mathbf{D},X)} &= \int_{\mathbf{D}} \left\|g^{(j)}(\varphi_j)\right\|^2_X |u_j|^2 \, dA_\alpha \\ &\lesssim \|g\|^2_{wA^2_\alpha(\mathbf{D},X)} \int_{\mathbf{D}} \frac{|u_j|^2}{(1-|\varphi_j|^2)^{\alpha+2+2j}} \, dA_\alpha.\end{aligned}$$

As a consequence, we conclude that (e) implies (d). The proof is complete. □

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(CHOE) DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 02841, KOREA
Email address: cbr@korea.ac.kr

(GUO) SCHOOL OF STATISTICS AND MATHEMATICS, ZHONGNAN UNIVERSITY OF ECONOMICS AND LAW, WUHAN 430073, CHINA
Email address: xguo.math@zuel.edu.cn

(HOSOKAWA) FACULTY OF ENGINEERING, IBARAKI UNIVERSITY, HITACHI 316-8511, JAPAN
Email address: takuya.hosokawa.com@vc.ibaraki.ac.jp

(KOO) DEPARTMENT OF MATHEMATICS, KOREA UNIVERSITY, SEOUL 02841, KOREA
Email address: koohw@korea.ac.kr

(OHNO) NIPPON INSTITUTE OF TECHNOLOGY, MIYASHIRO, MINAMI-SAITAMA 345-8501, JAPAN
Email address: ons767j@yahoo.co.jp

(WANG) SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN 430072, CHINA
Email address: mfwang.math@whu.edu.cn