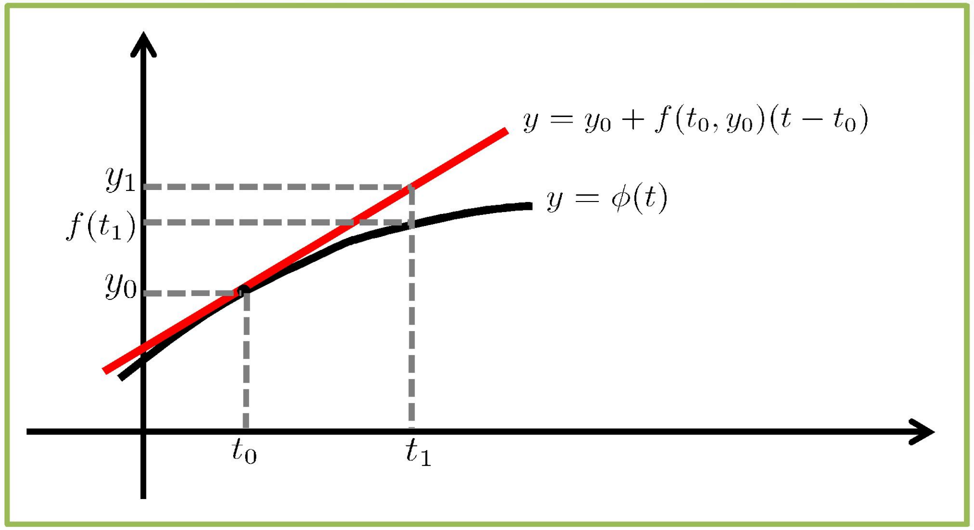
Euler's Method : Tangent Line Approximation

- For the initial value problem

$$y' = f(t, y), \;\; y(t_0) = y_0,$$

we begin by approximating solution $y = \phi(t)$ at initial point t_0.

- The solution passes through initial point (t_0, y_0) with slope $f(t_0, y_0)$. The line tangent to solution at initial point is thus $y = y_0 + f(t_0, y_0)(t - t_0)$



Euler's Formula

- For a point t_2 close to t_1, we approximate $\phi(t_2)$ using the line passing through (t_1, y_1) with slope $f(t_1, y_1)$:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

- Thus we create a sequence y_n of approximations to $\phi(t_n)$:

$$\begin{aligned} y_1 &= y_0 + f_0 \cdot (t_1 - t_0) \\ y_2 &= y_1 + f_1 \cdot (t_2 - t_1) \\ &\vdots \\ y_{n+1} &= y_n + f_n \cdot (t_{n+1} - t_n) \end{aligned}$$

where $f_n = f(t_n, y_n)$.

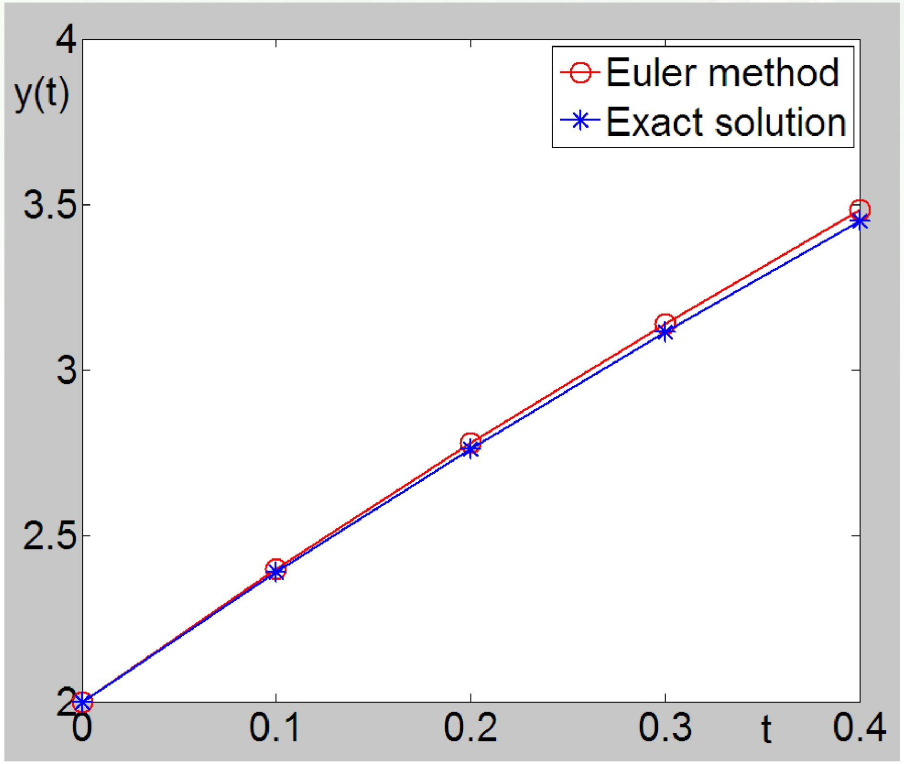
- For a uniform step size $h = t_n - t_{n-1}$, Euler's formula becomes

$$y_{n+1} = y_n + f_n h, \quad n = 0, 1, 2, \ldots$$

Euler Approximation

- To graph an Euler approximation, we plot the points $(t_0, y_0), (t_1, y_1), \cdots, (t_n, y_n)$, and then connect these points with line segments.

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n), \quad \text{where} \quad f_n = f(t_n, y_n)$$



Example

For the initial value problem

$$y' = 5 - 0.5y, \quad y(0) = 2$$

we can use Euler's method with $h = 0.1$ to approximate the solution at $t = 0.1,\ 0.2,\ 0.3,\ 0.4$, as shown below.

(1) By using Newton's method,

$$y_1 = y_0 + f_0 \cdot h = 2 + (5-1)(0.1) = 2.4$$

$$y_2 = y_1 + f_1 \cdot h = 2.4 + (5-4)(0.1) = 2.78$$

$$y_3 = y_2 + f_2 \cdot h = 2.78 + (5-(0.5)(2.78))(0.1) = 3.141$$

$$y_4 = y_3 + f_3 \cdot h = 3.141 + (5-(0.5)(3.141))(0.1) \approx 3.48$$

(2) Exact solution

- We can find the exact solution to our IVP, as in Ch. 1.2 :

$$y' = 5 - 0.5y, \quad y(0) = 2$$

$$y' = -0.5(y - 10)$$

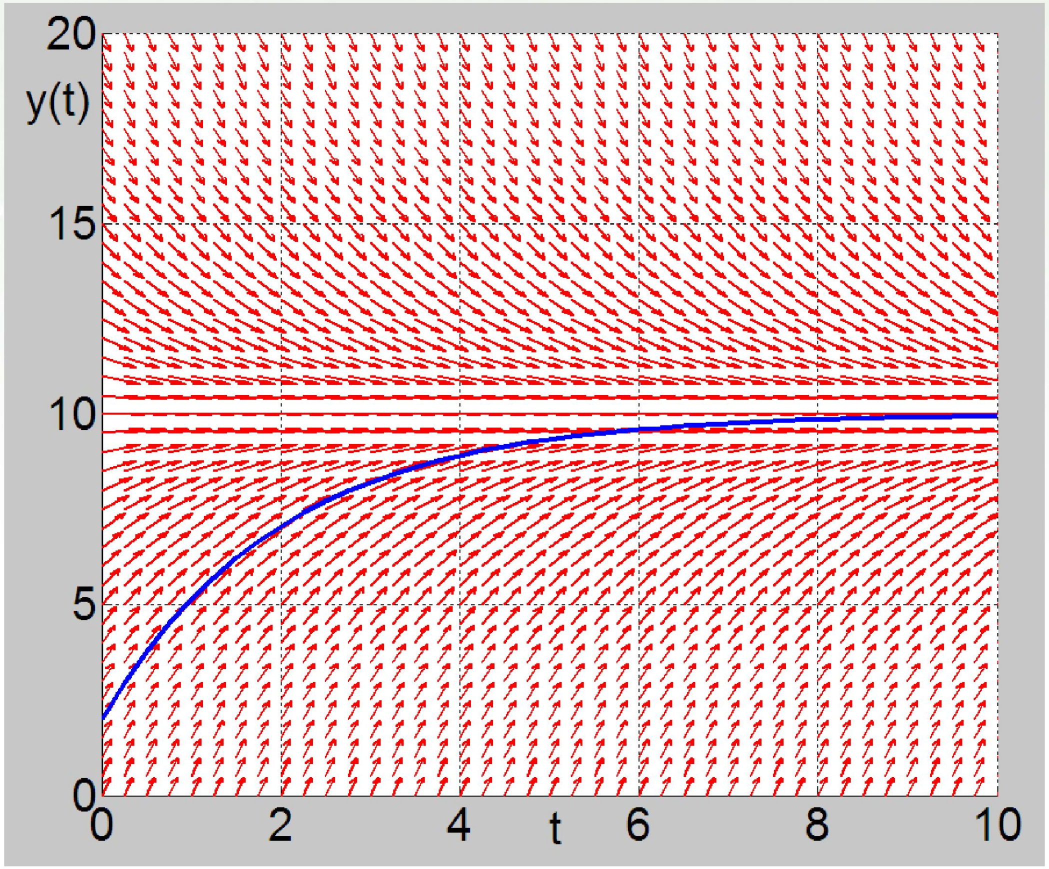
$$\frac{dy}{y-10} = -0.5dt$$

$$\ln|y - 10| = -0.5t + C$$

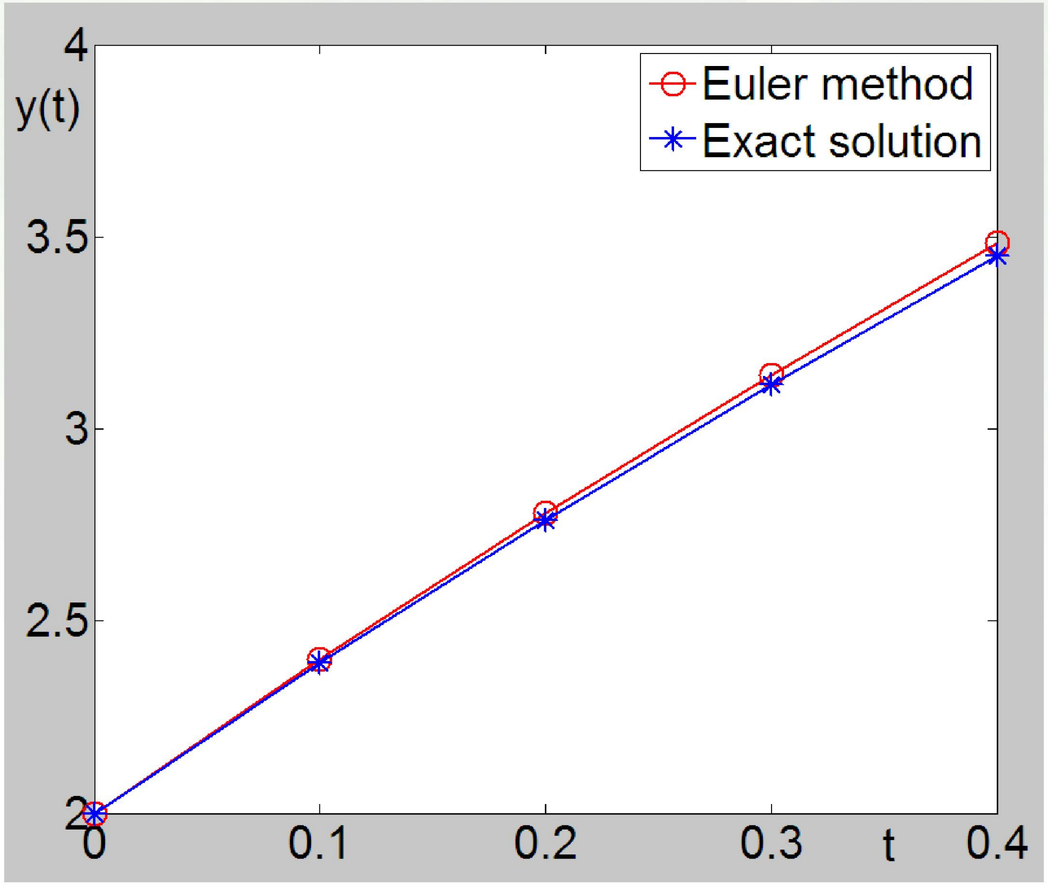
$$y = 10 + ke^{-0.5t}, \quad k = \pm e^{C}$$

$$y(0) = 2 \Rightarrow k = -8$$

$$\Rightarrow y = 10 - 8e^{-0.5t}$$



Euler approximation and Exact solution



(3) Error Analysis

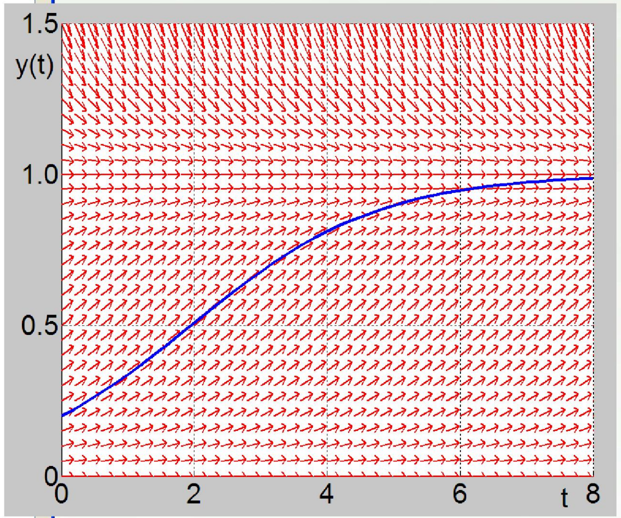
- From table below, we see that the errors are small. This is most likely due to round-off error and the fact that the exact solution is approximately linear on [0, 0.4]. Note:

$$\text{Percent Relative Error} = \frac{y_{exact} - y_{approx}}{y_{exact}} \times 100$$

t	Exact sol.	Approx. sol.	Error	% Rel. Error
0.00	2	2	0	0
0.10	2.3902	2.4	-0.0098	-0.41
0.20	2.7613	2.78	-0.0187	-0.6772
0.30	3.1143	3.141	-0.0267	-0.8573
0.40	3.4502	3.484	-0.0338	-0.9797

MATLAB Code (1)

```
clear all; clc; clf; hold on

t = 0:0.25:10;                 % define grid of values in t-direction
y = 0:0.5:20;                  % define grid of values in y-direction

[T,Y] = meshgrid(t,y);         % creates 2D matrices
dT = ones(size(T));            % dt = 1 for all points
dY = 5-0.5*Y;                  % dy = 5-0.5*y*dt; this is the ODE

N = sqrt(dT.^2 + dY.^2);       % magnitude of arrows
dT = dT./N; dY = dY./N;        % normalize arrows to get all same length
quiver(T,Y,dT,dY,'r');         % draw arrows (t,y) --> (t+dt,y+dy)

yy = 10 - 8*exp(-0.5*t);       % exact solution with y(0) = 2
plot(t,yy,'b-','LineWidth',2);

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
axis([0 10 0 20])
```



MATLAB Code (2)

```
clear all; clc; clf; hold on

t = 0.0:0.1:0.4;                         % interval [0 0.4]
% By using Euler's Method
h = 0.1;                                 % space step
y(1) = 2.0;                              % initial condition y(1) = 2.0
for  i = 1:4
    f(i) = 5-0.5*y(i);
    y(i+1) =  y(i)+f(i)*h;
end
y_exact = 10 - 8*exp(-0.5*t);            % exact solution

plot(t,y,'ro-','LineWidth',2);           % draw approximation solution
plot(t,y_exact,'b*-','LineWidth',2);     % draw exact solution

legend('Euler method','Exact solution','fontsize',30)
xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
axis([0 0.4 2 4])
```