Nonhomogeneous Equations

- We consider the **nonhomogeneous equation**

$$y'' + p(t)y' + q(t)y = g(t)$$

  where p, q, g are continuous functions on an open interval I.
- The associated **homogeneous equation** is

$$y'' + p(t)y' + q(t)y = 0$$

- In this section, to solve the nonhomogeneous equation, we will learn **the method of undetermined coefficients**, which relies on knowing solutions to homogeneous equation.

Theorem 1

- If Y_1, Y_2 are solutions of nonhomogeneous equation
$$y'' + p(t)y' + q(t)y = g(t)$$
then $Y_1 - Y_2$ is a solution of the homogeneous equation
$$y'' + p(t)y' + q(t)y = 0$$

- If y_1, y_2 form a fundamental solution set of homogeneous equation, then there exists constants c_1, c_2 such that
$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)$$

Theorem 2

- The general solution of nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

can be written in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

where y_1, y_2 form a fundamental solution set of homogeneous equation, c_1, c_2 are arbitrary constants and Y is a particular solution to the nonhomogeneous equation.

$$Y'' + p(t)Y' + q(t)Y = g(t)$$

Method of Undetermined Coefficients

- Recall the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with general solution

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

- In this section we use **the method of undetermined coefficients** to find a particular solution Y to the nonhomogeneous equation, assuming we can find solutions y_1, y_2 for the homogeneous case.

- The method of undetermined coefficients is usually limited to when p and q are constant, and $g(t)$ is a polynomial, exponential, sine or cosine function.

Example 1 : Exponential g(t)

Consider the nonhomogeneous equation

$$y'' - y' + 4y = 2e^{3t}$$

- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:

$$Y(t) = Ae^{3t} \Rightarrow Y'(t) = 3Ae^{3t},\ Y''(t) = 9Ae^{3t}$$

- Substituting these derivatives into differential equation,

$$9Ae^{3t} - 3Ae^{3t} + 4Ae^{3t} = 2e^{3t}$$

$$\Leftrightarrow \quad 10Ae^{3t} = 2e^{3t} \quad \Leftrightarrow \quad A = 1/5$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{1}{5}e^{3t}.$$

Example 2： Sine g(t)

Consider the nonhomogeneous equation

$$y'' - y' + 4y = 3\sin t$$

- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$

- Substituting these derivatives into differential equation,

$$-A\sin t - A\cos t + 4A\sin t = 3\sin t$$
$$\Leftrightarrow\ (3A-3)\sin t - 3A\cos t = 0$$
$$\Leftrightarrow\ c_1\sin t + c_2\cos t = 0$$

- Since $\sin(x)$ and $\cos(x)$ are linearly independent (they are not multiples of each other), we must have $c_1 = c_2 = 0$, and hence $3A - 3 = A = 0$, which is <u>impossible</u>.

Particular solution

- Our next attempt at finding a Y is

$$Y(t) = A\sin t + B\cos t$$

$$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$

- Substituting these derivatives into ODE, we obtain

$$(-A\sin t - B\cos t) - (A\cos t - B\sin t) + 4(A\sin t + B\cos t) = 3\sin t$$

$$\Leftrightarrow\ (3A + B)\sin t + (-A + 3B)\cos t = 3\sin t$$

$$\Leftrightarrow 3A + B = 3,\ -A + 3B = 0$$

$$\Leftrightarrow\ A = 9/10,\ B = 3/10$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{9}{10}\sin t + \frac{3}{10}\cos t$$

Example 3： Polynomial g(t)

Consider the nonhomogeneous equation

$$y'' - y' + 4y = 2t^2 - 1$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = At^2 + Bt + C \Rightarrow Y'(t) = 2At + B,\ Y''(t) = 2A$$

- Substituting these derivatives into differential equation,

$$2A - (2At + B) + 4\left(At^2 + Bt + C\right) = 2t^2 - 1$$
$$\Leftrightarrow\ 4At^2 - (2A + 4B)t + (2A - B + C) = 2t^2 - 1$$
$$\Leftrightarrow\ 4A = 2,\ -2A + 4B = 0,\ 2A - B + 4C = -1$$
$$\Leftrightarrow\ A = 1/2,\ B = 1/4,\ C = -7/16$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{1}{2}t^2 + \frac{1}{4}t - \frac{7}{16}.$$

Example 4： Product g(t)

Consider the nonhomogeneous equation

$$y'' - y' + 4y = -2e^{t}\cos 3t$$

- We seek Y satisfying this equation, as follows:

$$Y(t) = Ae^{t}\cos 3t + Be^{t}\sin 3t$$

$$Y'(t) = Ae^{t}\cos 3t - 3Ae^{t}\sin 3t + Be^{t}\sin 3t + 3Be^{t}\cos 3t$$

$$= (A + 3B)e^{t}\cos 3t + (-3A + B)e^{t}\sin 3t$$

$$Y''(t) = (A + 3B)e^{t}\cos 3t - 3(A + 3B)e^{t}\sin 3t + (-3A + B)e^{t}\sin 3t$$

$$+ 3(-3A + B)e^{t}\cos 3t$$

$$= (-8A + 6B)e^{t}\cos 3t + (-6A - 8B)e^{t}\sin 3t$$

- Substituting derivatives into ODE and solving for A and B :

$$A = \frac{5}{6},\ B = \frac{1}{6} \quad \Rightarrow Y(t) = \frac{5}{6}e^{t}\cos 3t + \frac{1}{6}e^{t}\sin 3t$$

Theorem 3 : Sum g(t)

- Consider again our general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Suppose that $g(t)$ is sum of functions:

$$g(t) = g_1(t) + g_2(t)$$

- If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$

$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

Example 5 : Sum g(t)

Consider the equation

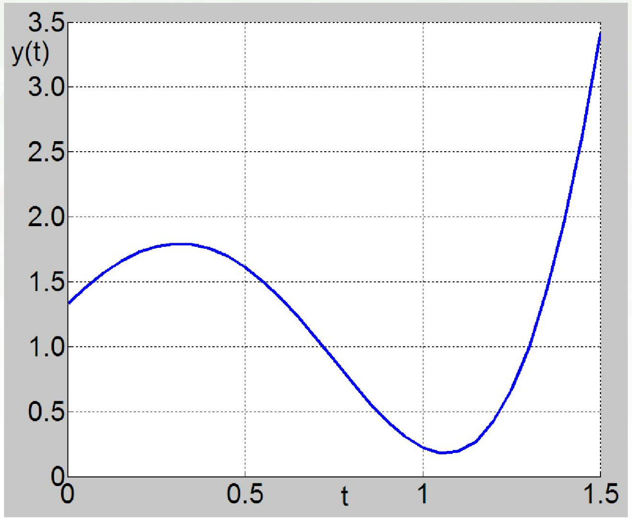
$$y'' - y' + 4y = 2e^{3t} + 3\sin t - 2e^{t}\cos 3t$$

- Our equations to solve individually are

$$y'' - y' + 4y = 2e^{3t}$$

$$y'' - y' + 4y = 3\sin t$$

$$y'' - y' + 4y = -2e^{t}\cos 3t$$



- Our particular solution is then

$$Y(t) = \frac{1}{5}e^{2t} + \frac{9}{10}\sin t + \frac{3}{10}\cos t + \frac{5}{6}e^{t}\cos 3t + \frac{1}{6}e^{t}\sin 3t.$$

MATLAB Code

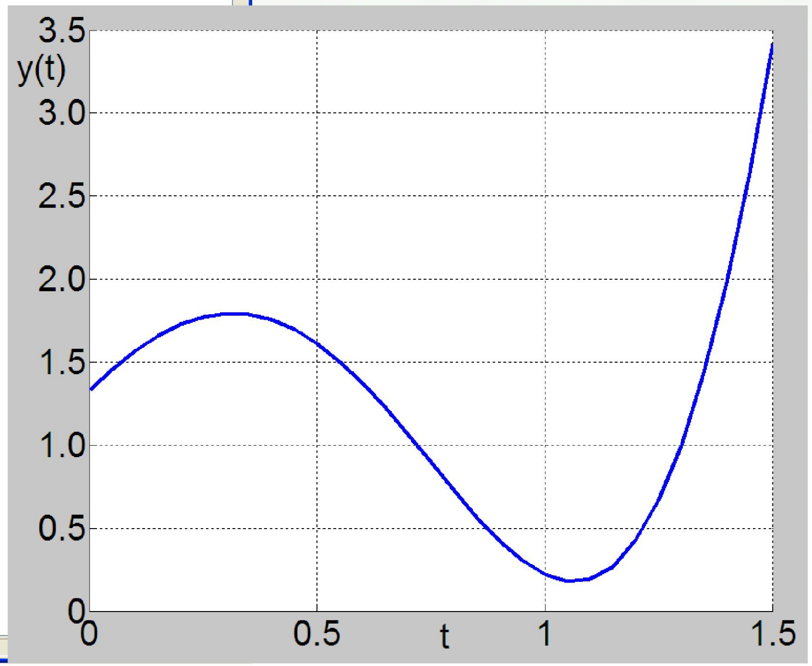
```
clear all; clc; clf; hold on

t = 0:0.05:1.5;                          % define grid of values in t-direction

y = 1/5*exp(2*t) + 9/10*sin(t) + 3/10*cos(t) ...
    + 5/6*exp(t).*cos(3*t) + 1/6*exp(t).*sin(3*t);  % particular solution
plot(t,y,'b-','LineWidth',2);                       % plot solution y(t)

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;

axis([0 1.5 0 3.5])
```



Example 6

Consider the equation

$$y'' + 9y = 4\cos 3t$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = A\sin 3t + B\cos 3t$$

$$\Rightarrow Y'(t) = 3A\cos 3t - 3B\sin 3t,\ Y''(t) = -9A\sin 3t - 9B\cos 3t$$

- Substituting these derivatives into ODE:

$$\left(-9A\sin 3t - 9B\cos 3t\right) + 9\left(A\sin 3t + B\cos 3t\right) = 4\cos 3t$$

$$\left(-9A + 9A\right)\sin 3t + \left(-9B + 9B\right)\cos 3t = 4\cos 3t$$

$$0 = 4\cos 3t$$

- Thus <u>no particular solution</u> exists of the form

$$Y(t) = A\sin 3t + B\cos 3t$$

Particular solution

- Thus no particular solution exists of the form

$$Y(t) = A\sin 3t + B\cos 3t$$

- To help understand why, recall that we found the corresponding homogeneous solution in Section 3.4 notes:

$$y'' + 9y = 0 \quad \Rightarrow \quad y(t) = c_1 \cos 3t + c_2 \sin 3t$$

- Thus our assumed particular solution solves homogeneous equation

$$y'' + 9y = 0$$

instead of the nonhomogeneous equation.

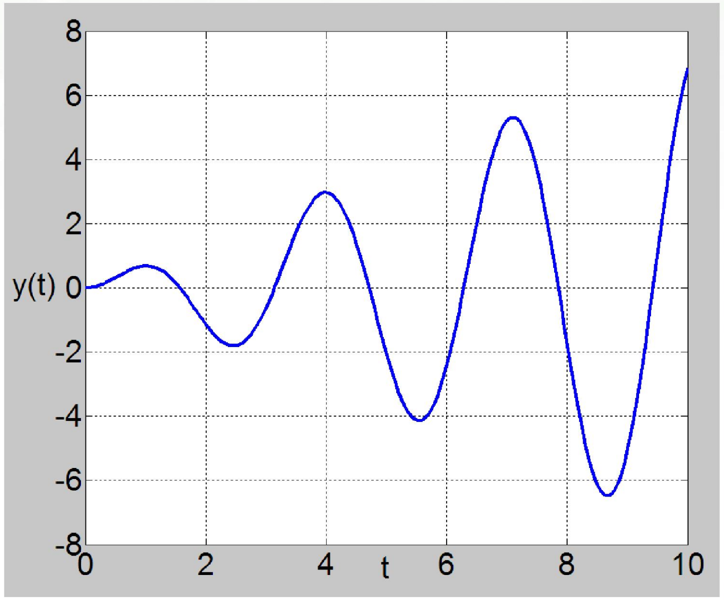
$$y'' + 9y = 4\cos 3t$$

- Our next attempt at finding a Y is:

$$Y(t) = At\sin 3t + Bt\cos 3t$$
$$Y'(t) = A\sin 3t + 3At\cos 3t + B\cos 3t - 3Bt\sin 3t$$
$$Y''(t) = 3A\cos 3t + 3A\cos 3t - 9At\sin 3t - 3B\sin 3t - 3B\sin 3t - 9Bt\cos 3t$$
$$= 6A\cos 3t - 6B\sin 3t - 9At\sin 3t - 9Bt\cos 3t$$

- Substituting derivatives into ODE,

$$6A\cos 3t - 6B\sin 3t = 4\cos 3t$$
$$\Rightarrow A = 2/3,\ B = 0$$
$$\Rightarrow Y(t) = \frac{2}{3}t\sin 3t.$$



MATLAB Code

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```
clear all; clc; clf; hold on

t = 0:0.05:10;                    % define grid of values in t-direction

y = 3/4*t.*sin(2*t);              % particular solution y(t)
plot(t,y,'b-','LineWidth',2);     % plot solution y(t)

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;
box on

axis([0 10 -8 8])
```