Repeated Roots

- Consider 2nd order linear homogeneous

$$ay'' + by' + cy = 0$$

(1) Assume exponential solution and characteristic equation :

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

(2) Then we have two solutions, r_1 and r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, $r_1 = r_2 = -b/2a$, there exists one solution:

$$y_1(t) = ce^{-bt/2a}$$

Second Solution : Multiplying Factor

- By the linear algebra, when

$$y_1(t) \text{ a solution} \Rightarrow y_2(t) = cy_1(t) \text{ a solution}$$

- Since y_1 and y_2 are linearly dependent, multiply function v. Then try y_2 as a solution:

$$y_1(t) = e^{-bt/2a} \text{ a solution} \Rightarrow \text{ try } y_2(t) = v(t)e^{-bt/2a}$$

- We have

$$y_2(t) = v(t)e^{-bt/2a}$$

$$y_2'(t) = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y_2''(t) = v''(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

Finding Multiplying Factor v(t)

- Substituting derivatives into $ay'' + by' + cy = 0$.

Then,

$$e^{-bt/2a}\left\{a\left[v''(t) - \frac{b}{a}v'(t) + \frac{b^2}{4a^2}v(t)\right] + b\left[v'(t) - \frac{b}{2a}v(t)\right] + cv(t)\right\} = 0$$

$$av''(t) - bv'(t) + \frac{b^2}{4a}v(t) + bv'(t) - \frac{b^2}{2a}v(t) + cv(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v(t) = 0$$

$$av''(t) + \left(\frac{b^2}{4a} - \frac{2b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0 \quad \Leftrightarrow \quad av''(t) + \left(\frac{-b^2}{4a} + \frac{4ac}{4a}\right)v(t) = 0$$

$$av''(t) - \left(\frac{b^2 - 4ac}{4a}\right)v(t) = 0$$

$$v''(t) = 0 \quad \Rightarrow \quad v(t) = k_3 t + k_4$$

General Solution

- We have:

$$y(t) = k_1 e^{-bt/2a} + k_2 v(t) e^{-bt/2a}$$

$$= k_1 e^{-bt/2a} + (k_3 t + k_4) e^{-bt/2a} = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

- The general solution for repeated roots is :

$$y(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

Wronskian

- The general solution is

$$y(t) = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}$$

- Solution is a linear combination of

$$y_1(t) = e^{-bt/2a},\ y_2(t) = te^{-bt/2a}$$

- The Wronskian of the two solutions is

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} e^{-bt/2a} & te^{-bt/2a} \\ -\dfrac{b}{2a} e^{-bt/2a} & \left(1 - \dfrac{bt}{2a}\right) e^{-bt/2a} \end{vmatrix} \\ &= e^{-bt/a}\left(1 - \frac{bt}{2a}\right) + e^{-bt/a}\left(\frac{bt}{2a}\right) \\ &= e^{-bt/a} \neq 0 \quad \text{for all } t \end{aligned}$$

- Thus y_1 and y_2 form a fundamental solution set for equation.

Example

Consider the initial value problem

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Assume exponential solution and characteristic equation :

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 4r + 4 = 0 \Leftrightarrow (r+2)^2 = 0 \Leftrightarrow r = -2$$

- The general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

- Using the initial conditions:

$$\begin{cases} c_1 & & = & 1 \\ -2c_1 & + \quad c_2 & = & -1 \end{cases}$$

$$\Rightarrow c_1 = 1, \; c_2 = 1$$

- Thus $y(t) = e^{-2t} + te^{-2t}$