Homogeneous Equations, Initial Values

- Consider homogeneous 2nd order equations :

$$ay'' + by' + cy = 0$$

- Initial conditions are :

$$y(t_0) = y_0, \; y'(t_0) = y_0'$$

- Solution passes through (t_0, y_0), and slope of solution at (t_0, y_0) is y_0'.

Characteristic Equation

- To solve the 2nd order equation with constant coefficients, $ay'' + by' + cy = 0,$

(1) Assume a solution of the form $y = e^{rt}$.

Substituting into the differential equation, we obtain

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

Since $e^{rt} \neq 0$, we have $ar^2 + br + c = 0.$

(2) We have the **characteristic equation** of the differential equation: $\boxed{ar^2 + br + c = 0.}$

(3) Solve r by factoring or using quadratic formula.

General Solution

$$ar^2 + br + c = 0,$$

we obtain two solutions, r_1 and r_2.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- Then:
  - The roots r_1, r_2 are real & $r_1 \neq r_2$.
  - The roots r_1, r_2 are real & $r_1 = r_2$.
  - The roots r_1, r_2 are complex.
- We assume r_1, r_2 are real, and $r_1 \neq r_2$.
- The **general solution** has the form

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Initial Conditions

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

We find c_1 and c_2 using the general form & initial:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},$$

$$\left.\begin{aligned} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} &= y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} &= y'_0 \end{aligned}\right\} \Rightarrow c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}, \; c_2 = \frac{y_0 r_1 - y'_0}{r_1 - r_2} e^{-r_2 t_0}$$

- As $r_1 \neq r_2$, a solution with form $y = e^{rt}$ exists, for any initial conditions.

Example

Consider the initial value problem

$$3y'' - y' - 2y = 0,\ \ y(0) = 1,\ y'(0) = 0$$

(1) Assume exponential: $y(t) = e^{rt}$

(2) We have characteristic equation

$$3r^2 - r - 2 = 0 \iff (3r + 2)(r - 1) = 0$$

(3) two solutions, r_1 = -2/3 and r_2 = 1.

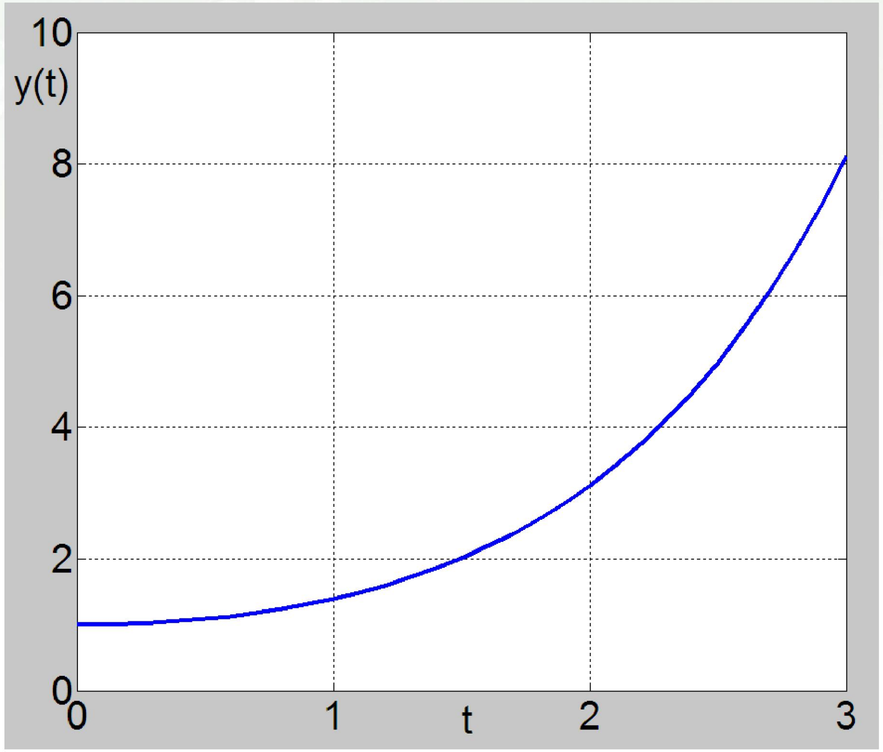
Thus $y(t) = c_1 e^{-\frac{2}{3}t} + c_2 e^{t}$

(4) Use the initial condition:

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -\frac{2}{3}c_1 + c_2 &= 0 \end{aligned}\right\} \Rightarrow c_1 = \frac{3}{5},\ c_2 = \frac{2}{5}$$

- Thus

$$y(t) = \frac{3}{5}e^{-\frac{2}{3}t} + \frac{2}{5}e^{t}.$$



MATLAB Code

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```
clear all; clc; clf; hold on

t = 0:0.1:3;
y = 3/5*exp(-2/3*t) + 2/5*exp(t);

plot(t,y,'b-','LineWidth',2);

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;

set(gca,'fontsize',30)
axis([0 3 0 10])
box on
```