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Phase-field method for mean curvature flow

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Phase-field method for mean curvature flow

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Phase-field method for mean curvature flow

By

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Abstract

Mean curvature flow is a geometric deformation process for hypersurface of Euclidean space. We delve into studying mean curvature flow of smooth $n-$dimensional hypersurface in $\mathbb{R}^{n+1}$. Let M be an n-dimensional smooth manifold. We wish to study theoretical and numerical the behavior of smooth family $X(\cdot,t) : M \to \mathbb{R}^{n+1}$ moving by mean curvature flow. Thus, we first consider the initial value problem:

$$\begin{aligned} \frac{\partial}{\partial t} X(p,t) &= \vec{H}(p,t) \\ X(p,0) &= X_0(p) \end{aligned}$$

where $t > 0$ and $p = (x_1, \ldots, x_n)$ is a local coordinate on M and $\vec{H}(p,t)$ denotes the mean curvature vector at the point $X(p,t)$ of the embedded n-dimensional manifold $X(M,t)$, and $X_0 : M \to \mathbb{R}^{n+1}$ is a given embedding.

From a different point of view, the flow by mean curvature can be constructed as the singular limit of the parabolic Allen–Cahn equation. For mean curvature flow, we consider the semi-linear heat equation

$$\frac{\partial u}{dt} - \triangle u + \frac{W'(u)}{\epsilon^2} = 0 \quad \text{in } \mathbb{R}^{n+1} \times (0,\infty)$$

where u, W and ϵ are the order parameter, double well function and small real number. It is formulated that for $\epsilon \to 0$ the solution u^ϵ becomes ± 1 in an interior $u^\epsilon = 1$ and exterior region $u^\epsilon = -1$, respectively, and the interface between the regions is moved by mean curvature.

To see the behavior of the Allen–Cahn equation for mean curvature flow, we take an example of the self-similar solutions, and observe its convergence to mean curvature.

We refer finally to the works of Angenent [2], Huisken [8], Ilmanen [10], Evans, Soner and Souganidis [7], for a detailed analysis of mean curvature flow, self-similar solutions and asymptotic behaviors of the Allen–Cahn equation.

Key work :Allen-Cahn equation, mean curvature flow, phase-field.

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Chapter 1

Introduction

We begin with some basic settings. The following background materials can be found in [4, 5, 13, 15, 18, 19, 21]. Note that all manifolds M are smooth and n-dimensional in this and next chapters. It admits that for every point $p \in M$, there exists a coordinate neighborhood U of p in M such that in the local coordinates x_i corresponding to U.

1.1. The fundamental equations

1.1.1. The first fundamental form. We shall express the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to a parametrization $\mathbf{x}(u,v)$ at p. Then the parameter (u,v) is mapped to the point of $\mathbb{R}^{n+1}$. In coordinates $\mathbf{x}(u,v)$, the first fundamental form is described by the following symmetric, positive definite matrix g_{ij}

$$(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial u}\right) & \left(\frac{\partial \mathbf{x}}{\partial u}, \frac{\partial \mathbf{x}}{\partial v}\right) \\ \left(\frac{\partial \mathbf{x}}{\partial v}, \frac{\partial \mathbf{x}}{\partial u}\right) & \left(\frac{\partial \mathbf{x}}{\partial v}, \frac{\partial \mathbf{x}}{\partial v}\right) \end{pmatrix} \tag{1.1}$$

In terms of these parameters, one writes the first fundamental form as a quadratic differential

$$(ds)^2 = E(u,v)(du)^2 + 2F(u,v)dudv + G(u,v)(dv)^2,$$

where ds is called the element measure.

1.1.2. Orthogonal coordinates. Let U be a surface patch, and assume that the metric is given by

$$(ds)^2 = E(u,v)(du)^2 + G(u,v)(dv)^2,$$

so that $F(u,v)=0$. Moreover, if we consider the coordinate curve $u = const$, $v=v(s)$ or $u=u(s)$, $u=const$ with the parameter s, then the direction coefficients of the parametric curves are given by

$$\left(0, \frac{1}{\sqrt{G}}\right), \text{ or } \left(\frac{1}{\sqrt{E}}, 0\right),$$

since along the curve $v=const$, u alone varies, so that u can be taken as $ds^2 = Edu^2$, dv being zero. At each point on the surface, let us have a vector $\alpha(s)$ and its tangential vector $\alpha'(s)$. If we denote the direction coefficients of two directions $\mathbf{x}_u$, $\mathbf{x}_v$ at the same point by (l,m) and (l',m'), then the angle θ between these two directions is given by

$$\cos\theta = \alpha\cdot\alpha' = (l\mathbf{x}_u + m\mathbf{x}_v)\cdot(l'\mathbf{x}_u + m'\mathbf{x}_v) = Ell' + F(lm' + l'm+) + Gmm'.$$

Since we have the normal vector N of α and $\sin\theta$ such that

$$\begin{aligned} N\sin\theta &= \alpha\times\alpha' = (l\mathbf{x}_u + m\mathbf{x}_v)\times(l'\mathbf{x}_u + m'\mathbf{x}_v) \\ &= (\mathbf{x}_u\times\mathbf{x}_v)(lm' - l'm) = \sqrt{EG-F^2}N(lm'-l'm), \end{aligned}$$

we get

$$\sin\theta = \sqrt{EG-F^2}(lm'-l'm).$$

If we have the parametric curve C for $v=const$, then the direction coefficients are given by

$$l = \frac{1}{\sqrt{E}},\ m=0,\ l'=u',\ m'=v'.$$

Further $F=0$, we have that

$$\cos\theta = \sqrt{E}u', \quad \sin\theta = \sqrt{G}v'. \tag{1.2}$$

1.2. Geodesics

Let C be an oriented regular curve on surface S, and let $\alpha(s)$ be a parametrization of C, in neighborhood of $p\in S$, by the arc length s. When T is the unit tangent vector of a curve C on the surface S, this curvature vector $\vec{\kappa}$ is decomposed into dT/ds, i.e.,

$$\frac{dT}{ds} = \vec{\kappa} = \vec{\kappa_n} + \vec{\kappa_g}.$$

where $\vec{\kappa}$, $\vec{\kappa_n}$ and $\vec{\kappa_g}$ are curvature, normal curvature and tangential curvature vectors, respectively. The tangential curvature vector is also called the geodesic curvature vector. In word, the curvature vector is the sum of the normal and the tangential curvature vectors. Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \to S$ on an oriented surface S in some interval $I \subset \mathbb{R}$. Since $w(t),\ t \in I$ is a unit vector field, $(dw/dt)(t)$ is normal to $w(t)$, and

$$\frac{Dw}{dt} = \lambda(N \times w(t)), \tag{1.3}$$

where λ is the algebraic value of the covariant derivative of w at t, and is defined by

$$\lambda(t) = \left[\frac{Dw}{dt}\right] = \left(\frac{dw}{dt}, N \times w\right). \tag{1.4}$$

We now consider the vector product. The following rules are readily checked.

$$\begin{aligned}
&u \times v = -v \times u,\\
&(au + bw) \times v = au \times v + bw \times v,\\
&u \times v = 0 \text{ if and only if } u \text{ and } v \text{ are linearly independent},\\
&(u \times v) \cdot u = 0,\ (u \times v) \cdot v = 0,\\
&(u \times v) \times w = (u, w)v - (v, w)u
\end{aligned} \tag{1.5}$$

where u, v, w are vectors and a, b are scalars.

Lemma 1.2.1. *Let a and b be differentiable functions in I with $a^2 + b^2 = 1$ and ϕ_0 be such that $a(t_0) = \cos\phi_0,\ b(t_0) = \sin\phi_0$. Define the differentiable function*

$$\phi = \phi_0 + \int_{t_0}^{t} (ab' - ba')dt, \tag{1.6}$$

then it is such that $\cos\phi(t) = a(t)$, $\sin\phi(t) = b(t),\ t \in I$, and $\phi(t_0) = \phi_0$.

PROOF. Note that $\cos\phi(t) = a(t)$ and $\sin\phi(t) = b(t)$ follow immediately from

$$\begin{aligned} 0 &= (a-\cos\phi)^2 + (b-\sin\phi)^2 \\ 1 &= a\cos\phi + b\sin\phi, \end{aligned} \tag{1.7}$$

since $a^2+b^2=1$. Differentiating equation (1.6), (1.7) and $a^2+b^2=1$ give that

$$\begin{aligned} 0 &= a'\cos\phi - a(\sin\phi)\phi' + b'\sin\phi + b(\cos\phi)\phi' \\ &= a'\cos\phi + \frac{bb'}{a'}(\sin\phi)(ab'-ba') + b'\sin\phi - \frac{aa'}{b'}(\cos\phi)(ab'-ba') \\ &= -b'(\sin\phi)(a^2+b^2) - a'(\cos\phi)(a^2+b^2) + a'\cos\phi + b'\sin\phi. \end{aligned}$$

If we set $h(\phi) := a\cos\phi + b\sin\phi$, then $h'(\phi) = 0$. Therefore $h(\phi) = const$ and $h(\phi_0) = 1$, as required. □

Let v and w be two differentiable vector fields along the parametrized curve $\alpha : I \to S$ with $|v(t)| = |w(t)| = 1,\ t \in I$. We define a differentiable function $\phi : I \to \mathbb{R}$ in such a way that $\phi(t),\ t \in I$, is a determination of the angle from $v(t)$ to $w(t)$ in the orientation of S. , we see the differentiable vector field $\bar{v}$ along α, defined by the condition that $\{v(t), \bar{v}(t)\}$ is an orthonormal positive basis for every $t \in I$. Thus, this gives that

$$w(t) = a(t)v(t) + b(t)\bar{v}(t). \tag{1.8}$$

where a and b are differentiable functions in I and $a^2+b^2=1$.

LEMMA 1.2.2. *Let v and w be two differentiable vector fields along the curve $\alpha : I \to S$, with $|w(t)| = |v(t)| = 1,\ t \in I$. Then we have*

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{\partial\phi}{dt},$$

where ϕ is the angle from v to w.

Proof. We set the vectors $\bar{v} = N \times v$ and $\bar{w} = N \times w$. From (1.2.1), (1.5) and (1.8),

$$w = (\cos\phi)v + (\sin\phi)\bar{v}, \tag{1.9}$$

$$\bar{w} = N \times w = (\cos\phi)N \times v + (\sin\phi)N \times \bar{v} = (\cos\phi)\bar{v} - (\sin\phi)v. \tag{1.10}$$

Differentiating (1.9) with respect to t gives

$$w' = -(\sin\phi)\phi' v + (\cos\phi)v' + (\cos\phi)\phi'\bar{v} + (\sin\phi)\bar{v}'.$$

Next we take the inner product $(w', \bar{w})$ using the fact that $(v, \bar{v}) = 0$, thus, $(v', \bar{v}) = -(v, \bar{v}')$. Further, we have that $(v, \bar{v}) = 0$, $(v, v') = 0$ and (1.10), then

$$\begin{aligned}(w', \bar{w}) &= (\sin^2\phi)\phi' + (\cos^2\phi)(v', \bar{v}) + (\cos^2\phi)\phi' - (\sin^2\phi)(\bar{v}', v)\\ &= \phi' + (\cos^2\phi)(v', \bar{v}) - (\sin^2\phi)(\bar{v}', v).\end{aligned}$$

We get

$$(w', \bar{w}) = \phi' + (\cos^2\phi + \sin^2\phi)(v', \bar{v}) = \phi' + (v', \bar{v}).$$

The rest follows easily by combining (1.3) with

$$(w', \bar{w}) = \left(\frac{dw}{dt}, \bar{w}\right) = \left[\frac{Dw}{dt}\right](N, \times w, \bar{w}) = \left[\frac{Dw}{dt}\right].$$

Thus, this concludes that

$$\begin{aligned}\left[\frac{Dw}{dt}\right] &= (w', \bar{w}) = \phi' + (v', \bar{v}) = \frac{d\phi}{dt} + \left[\frac{Dv}{dt}\right],\\ \left[\frac{Dw}{dt}\right] &- \left[\frac{Dv}{dt}\right] = \frac{d\phi}{dt}.\end{aligned}$$

□

Our proof has a geometrical interpretation. If we assume that C is a curve on surface S, and $\alpha(s)$ is parametrized by the arc length s of C at $p \in C$, and $v(s)$ a parallel field along $\alpha(s)$. Then by taking $w(s) = \alpha'(s)$, we get

$$\kappa_g(s) = \left[\frac{D\alpha'(s)}{ds}\right] = \frac{d\phi}{ds}. \tag{1.11}$$

Thus, the geodesic curvature is the rate of change of the angle that the tangent to the curve makes with a parallel direction along the curve.

LEMMA 1.2.3. *Let $\mathbf{x}(u,v)$ be an orthogonal parametrization of a neighborhood of an oriented surface S, and $w(t)$ be a differentiable field of unit vectors along the curve $\mathbf{x}(u(t),v(t))$. Then we have*

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) + \frac{d\phi}{dt}, \tag{1.12}$$

where $\phi(t)$ is the angle form x_u to $w(t)$.

Observe that $e_1 = \mathbf{x}_u/\sqrt{E}$, $e_2 = \mathbf{x}_v/\sqrt{G}$ be the unit vectors tangent to the coordinate curves. By (1.4), $e_1 \times e_2 = N$ and (1.2.2), we have that

$$\left[\frac{Dw}{dt}\right] = \left[\frac{De_1}{ddt}\right] + \frac{d\phi}{dt}, \tag{1.13}$$

where $e_1(t) = e_1(u(t),v(t))$. For the next, we find

$$\left[\frac{De_1}{dt}\right] = \left(\frac{de_1}{dt}, N \times e_1\right) = \left(\frac{de_1}{dt}, e_2\right) = ((e_1)_u, e_2)\frac{du}{dt} + ((e_1)_v, e_2)\frac{dv}{dt}. \tag{1.14}$$

We get $(\mathbf{x}_{uu}, \mathbf{x}_v) = -E_v/2$, owing to $F = 0$ and therefore

$$((e_1)_u, e_2) = \left(\left(\frac{\mathbf{x}_u}{\sqrt{E}}\right)_u, \frac{\mathbf{x}_v}{\sqrt{G}}\right) = -\frac{1}{2}\frac{E_v}{\sqrt{EG}}. \tag{1.15}$$

In the same fashion,

$$((e_1)_v, e_2) = \frac{1}{2}\frac{G_u}{\sqrt{EG}}. \tag{1.16}$$

Combining (1.14), (1.15) and (1.16) and plugging this equation into (1.13) give that

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) + \frac{d\phi}{dt}.$$

Chapter 2

Mean curvature flow

2.1. Geometry of hypersurface

We denote the embedding map by $X : M \to \mathbb{R}^{n+1}$. The tangent vectors $\partial_i X(p) \equiv \partial X(p)/\partial p_i$, $1 \le i \le n$ form a basis of the tangent space $T_x M$ at $x = X(p)$ at every $p \in M$. The metric is the same as (1.1) , but given differently by

$$g_{ij} = \partial_i X \cdot \partial_j X \quad \text{for } 1 \le i, j \le n, \tag{2.1}$$

the inverse metric by

$$g^{ij} = g_{ij}^{-1}, \tag{2.2}$$

and the area element is given by

$$\sqrt{g} = \sqrt{\det g_{ij}}. \tag{2.3}$$

Let ν be a choice of unit normal field to M. In particular, this satisfies $\nu \cdot \partial_i X = 0$ on M for $1 \le i \le n$. Note also that $\partial_i \nu$ is tangential vector field to M for $1 \le i \le n$ since ν has unit length. The second fundamental form of M is defined by

$$h_{ij} = \partial_i \nu \cdot \partial_j X = -\nu \cdot \partial_i \partial_j X. \tag{2.4}$$

The eigenvalues $\kappa_1, \ldots, \kappa_n$ of the Weingarten map given by $A_i^j = g^{ik} h_{kj}$ are called the principal curvatures of M. The mean curvature H can be expressed in terms of the values such that

$$H = \sum_i^n \kappa_i = g^{ij} h_{ij} = g^{ij} \partial_i \nu \cdot \partial_j F = \operatorname{div}^M \nu. \tag{2.5}$$

The mean curvature vector of M is given by

$$\vec{H} = -H\nu = -(\operatorname{div}^M \nu)\nu. \tag{2.6}$$

The tangential gradient of a function $h : M \to \mathbb{R}$ is denoted by $\nabla^M f$ and defined by

$$\nabla^M h = \nabla h - (\nabla h, \nu)\nu. \tag{2.7}$$

Thus, it is checked that $\nabla^M h$ is the projection of the standard Euclidean gradient ∇h on the hypersurface tangent to M. Given a vector field $V : M \to \mathbb{R}^{n+1}$, we denote the components of V and the vectors of the canonical basis in $\mathbb{R}^{n+1}$ by $V^1, \ldots, V^{n+1}$ and $e_1, \ldots, e_{n+1}$, respectively. Then $\operatorname{div}^M V : M \to \mathbb{R}$ is defined by

$$\operatorname{div}^M V = \sum_{\alpha=1}^{n+1} (\nabla^M V_\alpha) \cdot e_\alpha. \tag{2.8}$$

where ν_α is the α-component of ν. If we derive a more explicit form of $\operatorname{div}^M V$ using (2.7),

$$\operatorname{div}^M V = \sum_{\alpha=1}^{n+1} \left(\nabla V_\alpha - \left(\sum_{\beta}^{n+1} \frac{\partial V_\alpha}{\partial y_\beta} \nu_\beta \right) \nu \right) \cdot e_\alpha. \tag{2.9}$$

The divergence theorem states that if M is a smooth and closed compact support manifold, then for any $V : M \to \mathbb{R}$,

$$\int_M \operatorname{div}^M V d\mu = \int_M H(V\nu) d\mu. \tag{2.10}$$

To see that, we decompose V into its tangent and normal parts:

$$V = V^\top + V^\perp,$$

where $V^\perp = (V, \nu)\nu$. Then we have

$$\operatorname{div}^M V^\perp = (\nu \cdot V) \operatorname{div} \nu = (\operatorname{div}^M \nu \cdot \nu, V) = -(\vec{H}, V).$$

On the other hand, the tangential component $V^\top$ can be evaluated by the divergence theorem

$$\int_M \operatorname{div}^M V^\top = \int_{\partial M} (V^\top, \vec{n}),$$

where $\vec{n}$ is the inward pointing unit normal along ∂M, tangent to M. But because V has compact support, this reduces to zero and therefore

$$\int_M \operatorname{div}^M V = \int_M (\operatorname{div}^M V^\top + \operatorname{div}^M V^\perp) = -\int_M (\bar{H}, V). \tag{2.11}$$

2.2. Evolution of geometry

Recall that $X(\cdot, t)$ is a one-parameter family of smooth hypersurface as in the previous section. We say that it is a solution of mean curvature flow if

$$\frac{\partial}{\partial t} X(p,t) = -H(p,t)\nu(p,t), \quad p \in M^n,\ t > 0. \tag{2.12}$$

For the given family X evolving by mean curvature flow, we denote by Φ the function f evaluated on M, i.e.,

$$\Phi(p,t) = f(X(p,t),t), \quad p \in M,\ t \in [0,T).$$

Then we have

$$\frac{\partial}{\partial t}\Phi = \nabla f \cdot \frac{\partial F}{\partial t} + \frac{\partial f}{\partial t}. \tag{2.13}$$

In order to prove the following, we use the fact that for a non-singular metric g_{ij} with its inverse g^{ij}, the determinant g evolves according to

$$\frac{d}{dt} g = \frac{d}{dt}(\det g) = \det g \operatorname{Tr}\left(g^{-1}\frac{d}{dt} g\right) = (\det g) g^{ij} \frac{d}{dt} g_{ij}. \tag{2.14}$$

Lemma 2.2.1. *(Evolution equations* [9]*) Under mean curvature flow,*

$$\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}, \tag{2.15}$$

$$\frac{\partial}{\partial t}\sqrt{g} = -H^2\sqrt{g}. \tag{2.16}$$

Proof. Since X_t with time variable t, moves by mean curvature $\partial X/\partial t = H\nu$, we can commute the mixed partial derivatives below to get

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t} =& \frac{\partial}{\partial t}\left(\frac{\partial X}{\partial x_i}, \frac{\partial X}{\partial x_j}\right) = \left(\frac{\partial}{\partial x_i}(-H\nu), \frac{\partial X}{\partial x_j}\right) + \left(\frac{\partial X}{\partial x_i}, \frac{\partial}{\partial x_j}(-H\nu)\right) \\ =& -H\left(\frac{\partial \nu}{\partial x_i}, \frac{\partial X}{\partial x_j}\right) - H\left(\frac{\partial X}{\partial x_i}, \frac{\partial \nu}{\partial x_j}\right) = -2Hh_{ij}.\end{aligned}$$

To see the second equation, we use equations (2.14), (2.5) and (2.15). Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}\sqrt{g} &= \frac{1}{2\sqrt{g}}\frac{\partial}{\partial t}g = \frac{1}{2\sqrt{g}}gg^{ij}\frac{\partial}{\partial t}g_{ij} = \frac{1}{2}\sqrt{g}g^{ij}(-2Hh_{ij}) = -Hg^{ij}h_{ij}\sqrt{g}\\ &= -H^2\sqrt{g}\end{aligned}$$

as required. □

If $d\mu_t$ is the measure on M_t, then $\mu = \sqrt{\det g_{ij}}$, and the area element of a solution of mean curvature flow satisfies the evolution equation

$$\frac{\partial}{\partial t}d\mu_t = -|\vec{H}|^2 d\mu_t \tag{2.17}$$

for all $t \in I$ the interval of times.

LEMMA 2.2.2. *Let M_t move by mean curvature flow in an open subset $U \subset R^{n+1}$ there holds*

$$\frac{d}{dt}\int_{M_t} f d\mu = \int_{M_t}\left(\frac{\partial f}{\partial t} - H\nabla f\cdot\nu - H^2 f\right)d\mu \tag{2.18}$$

for all the time interval I and $f \in C_0^1$.

PROOF. Let us see that for a test function f, we have, by definition of mean curvature flow and by (2.13) and (2.17)

$$\begin{aligned}\frac{d}{dt}\int_{M_t} f d\mu &= \int_{M_t}\left(\frac{\partial f}{\partial t} + \nabla f\cdot\frac{\partial X}{\partial t}\right)d\mu + \int_{M_t} f\frac{\partial}{\partial t}d\mu\\ &= \int_{M_t}\left(\frac{\partial f}{\partial t} - H\nabla f\cdot\nu - H^2 f\right)d\mu.\end{aligned}$$

□

2.2.1. Monotonicity formula. Let $\rho(y,t)$ be the backward heat kernel for any $\bar{t} \geq t_1$ with $t \in (t_0, t_1)$,

$$\rho(y,t) = \frac{1}{(4\pi(\bar{t}-t))^{n/2}}\exp\left(-\frac{|y|^2}{4(\bar{t}-t)}\right), \quad y \in \mathbb{R}^{n+1},\ t < \bar{t}.$$

THEOREM 2.2.1. *(Huisken's theorem from [8]) If a surface M_t satisfies for $t < 0$ then we have the monotonicity formula*

$$\frac{d}{dt}\int_{N_t} \rho(x,t)d\mu_t = -\int_{N_t} \rho(x,t)\left|\vec{H} - \frac{1}{2t}\vec{X}^{\perp}\right| d\mu_t,$$

where $X^{\perp}$ is the normal component of X.

PROOF. Let us introduce the vector field $V : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$V = y\exp\left(\frac{|y|^2}{4\tau}\right)$$

where $y = (y_1, \ldots, y_{n+1})$ and $\tau = \bar{t} - t$. Applying the tangential divergence of V and considering the below equation

$$\frac{\partial V_\alpha}{\partial y_\beta} = \left(\delta_{\alpha\beta} - \frac{y_\alpha y_\beta}{2\tau}\right)\exp\left(-\frac{|y|^2}{4\tau}\right)$$

give that

$$\begin{aligned}\operatorname{div}^M V &= \left(n+1-\frac{|y|^2}{2\tau}\right)\exp\left(-\frac{|y|^2}{4\tau}\right) - \left(|\nu|^2 - \frac{(y,\nu)^2}{2\tau}\right)\exp\left(-\frac{|y|^2}{4\tau}\right)\\ &= \left(n - \frac{|y^{\perp}|^2}{2\tau}\right)\exp\left(-\frac{|y|^2}{4\tau}\right),\end{aligned}$$

where $y^{\perp}$ denotes the normal component of y. Together with the divergence theorem (2.11), we get

$$\int_M \left(n - \frac{|y^T|^2}{2\tau}\right)\exp\left(-\frac{|y|^2}{4\tau}\right)d\mu = \int_M H(V\cdot\nu)d\mu. \tag{2.19}$$

We start off again by introducing $\tau = \bar{t} - t$ and calculate

$$\frac{\partial\rho}{\partial t} = \left(\frac{n}{2\tau} - \frac{|y|^2}{4\tau^2}\right)\rho, \quad \nabla\rho = -\frac{y}{2\tau}\rho. \tag{2.20}$$

From this, we get

$$\frac{\partial\rho}{\partial t} - H\nabla\rho\cdot\nu - H^2\rho = \frac{n}{2\tau}\rho - \left|\frac{y}{2\tau} - H\nu\right|^2\rho + H\nabla\rho\cdot\nu.. \tag{2.21}$$

From (2.20),

$$\nabla\rho = -\frac{\rho}{2\tau}\cdot y\exp\left(-\frac{|y|^2}{4\tau}\right)/\exp\left(-\frac{|y|^2}{4\tau}\right) = -\frac{\rho}{2\tau}\frac{V}{\exp\left(-\frac{|y|^2}{4\tau}\right)}$$

Observe that V can be expressed by $\nabla\rho$ and a space independent variable such that

$$V = \nabla\rho \frac{\exp\left(-\frac{|y|^2}{4\tau}\right)}{-\frac{\rho}{2\tau}} = -\frac{(4\pi\tau)^{n/2}}{2\tau}.$$

Therefore, applying this with equation (2.19), we have

$$\int_M H(\nabla\rho\cdot\nu)d\mu = -\int_M \left(n - \frac{|y|^2}{2\tau}\right)\frac{\rho}{2\tau}d\mu. \tag{2.22}$$

Putting together with (2.18), (2.21) and (2.22) gives

$$\frac{d}{dt}\int_M \rho(y,t)d\mu = \int_M \left(-\left|\frac{y}{2\tau} - H\nu\right|^2\rho + \frac{|y^\top|^2}{4\tau^2}\rho\right)d\mu. \tag{2.23}$$

Note that

$$\left|\frac{y}{2\tau} - H\nu\right|^2 = \left|\left(\frac{y^\perp}{2\tau} - H\nu\right) + \frac{y^\top}{2\tau}\right|^2 = \left|\frac{y^\perp}{2\tau} - H\nu\right|^2 + \frac{|y^\top|^2}{4\tau^2}. \tag{2.24}$$

Combining (2.24) with (2.23) and $y \equiv X$ give

$$\frac{d}{dt}\int_{M_t}\rho(x,t)d\mu_t = -\int_{M_t}\rho(x,t)\left|\vec{H} - \frac{1}{2t}\vec{X}^\perp\right|d\mu_t. \tag{2.25}$$

$\square$

2.3. Self-similar solutions

A self-similar solution forms the surface that translates or shrinks under homothetically. In this section, self-shrinking doughnut and self-translating grim reaper solutions are considered. Note that there are other examples such as plane, sphere, cylinder, Tom Ilmanen's shrinker of genus 8 [10] and [11], self- shrinking trumpet ends [12], toruspheres [16] and desingularizing the intersection of grim reapers [17]. We first deduce the equation that holds for any self-similar solution. For convenience, we assume that the surface X_t becomes singular at $T = 0$. We further impose the ansatz as the self-similar shrinker

$$P_t = \lambda(t)\cdot P_0 \tag{2.26}$$

where P_1 is the surface at time $t = -1$ and $\lambda(t) : \mathbb{R}^- \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is the homothety factor such that $\lambda(-1) = 1$ and $\lambda(0) = 0$. Suppose X be the embedding of M and $\bar{X} = \lambda \cdot X$, then the metric $\bar{g}_{ij}$ and second fundamental form $\bar{h}_{ij}$ of the X_t change accordingly

$$\bar{g}_{ij} = \partial_i X_t \cdot \partial_j X_t = \lambda^2 g_{ij}$$

$$\bar{h}_{ij} = -\nu \cdot \partial_i \partial_j X_t = \lambda h_{ij}.$$

Note that we get

$$\bar{H} = \bar{g}^{ij} \bar{h}_{ij} = \frac{1}{\lambda^2} g^{ij} \cdot \lambda h_{ij} = \frac{1}{\lambda} H. \tag{2.27}$$

Now we apply the ansatz $P_t = \lambda(t) \cdot P_0$ into the mean curvature flow equation

$$\frac{\partial}{\partial t}(\lambda(t) \cdot X) = \frac{\vec{H}}{\lambda(t)}.$$

Now working on,

$$\frac{d\lambda}{dt} \cdot \lambda \cdot X = \vec{H}, \tag{2.28}$$

we get the following ordinary difference equation

$$\frac{d\lambda}{\lambda} \cdot \lambda = C,$$

where C is a constant, and the equation is independent of $\vec{H}$ and X since it is the self-shrinking surface. Thus, solve it, we have $\lambda(t) = \sqrt{C \cdot t}$. Applying the conditions $\lambda(-1) = 1$ and $\lambda(0) = 0$, we have

$$\lambda(t) = \sqrt{-t}. \tag{2.29}$$

This leads to

$$\lambda(t) = -t^{\frac{1}{2}}, \quad \lambda'(t) = -\frac{1}{2}(-t)^{-\frac{1}{2}}, \quad \lambda(t) \cdot \lambda'(t) = -\frac{1}{2}. \tag{2.30}$$

If we plug (2.29) into (2.28) and use (2.30), we get the the following elliptic parametric equation

$$H + \frac{X \cdot \nu}{2} = 0. \tag{2.31}$$

2.3.1. Liouville's formula.

LEMMA 2.3.1. *If $\alpha(s)$ is a parametrization on the surface S. Let $\phi(s)$ be the angle between x_u and $\alpha'(s)$. Then*

$$\kappa_g = \frac{d\phi}{ds} - \frac{(\sqrt{E})_v}{\sqrt{EG}}\cos\phi + \frac{(\sqrt{G})_u}{\sqrt{EG}}\sin\phi. \tag{2.32}$$

This can be written as

$$\kappa_g = \frac{d\phi}{ds} + \kappa_u\cos\phi + \kappa_v\sin\phi,$$

where

$$\kappa_u = -\frac{(\sqrt{E})_v}{\sqrt{EG}}, \quad \kappa_v = \frac{(\sqrt{G})_u}{\sqrt{EG}}.$$

Thus, κ_u and κ_v are the geodesic curvatures of the curves $\{v = const\}$, and the curves $\{u = const\}$, respectively.

PROOF. If we let $w = \alpha'(s)$ in (1.12) and use (1.11), we have

$$\kappa_g = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{ds} - E_v\frac{du}{ds}\right) + \frac{d\phi}{ds}. \tag{2.33}$$

For the orthogonal coordinate, we have $dv/ds = 0$ and $du/ds = 1/\sqrt{E}$ in the case of $v = const$ and $u = u(s)$. For the each case, we plug these equations into (2.33). Then we obtain the following equations, respectively.

$$(\kappa_g)_1 = -\frac{E_v}{2E\sqrt{G}},$$
$$(\kappa_g)_2 = \frac{G_u}{2G\sqrt{E}}.$$

If we put these equations in (2.33), then

$$\kappa_g = (\kappa_g)_1\sqrt{E}\frac{du}{ds} + (\kappa_g)_2\sqrt{G}\frac{dv}{ds} + \frac{d\phi}{ds}. \tag{2.34}$$

Since (1.2), we finally have the following

$$\kappa_g = (\kappa_g)_1\cos\phi + (\kappa_g)_2 + \frac{d\phi}{ds}.$$

□

2.3.2. Self-shrinking doughnut. By definition a self-similar shrinker (2.26), one finds that self similar solution satisfies (2.31). Indeed, the solutions to (2.31) are exactly the immersions $X_0 : M^n \to R^{n+1}$ at which the functional

$$A(X) = \int_{M^n} e^{-|X(p)|^2/4} d\sigma_X^n (p$$

is stationary. The measure $d\sigma_X^n$ is called the n-dimensional volume element which $X : M \to R^{n+1}$ induces on M. A direct calculation verifies that first variation of $A(X)$ under a normal variation $X(\epsilon, p) = X_0(p) + \epsilon u(p)\nu_{X_0}(p)$ for any given $C^\infty(M^n)$ is given by

$$\left.\frac{dA(X_0 + \epsilon u \nu_{X_0})}{d\epsilon}\right|_{\epsilon=0} = -\int_{M^n} e^{-|X(p)|^2/4} \left\{ nH_{X_0}(p) + \frac{1}{2}(X_0(p), \nu_{X_0}) \right\} d\sigma_X^n(p).$$

Thus, the solutions to (2.31) are the minimal hypersurfaces in $\mathbb{R}^{n+1}$ with respect to the metric $ds^2 = e^{-|x|^2/4n}\{(dx_0)^2 + \ldots + (dx_n)^2\}$. If we consider hypersurfaces of revolution, the X_0 has the form

$$X_0(s, \omega) = x(s)\vec{e}_0 + r(s)\vec{\omega} : \quad (a, b) \times \mathbb{S}^{n-1} \to \mathbb{R}^{n+1},$$

where we rotate an $x_0 - x_1$ plane curve $x(s), r(s)$ with $r(s) > 0$ ($s \in (a, b)$) around the x_0 axis. We note that $\mathbb{S}^{n-1}$ is the standard unit sphere and $\vec{e}_0$ is the first unit basis vector $(1, 0, \cdots, 0)$ in $\mathbb{R}^{n+1}$.

To construct a self-similar doughnut in $\mathbb{R}^3$, we are required to find the curve $\{(x(s), r(s) | s \in [a, b])\}$, $a, b \in \mathbb{R}$, is geodesic in the upper half plane $r > 0$ with metric $ds^2 = r^2 e^{(-x^2 + r^2)}(dx^2 + dr^2)$. In fact, the volume element $d\sigma_X^n$ is given by

$$d\sigma_X^n = r(s)^{n-1}\sqrt{x'(s)^2 + r'(s)^2} \cdot ds \ d\omega^{n-1},$$

where $d\omega^{n-1}$ is the volume element on the $n-1$ sphere. Thus, we have

$$\begin{aligned} A(x_0) &= \int_a^b \int_{\mathbb{S}^{n-1}} r^{n-1}(s)\sqrt{x'(s)^2 + (r'(s))^2} \cdot e^{-(x^2(s)+r^2(s))/4} ds d\omega^{n-1} \\ &= vol(\mathbb{S}^{n-1}) \int_a^b r^{n-1}(s)\sqrt{x'(s)^2 + r'(s)^2} ds. \end{aligned}$$

The functional $A(X)$ will be stationary at X corresponding to $x(s), r(s)$ if and only if the curve $\{(x(s), r(s)) : s \in (a, b)\}$ is a geodesic in the upper half plane $\{r > 0\}$

with metric

$$(ds)^2 = r(s)^{2(n-1)} e^{-(x^2+r^2)/4}((dx)^2 + (dr)^2).$$

Then the Liouville's formula 2.32 induces the following ordinary differential equations and the curve forms a torus by rotating a simple closed curve around an r-axis. We denote the angle of the unit tangent vector $(dx/ds, dr/ds)$ by θ and s is the arc-length parameter. Plugging into the formula, we get

$$\begin{aligned}
\frac{d\theta}{ds} &= \frac{1}{2}\left\{\frac{\partial}{\partial r}\log(r^{2(n-1)} \cdot e^{-(x^2+r^2)}) \cdot \cos\theta - \frac{\partial}{\partial x}\log(r^{2(n-1)} \cdot e^{-(x^2+r^2)}) \cdot \sin\theta\right\} \\
&= \frac{1}{2}\left[\left\{\frac{2(n-1)}{r} - 2r\right\}\cos\theta + 2x\sin\theta\right].
\end{aligned}$$

Since the geodesic must be zero, we have that

$$\begin{aligned}
\dot{x} &= \cos\theta, \\
\dot{r} &= \sin\theta, \\
\dot{\theta} &= \frac{x}{2}\sin\theta + \left(\frac{n-1}{r} - \frac{r}{2}\right)\cos\theta,
\end{aligned} \tag{2.35}$$

where prime denotes a derivative with respect to the parameter s. To find such a curve, we are allowed to use a shooting method.

2.3.3. Self-translating solution. Consider a translating solution of mean curvature flow given by that

$$u(x,t) = \alpha t - \frac{\log\cos(\alpha x)}{\alpha}, \tag{2.36}$$

for time $t \in I = [0, \infty)$, $x \in (-\pi/2, \pi/2)$. We see that the solutions move at the speed α. We refer the reader to [2, 5, 21] for an explanation.

Let us consider that our surface is an entire graph $u(\cdot, t)$ on $\mathbb{R}^n$, and time $t \in I$. Then we can express the graph of the first n components such that

$$X(p,t) = (\mathbf{x}, u(\mathbf{x}), t)) \tag{2.37}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Now taking the derivative with respect to time t of the equation (2.37), we have

$$\frac{\partial X}{\partial t} = \left(\frac{\partial \mathbf{x}}{\partial t}, \nabla u \cdot \frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u}{\partial t}\right). \tag{2.38}$$

The unit normal vector field is given by $\nu = (-\nabla u, 1)/\sqrt{1+|\nabla u|^2}$ and the mean curvature by

$$-H = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{2.39}$$

Since M_t is moving by the mean curvature $\partial X/\partial t = -H(X(p,t))\nu$, we calculate

$$\begin{aligned} -H(X(p,t)) = \frac{\partial X}{\partial t}(p,t) \cdot \nu(X(p,t)) &= \left(\frac{\partial \mathbf{x}}{\partial t}, \nabla u \cdot \frac{\partial \mathbf{x}}{\partial t} + \frac{\partial u}{\partial t}\right) \cdot \nu(X(p,t)) \\ &= -\frac{1}{\sqrt{1+|\nabla u|^2}} \frac{\partial u}{\partial t}. \end{aligned}$$

Together with (2.39), we obtain the nonlinear and parabolic partial differential equation

$$\frac{\partial u}{\partial t} = \sqrt{1+|\nabla u|^2}\ \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{2.40}$$

2.3.3.1. *Curves in* $\mathbb{R}^2$. Now we want to study the evolution under mean curvature flow of graphs. If we consider the flow of planar curves in dimension $n = 1$, we can describe the translating solution with the initial curve and speed 1 as $u(\mathbf{x}, t) = u_0(\mathbf{x}) + t$. Consider the equation of mean curvature flow in $\mathbb{R}^2$

$$\alpha = \frac{du}{dt} = \sqrt{1+|u'|^2}\ \left(\frac{u'}{\sqrt{1+|u'|^2}}\right)'. \tag{2.41}$$

where primes above denote derivatives with respect to x. Then we solve the differential equation with the initial profile u_0 when $t = 0$, as follows

$$\alpha = \left(1 - \frac{u_0'^2}{1+u_0'^2}\right) u_0'' = \frac{u_0''}{1+u_0'^2} = (\arctan u_0')'$$

This gives the graph of $u(\mathbf{x}, t) = -\log\cos x + \alpha t$ where $x \in (0, \pi)$.

Chapter 3

Asymptotic analysis

The phase field approach to interface evolution is based on physical models for problems involving phase transitions. If Ω is a bounded domain, and $\Gamma(t)$ is a hypersurface moving through, then one of two phases has the notion of an order parameter $u : \Omega \times (0, T) \to \mathbb{R}$, which means the phase of a material by associating with the phases the minima of a C^2 double well bulk energy function $W(\cdot) : \mathbb{R} \to \mathbb{R}$. For convenience, we suppose that minima of $W(\cdot)$ are at ± 1 and $W(s) = W(-s)$ such that

$$W(s) = \frac{1}{4}(s^2 - 1)^2, \quad s \in \mathbb{R}.$$

Consider the gradient energy functional

$$E(u) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla u|^2 + \frac{W(s)}{\epsilon} \right) dx,$$

where ϵ is a small parameter. Gradient flow for this functional leads to the Allen–Cahn equation for singular mean curvature flow

$$\begin{gathered} u_t^\epsilon - \triangle u^\epsilon + \frac{1}{\epsilon^2} f(u^\epsilon) = 0 \quad \text{in } \mathbb{R}^n \times (0, T) \\ u^\epsilon = u_0^\epsilon, \end{gathered}$$

with Neumann boundary conditions. The equation was originally introduced by Allen and Cahn [1] to describe the motion of anti-phase boundaries. In some contexts in the literature this equation is also referred as the parabolic Ginzburg-Landau equations. Here and from now on, we consider $\mathbb{R}^n$, instead of previous $\mathbb{R}^{n+1}$.

3.1. The distance function

In this part we again assume that Γ is a smooth interface of Ω. Then the distance function of $\Gamma \subset \mathbb{R}^n$ is defined by

$$\operatorname{dist}(x, \Gamma) = \inf_{y \in \Gamma} |x - y|, \quad x \in \mathbb{R}^n,$$

and define the inside and outside of the evolution at time t

$$I_t \equiv \{x \in \mathbb{R}^n | u(x,t) > 0\}, \text{ and } O_t \equiv \{x \in \mathbb{R}^n | u(x,t) < 0\}.$$

From this, we write that

$$(p,t) = \left\{ \begin{array}{ll} \operatorname{dist}(p, \Gamma_t) & \text{if } p \in I_t, \\ -\operatorname{dist}(p, \Gamma_t) & \text{if } p \in O_t, \end{array} \right\},$$

for $p \in \mathbb{R}^n,\ 0 \le t \le t^*$. Moreover, geometric properties of the distance function d imply

$$\triangle d(p,t) = \sum_{i=1}^{n-1} \frac{-\kappa_i(p,t)}{1 - d(p,t)\kappa_i(p,t)} = -(\kappa_1 \ldots \kappa_{n-1}) - d(p,t)h(p,t) + O(d(p,y)^2), \tag{3.1}$$

where $h(p,t) = \sum_{i=1}^{n-1} \kappa_i^2(p,t)$.

THEOREM 3.1.1. *Let d be the signed distance function as above, then*

$$d_t - \triangle d \ge 0 \quad \textit{in } I \cap (R^n \times (0, t^*])$$
$$d_t - \triangle d \le 0 \quad \textit{in } O \cap (R^n \times (0, t^*]).$$

In other words,

$$d_t - \triangle d = 0 \quad \textit{on } \Gamma_t. \tag{3.2}$$

If Γ is a smooth evolution via mean curvature, a direct calculation [6] and limiting behavior of solutions [7] verify $d_t - \triangle d \ge 0$ and $d_t - \triangle d \le 0$ inside and outside of γ.

3.2. The Allen–Cahn equation

Consider the Allen–Cahn equation

$$u_t^\epsilon - \triangle u^\epsilon + \frac{1}{\epsilon^2} w(u^\epsilon) = 0, \tag{3.3}$$

in $\mathbb{R}^n \times (0,T)$ with initial condition $u^\epsilon = u_0^\epsilon$. The potential W is to satisfy

$$w(s) = W'(s),$$

where

$$\begin{aligned} & w(-1) = w(1) = w(1) = 0, \quad w > 0 \text{ on } (-1,0), \quad f < 0 \text{ on } (0,1), \\ & w'(-1) > 0, \; w'(1) > 0 \quad w'(0) < 0, \\ & W(-1) = W(1) = 0 \quad \text{and} \quad W > 0 \text{ on } (-1,1). \end{aligned}$$

Thus, w forms a double well potential

$$W(s) = \frac{1}{2}(s^2-1)^2, \quad s \in \mathbb{R}.$$

The potential W has exactly two minima at ± 1 which are both stable. As the minima of W are stable, the equation forces the solution to get close to ± 1 except on a small intermediate layer of width $O(\epsilon)$. This intermediate layer represents the sharp free boundary we are going to approximate. Let Γ_t be a free boundary evolving and consider its signed distance function $d(x, \Gamma_t)$ to be positive in the interior of Γ_t and negative outside. Let $q(s) = \tanh(s)$, it follows that

$$\begin{aligned} q'(s) &= \operatorname{sech}^2(s) \\ q''(s) &= -2 \operatorname{sech}^2(s) \tanh(s). \end{aligned}$$

Consequently,

$$q''(s) = -2q'(s) \cdot q(s) = w(q(s)). \tag{3.4}$$

We then introduce a new function u^ϵ by

$$u^\epsilon \approx q(\frac{d}{\epsilon}),$$

where $q : \mathbb{R} \to]-1, 1[$ is the profile of the solution around the free boundary which has to be determined. We compute that

$$\begin{aligned}
\partial_t u^\epsilon &= q'\left(\frac{d}{\epsilon}\right) \cdot \frac{\partial_t d}{\epsilon}, \\
\nabla u^\epsilon &= q'\left(\frac{d}{\epsilon}\right) \cdot \frac{\nabla}{\epsilon}, \\
\triangle u^\epsilon &= q''\left(\frac{d}{\epsilon}\right) \cdot \frac{|\nabla d|^2}{\epsilon^2} + q'\left(\frac{d}{\epsilon}\right) \cdot \frac{\triangle d}{\epsilon}.
\end{aligned}$$

Combining with (3.3) and (3.4),

$$\begin{aligned}
0 &= q'\left(\frac{d}{\epsilon}\right) \cdot \frac{\partial_t d}{\epsilon} - \left\{ q''\left(\frac{d}{\epsilon}\right) \cdot \frac{|\nabla d|^2}{\epsilon^2} + q'\left(\frac{d}{\epsilon}\right) \cdot \frac{\triangle d}{\epsilon} \right\} + \frac{1}{\epsilon} w(u^\epsilon) \\
&= \frac{1}{\epsilon}\{\partial_t d - \triangle d + \frac{1}{\epsilon^2} \left\{ -2q'\left(\frac{d}{\epsilon}\right) q\left(\frac{d}{\epsilon}\right)(1 - |\nabla d|^2) \right\} \\
&= d - \triangle d + \frac{2d}{\epsilon}(|\nabla d|^2 - 1|).
\end{aligned}$$

Since the above equation suggests that $|\nabla d| = 1$, we get

$$d_t^\epsilon - \triangle d^\epsilon = 0.$$

For the smooth case, we set

$$u_\pm^\epsilon = q\left(\frac{d}{\epsilon}\right) + \epsilon \nu_\pm^\epsilon$$

and observe that sub and supersolutions

$$\begin{aligned}
\epsilon \partial_t u_\pm^\epsilon - \epsilon \triangle u_\pm^\epsilon + \frac{1}{\epsilon} W'(u_\pm^\epsilon) \le 0 \\
\epsilon \partial_t u_\pm^\epsilon - \epsilon \triangle u_\pm^\epsilon + \frac{1}{\epsilon} W'(u_\pm^\epsilon) \ge 0,
\end{aligned}$$

respectively. Then the comparison principle gives

$$u_-^\epsilon \le u^\epsilon \le u_+^\epsilon.$$

Since

$$u^\epsilon = \begin{cases} +1 & \text{if } d(p,t) > 0, \\ -1 & \text{if } d(p,t) < 0, \end{cases}$$

u^ϵ approximates Γ_t. Finally, (2.12), (3.1) and (3.2) imply that the evolution of Γ_t follows mean curvature flow.

3.3. Discretization of the Allen–Cahn equation

In this section, we describe an unconditionally stable and hybrid numerical method for the Allen–Cahn equation. Let a computational domain be partitioned into a uniform mesh with spacial step h. The center of each cell, Ω_{ij}, is located at $\mathbf{x}_{ij} = (x_i, y_j) = (a + (i - 0.5)h, b + (j - 0.5)h)$ for $i = 1, \cdots, N_x$ and $j = 1, \cdots, N_y$. Here, N_x and N_y are the numbers of cells in x- and y-directions, respectively. Let ϕ_{ij}^n be approximations of $\phi(x_i, y_j, n\Delta t)$, where $\Delta t = T/N_t$ is the time step, T is the final time, and N_t is the total number of time steps. In this paper, we use an operator splitting method, which is to split the Allen–Cahn equation into a sequence of simpler problems for governing equations:

$$\phi_t = \Delta\phi, \tag{3.5}$$

$$\phi_t = \frac{\phi - \phi^3}{\epsilon^2}. \tag{3.6}$$

As first step, we solve equation (3.5) by applying the Crank–Nicolson method, that is,

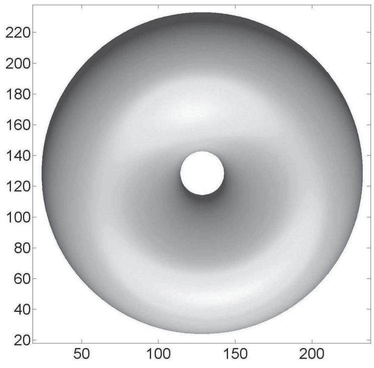
$$\frac{\phi_{ij}^* - \phi_{ij}^n}{\Delta t} = \frac{1}{2}(\Delta_h \phi_{ij}^* + \Delta_h \phi_{ij}^n). \tag{3.7}$$

And we use the multigrid method [3, 20] for the numerical solver. The next step is to solve equation (3.6) analytically. Then, the solution is found by

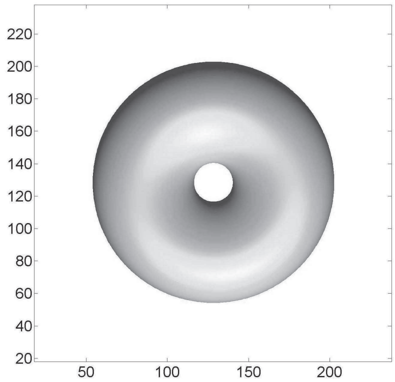
$$\phi_{ij}^{n+1} = \frac{\phi_{ij}^*}{\sqrt{e^{-\frac{2\Delta t}{\epsilon^2}} + (\phi_{ij}^*)^2 \left(1 - e^{-\frac{2\Delta t}{\epsilon^2}}\right)}}. \tag{3.8}$$

The readers can refer to [14] for more details of the unconditionally stable hybrid scheme.

3.3.1. Self-shrinking doughnut. Figure 3.1 shows the evolution of the shrinking doughnut's interface in a curvature-driven flow on the computational domain, $\Omega = (-4, 4) \times (-4, 4) \times (-4, 4)$, with a $128 \times 128 \times 128$ mesh. The computation is run up to $T = 0.6$ with $\Delta t = 0.002$.



(a) $t = 0$



(b) $t = 0.06$

FIGURE 3.1. Shrinking doughnut profile

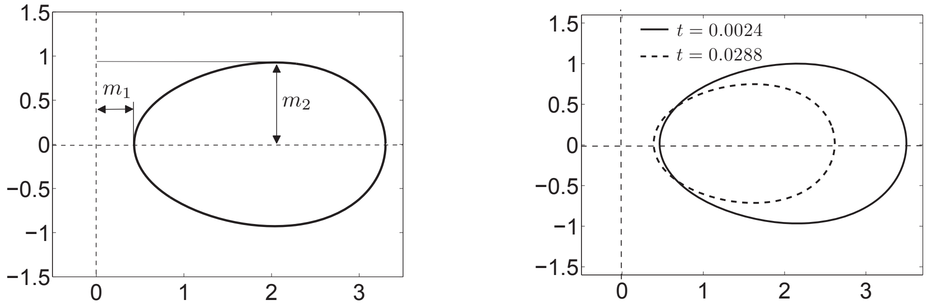


FIGURE 3.2. initial shrinking doughnut and its cross section.

Figures 3.1 and 3.2 show the evolution of mean curvature flow. In figure 3.1, we verify that the torus is shrinking, and figure 3.2 shows the shape of the cross-section of shrinking doughnut, and (a) is a simple curve which starts and ends on the x-axis, and whose tangents on the x-axis are orthogonal to the r-axis. The dot line in (b) is the evolution of the shrinking doughnut. We define the ratio such as

$$\rho(0) = \frac{m_2}{m_1}. \tag{3.9}$$

Then $\rho(t)$ remains constant while the evolution goes on. However, the absence of the rescaling procedure [8] limits the numerical simulations.

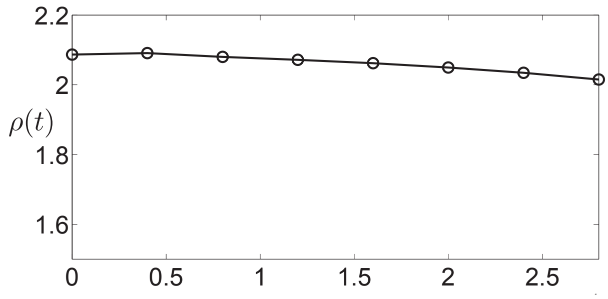


FIGURE 3.3. $\rho(t)$ remains constant with respect to time t

3.3.2. Grim reaper. On the computational domain $[0, 3\pi] \times [0, \pi]$ numerical test is performed with a spatial step size 512×1024 and a temporal step size $\Delta =$ 3.52e$-$4. Figure 3.4 visualizes the graph of analytic and numerical solutions by mean curvature flow. In this graph, we translate by $\pi/2$ to the x-axis for computational convenience.

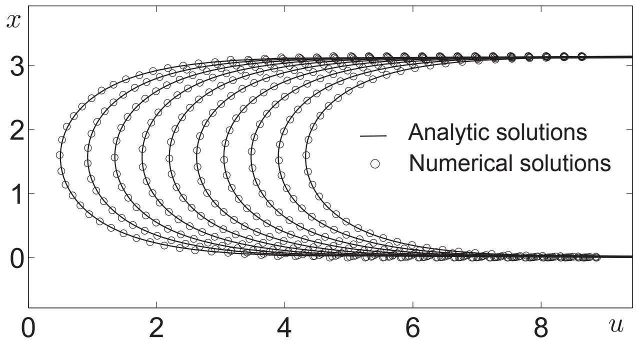


FIGURE 3.4. Graph $u(\mathbf{x}, t) = -\log\cos(x + \pi/2) + t$ by mean curvature flow with speed $\alpha = 1.0$.

Chapter 4

Conclusion

We now finished derivations of self-similar solutions and motion by mean curvature as the singular limit of the Allen-Cahn equation. Furthermore, we applied the phase-field model for simulating the self-similar solutions. In this thesis I theoretically and numerically studied mean curvature flow and self-similar solutions. In theoretical aspect, we showed that the interface motion of the Allen–Cahn equation is driven by mean curvature flow. In numerical aspect, the results are coincided with the theoretical meaning. Actually, many self-similar solutions are not known as well as only indicated by various computations, so that this work benefits us to simulate the motion by mean curvature flow in the phase-field method.

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