Binomial Methods

경원대학교 수학정보학과
문 경숙

목차

Introduction

- Asset price model

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ S(0) = S_0$$

- Option price
  - Black-Scholes PDE

$$\partial_t V + rS\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V = rV,\ 0 < t < T$$

$$V(S,T) = \Lambda(S),\ r: \text{riskless interest rate},\ \sigma: \text{volatility}$$

→ FDM, FEM

  - Risk-neutral measure

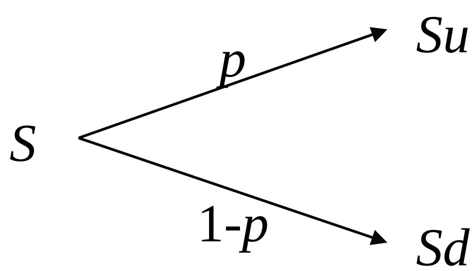
$$V(s,t) = E[e^{-r(T-t)}\max(S(T) - E, 0) \mid S(t) = s]$$

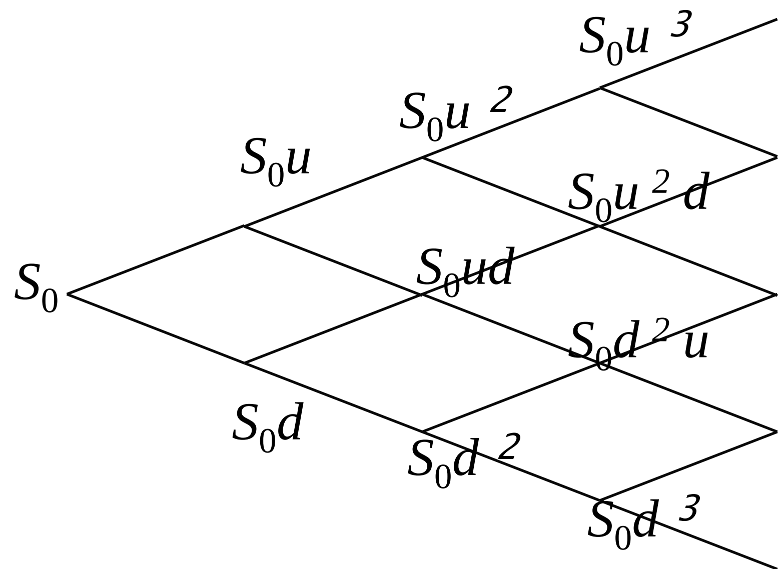
$$dS = rSdt + \sigma SdW(t)$$

→ Monte Carlo, Binomial Trees

Binomial method

- Idea: Asset price moves either up or down.





$$S_n^i = S_0 d^{i-n} u^n,\ 0 \le n \le i,\ 0 \le i \le M$$

Binomial mehod: multi-period

- Step1: Compute the asset price at $t = M\Delta t = T$

$$S_n^M = S_0 d^{M-n} u^n, \quad n = 0,1,\ldots,M$$

- Step2: Compute the terminal payoffs

$$V_n^M = \Lambda(S_n^M) = \max(S_n^M - E, 0), \quad n = 0,1,\ldots,M$$

- Step3: Backward expectation

$$V_n^M = e^{-r\Delta t}(pV_{n+1}^{i+1} + (1-p)V_n^{i+1}), \quad n = 0,1,\ldots,i, \; i = M-1,\ldots,0$$

- Option price V_0^0

Parameters

- From the original model : 3 unknowns, 2 eqs

$$pSu + (1-p)Sd = Se^{r\Delta t}$$

$$pS^2u^2 + (1-p)S^2d^2 = S^2e^{2r\Delta t+\sigma^2\Delta t}$$

  - Jarrow–Rudd approach

$$p = \frac{1}{2},\ u + d = 2e^{r\Delta t},\ u^2 + d^2 = 2e^{(2r+\sigma^2)\Delta t} \Rightarrow \begin{array}{l} u = e^{r\Delta t}(1+\sqrt{e^{\sigma^2\Delta t}-1}) \\ d = e^{r\Delta t}(1-\sqrt{e^{\sigma^2\Delta t}-1}) \end{array}$$

  - Cox–Ross–Rubinstein(CRR) approach

$$ud = 1,\ pu + (1-p)\frac{1}{u} = e^{r\Delta t},\ pu^2 + (1-p)\frac{1}{u^2} = e^{(2r+\sigma^2)\Delta t}$$

$$\Rightarrow \quad p = \frac{e^{r\Delta t} - d}{u-d}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

Parameters: general tree

- General tree with constant ν

$$ud = e^{2\nu\Delta t},\ u = e^{\nu\Delta t + \sigma\sqrt{\Delta t}},\ d = e^{\nu\Delta t - \sigma\sqrt{\Delta t}}$$

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{r-\nu}{\sigma}\right)\sqrt{\Delta t}$$

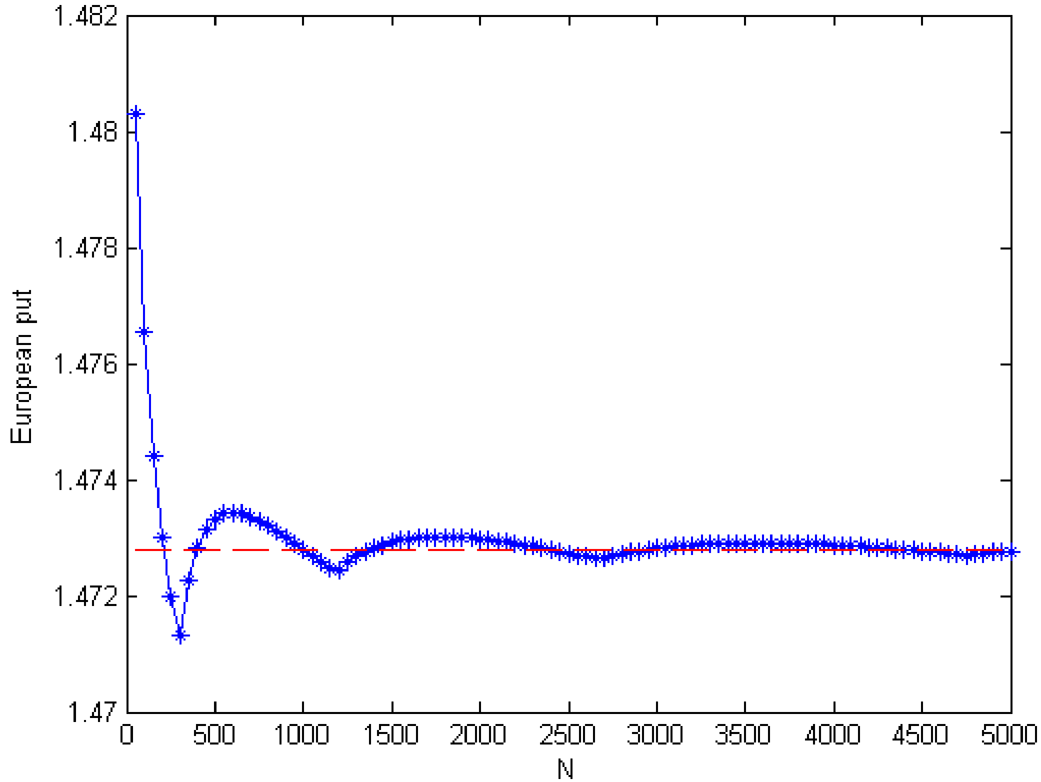
- CRR : $\nu = 0$
- JR : $\nu = r - \frac{1}{2}\sigma^2$

Program: European put option

- An introduction to financial option valuation, D. J. Higham (chapter 16)
- MATLAB code

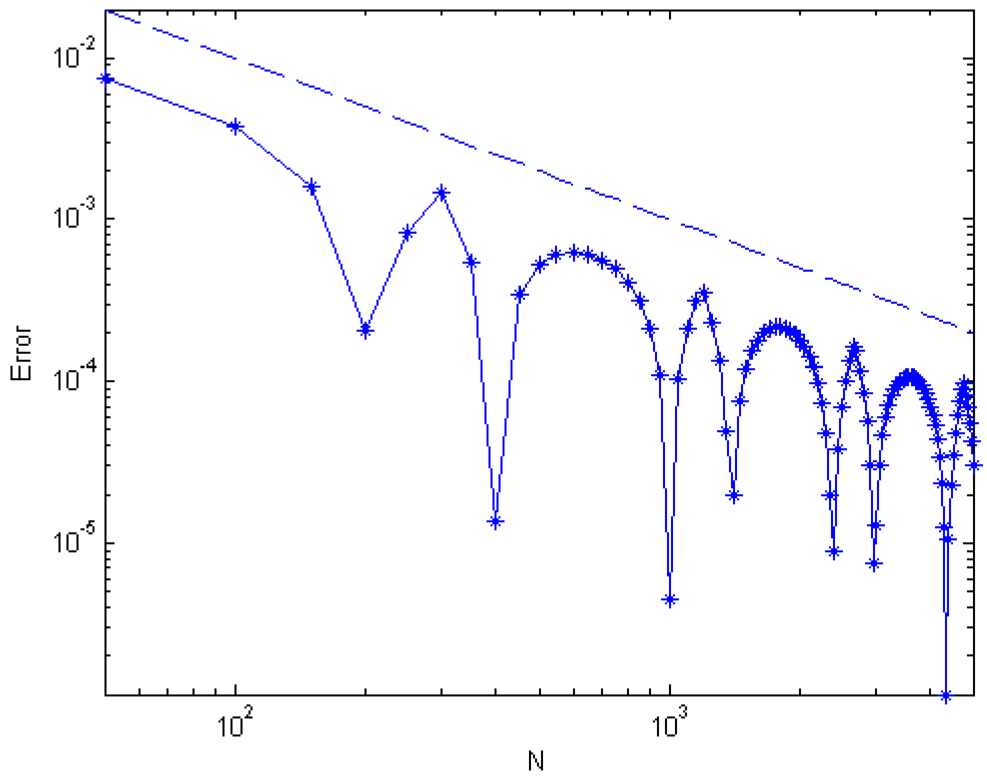
```
%%%%%%%%%% Problem and method parameters %%%%%%%%%%%%%
S = 9; E = 10; T = 3; r = 0.06; sigma = 0.3; M = 100; dt = T/M; p =0.5;
u = exp(sigma*sqrt(dt) + (r-0.5*sigma^2)*dt);
d = exp(-sigma*sqrt(dt) + (r-0.5*sigma^2)*dt);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Time T option values
W = max(E-S*d.^([M:-1:0]').*u.^([0:M]'),0);

% Work back to option value at time zero
for i = M:-1:1
   W = exp(-r*dt)*(p*W(2:i+1) + (1-p)*W(1:i));
end

disp('Option value is'), disp(W)
```

Result: option value



M	Option Value
100	1.4766
200	1.4730
400	1.4728
BS	1.4728

Result: computational error



M	Error
100	3.8e−3
200	2.1e−4
400	1.8e−5

Greeks and convergence

- Finite difference approxiamtion

$$\Delta = \frac{\partial V}{\partial S} \cong \frac{V_1^1 - V_0^1}{S_1^1 - S_0^1}$$

- Let the error $e_M = |V_0^0 - C(S_0, 0)|$ with $\Delta t = T / M$

⇒ There is a constant K such that $e_M \leq \frac{K}{M}$

American options

- American put: holder has the right to sell a prescribed asset at any time between the start date and a prescribed expiry date.
- Complementarity problem

$$\partial_t V + rS\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_{SS} V - rV \le 0,\ \ 0 < t < T \qquad (1)$$

$$V(S,t) \ge \Lambda(S(t)),\ \ 0 \le t \le T, S \ge 0 \qquad (2)$$

For each S, t one of (1) and (2) is at equality

$$V(S,T) = \Lambda(S(T)),\ \ S \ge 0$$

Binomial method for American

- American put

$$P(0,S_0) = \max_{0\le\tau\le T} E[e^{-r\tau}\max(E-S_\tau,0)]$$

$$dS_t = rS_t dt + \sigma S_t dW_t$$

- Idea: choose the best of two possibilities

$$V_n^i = \max[\max(E-S_n^i,0), e^{-r\Delta t}(pV_{n+1}^{i+1} + (1-p)V_n^{i+1})],$$

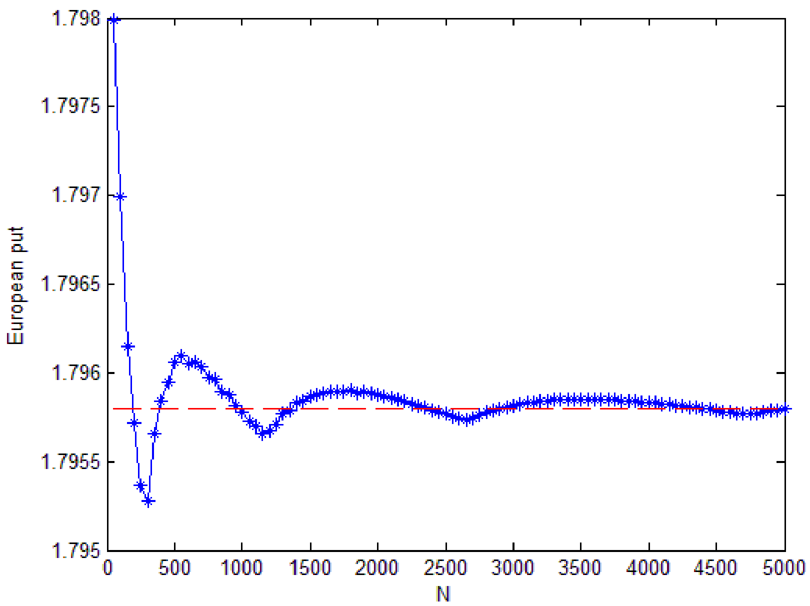
$$0\le n\le i, \quad 0\le i\le M\text{-}1$$

Program : American put

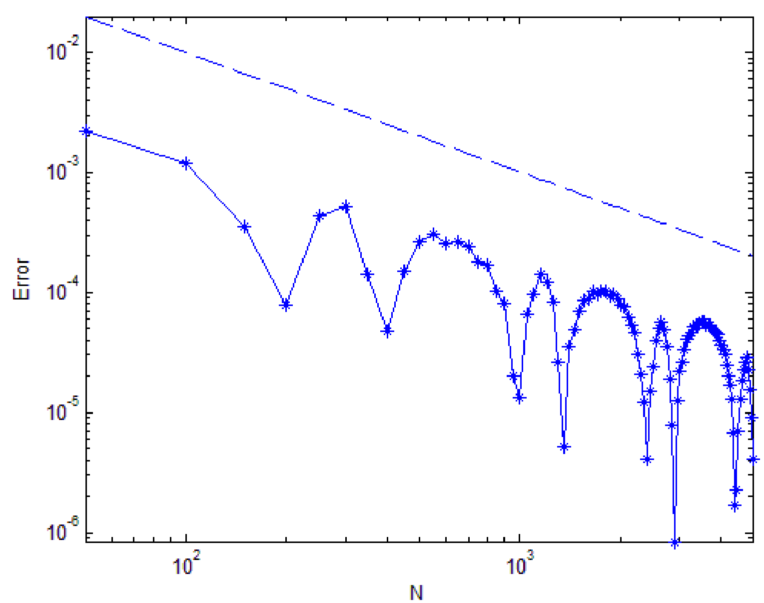
- MATLAB code (Higham chapter 18)

```
% Initial computations
dpowers = d.^([M:-1:0]');
upowers = u.^([0:M]');
% Time T option values
W = max(E-S*dpowers.*upowers,0);
% Work back to option value at time zero
for i = M:-1:1
    Si = S*dpowers(M-i+2:M+1).*upowers(1:i);
    W = max(max(E-Si,0),...
              exp(-r*dt)*(p*W(2:i+1)+(1-p)*W(1:i)));
end
```

Result : American put



Approx. Value with M=5000 = 1.7958



Exercises

1. Compute the ratio $(V_1^1 - V_0^1)/(S_1^1 - S_0^1)$ and explain why this can be regarded as an approximation to the time-zero delta.
2. Modify the program to compute the value of other exotic options, such as digital, barrier options.
3. Consider a case that a constant dividend yield D_0 paid on the underlying. Modify the parameters (u, d, p) for this case.

Reference

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