Homogeneous Equations
with Constant Coefficients

- Consider the n-th order linear homogeneous differential equation with constant, real coefficients :

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- As with second order linear equations with constant coefficients, $y = e^{rt}$ is a solution for values of r that make characteristic polynomial $Z(r)$ zero:

$$L\left[e^{rt}\right] = e^{rt} \underbrace{\left[a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right]}_{\text{characteristic polynomial } Z(r)} = 0$$

- By the fundamental theorem of algebra, a polynomial of degree n has n roots r_1, r_2, …, r_n, and hence

$$Z(r) = a_0 (r - r_1)(r - r_2) \cdots (r - r_n)$$

Real and Unequal Roots

- If roots of characteristic polynomial $Z(r)$ are real and unequal, then there are n distinct solutions of the differential equation:

$$e^{r_1 t},\ e^{r_2 t},\ \ldots,\ e^{r_n t}$$

- If these functions are linearly independent, then general solution of differential equation is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \ldots + c_n e^{r_n t}$$

- The Wronskian can be used to determine linear independence of solutions.

Example 1

Consider the initial value problem

$$y^{(4)} - 8y''' + 9y'' + 38y' - 40y = 0$$

$$y(0) = 1,\ y'(0) = -1, y''(0) = 2,\ y'''(0) = -3$$

- Assuming exponential solution leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 - 8r^3 + 9r^2 + 38r - 40 = 0$$

$$\Leftrightarrow \quad (r-1)(r+2)(r-4)(r-5) = 0$$

- Thus the general solution is

$$y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{4t} + c_3 e^{5t}$$

- The initial conditions

$$y(0)=1, y'(0)=-1, y''(0)=2, y'''(0)=-3$$

yield

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 1 \\ c_1 - 2c_2 + 4c_3 + 5c_4 &= -1 \\ c_1 + 4c_2 + 16c_3 + 25c_4 &= 2 \\ c_1 - 8c_2 + 64c_3 + 125c_4 &= -3 \end{aligned}$$

- Solving,

$$c_1 = \frac{7}{12}, c_2 = \frac{4}{7}, c_3 = -\frac{1}{3}, c_4 = \frac{5}{28}$$

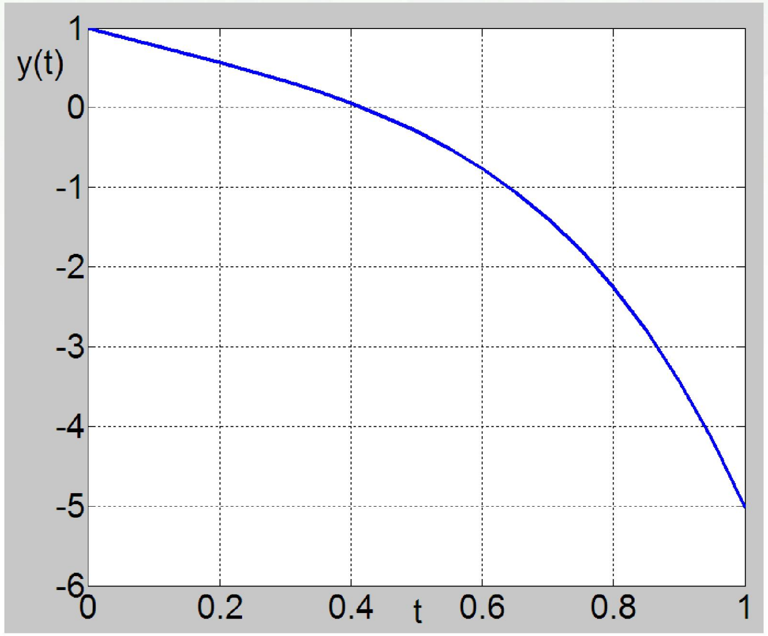
- Hence

$$y(t) = \frac{7}{12}e^{t} + \frac{4}{7}e^{-2t} - \frac{1}{3}e^{3t} + \frac{5}{28}e^{-4t}$$

Graph of Solution

- The graph of the solution is given below. Note the effect of the largest root of characteristic equation.

$$y(t) = \frac{7}{12}e^{t} + \frac{4}{7}e^{-2t} - \frac{1}{3}e^{3t} + \frac{5}{28}e^{-4t}$$



MATLAB Code

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clear all; clc; clf; hold on

t = 0:0.05:1;

y = 7/12*exp(t) + 4/7*exp(-2*t) - 1/3*exp(3*t) + 5/28*exp(-4*t);
plot(t,y,'b-','LineWidth',2);

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;

axis([0 1 -6 1])
box on
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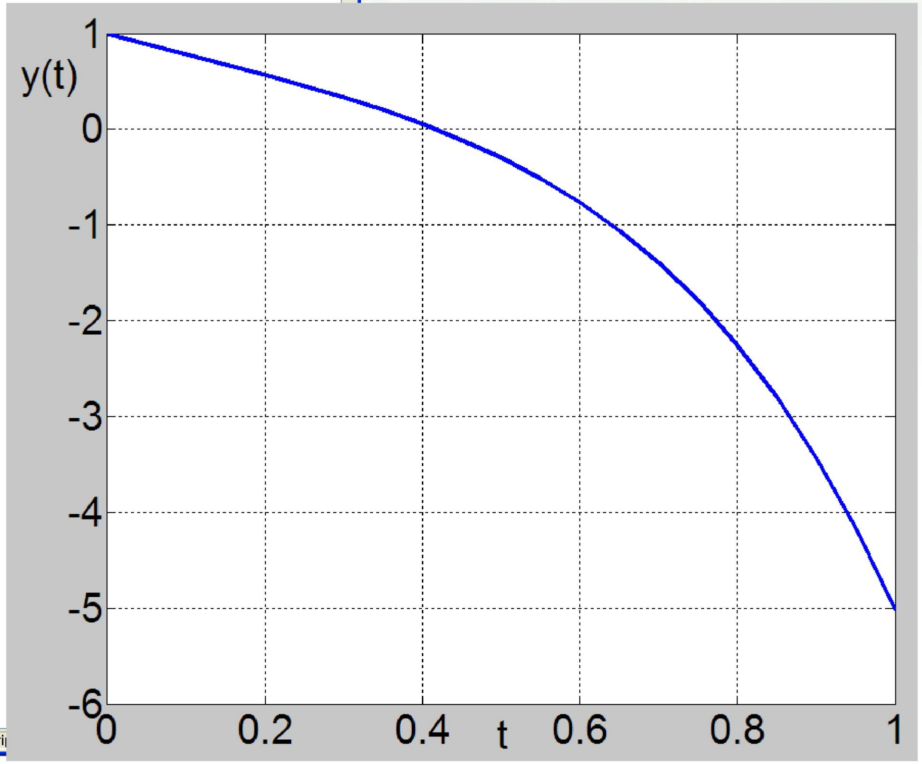


Complex Roots

- If the characteristic polynomial $Z(r)$ has complex roots, then they must occur in conjugate pairs, $\lambda \pm i\mu$.
- Note that not all the roots need be complex.
- Solutions corresponding to complex roots have the form

$$e^{(\lambda+i\mu)t} = e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t$$
$$e^{(\lambda-i\mu)t} = e^{\lambda t}\cos\mu t - ie^{\lambda t}\sin\mu t$$

- As in Chapter 3.4, we use the real-valued solutions

$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t$$

Example 2

Consider the equation

$$y''' - 8y = 0$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad r^3 - 8 = 0 \Leftrightarrow \quad (r-2)(r^2 + 2r + 4) = 0$$

- Now

$$r^2 + 2r + 4 = 0 \quad \Leftrightarrow \quad r = -1 \pm \sqrt{1-4} = -1 \pm \sqrt{3}\, i$$

- Thus the general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-t} \cos\left(\sqrt{3}t\right) + c_3 e^{-t} \sin\left(\sqrt{3}t\right)$$

Example 3

Consider the initial value problem

$$y^{(4)} - 16y = 0, \quad y(0) = 0, y'(0) = 2, y''(0) = 4, y'''(0) = -8$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 - 16 = 0 \Leftrightarrow \left(r^2 - 4\right)\left(r^2 + 4\right) = 0$$

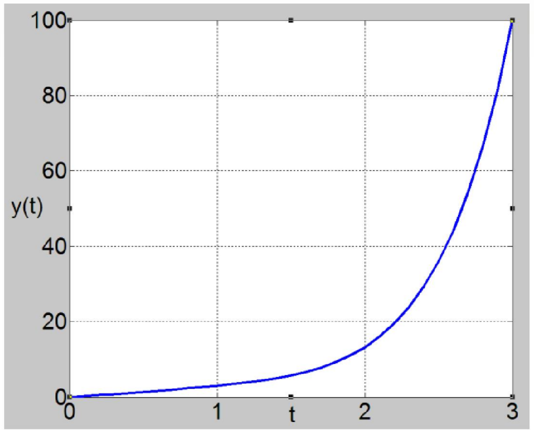
- The roots are 2, -2, *i*, -*i*. Thus the general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos(2t) + c_4 \sin(2t)$$

- Using the initial conditions, we obtain

$$y(t) = \frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} - \frac{1}{2}\cos(2t) + \sin(2t)$$

- The graph of solution is given on right.



Small change in an initial condition

- Note that if one initial condition is slightly modified, then the solution can change significantly. For example, replace

$$y(0) = 0, y'(0) = 2, y''(0) = 4, y'''(0) = -8$$

with

$$y(0) = 0, y'(0) = 2, y''(0) = 4, y'''(0) = 0$$

then

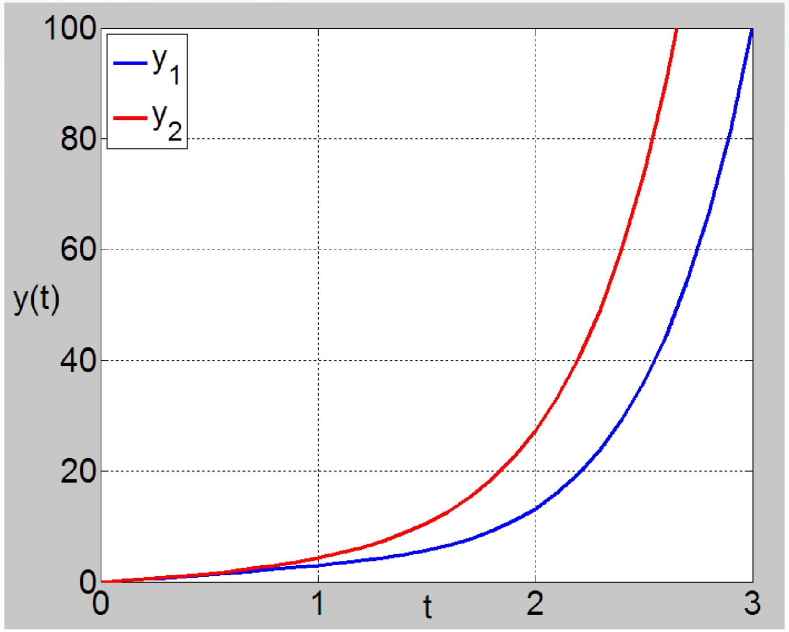
$$y(t) = \frac{1}{2}e^{2t} + 0 \cdot e^{-2t} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$$

Graph of Solutions

- The graph of this solution y_2 and original solution y_1 are given below.

$$y_1(t) = \frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} - \frac{1}{2}\cos(2t) + \sin(2t) \quad \text{(Blue line)}$$

$$y_2(t) = \frac{1}{2}e^{2t} + 0 \cdot e^{-2t} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t) \quad \text{(Red line)}$$



MATLAB Code

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clear all; clc; clf; hold on

t = 0:0.1:3;

y1 = 1/4*exp(2*t) + 1/4*exp(-2*t) - 1/2*cos(2*t) + sin(2*t);
y2 = 1/2*exp(2*t) + 0*exp(-2*t) - 1/2*cos(2*t) + 1/2*sin(2*t);

plot(t,y1,'b-','LineWidth',2);
plot(t,y2,'r-','LineWidth',2);

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
legend('y_1','y_2','fontsize',30,2)
grid on;

set(gca,'fontsize',30)
axis([0 3 0 100])
box on
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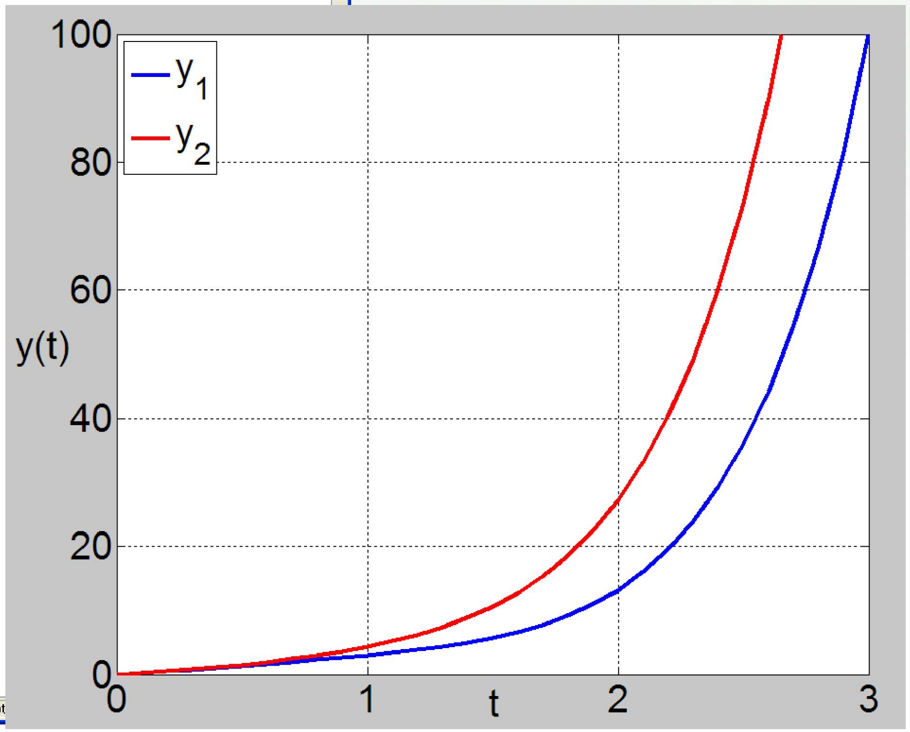


Repeated Roots

- Suppose a root r_k of characteristic polynomial $Z(r)$ is a repeated root with multiplicty s. Then linearly independent solutions corresponding to this repeated root have the form

$$e^{r_k t},\ te^{r_k t},\ t^2 e^{r_k t},\ \ldots,\ t^{s-1} e^{r_k t}$$

- If a complex root $\lambda + i\mu$ is repeated s times, then so is its conjugate $\lambda - i\mu$. There are $2s$ corresponding linearly independent solutions, derived from real and imaginary parts of

$$e^{(\lambda+iu)t},\ te^{(\lambda+iu)t},\ t^2 e^{(\lambda+iu)t},\ \ldots,\ t^{s-1} e^{(\lambda+iu)t}$$

or

$$e^{\lambda t}\cos\mu t,\ e^{\lambda t}\sin\mu t,\ te^{\lambda t}\cos\mu t,\ te^{\lambda t}\sin\mu t, \ldots,$$

$$t^{s-1} e^{r_k t}\cos\mu t,\quad t^{s-1} e^{r_k t} e^{\lambda t}\sin\mu t,$$

Example 4

Consider the equation

$$y^{(4)} + 4y'' + 4y = 0$$

- Then

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 + 4r + 4 = 0 \Leftrightarrow \left(r^2 + 2\right)\left(r^2 + 2\right) = 0$$

- The roots are

$$r = \sqrt{2}i\,,\, \sqrt{2}i\,,\, -\sqrt{2}i\,,\, -\sqrt{2}i$$

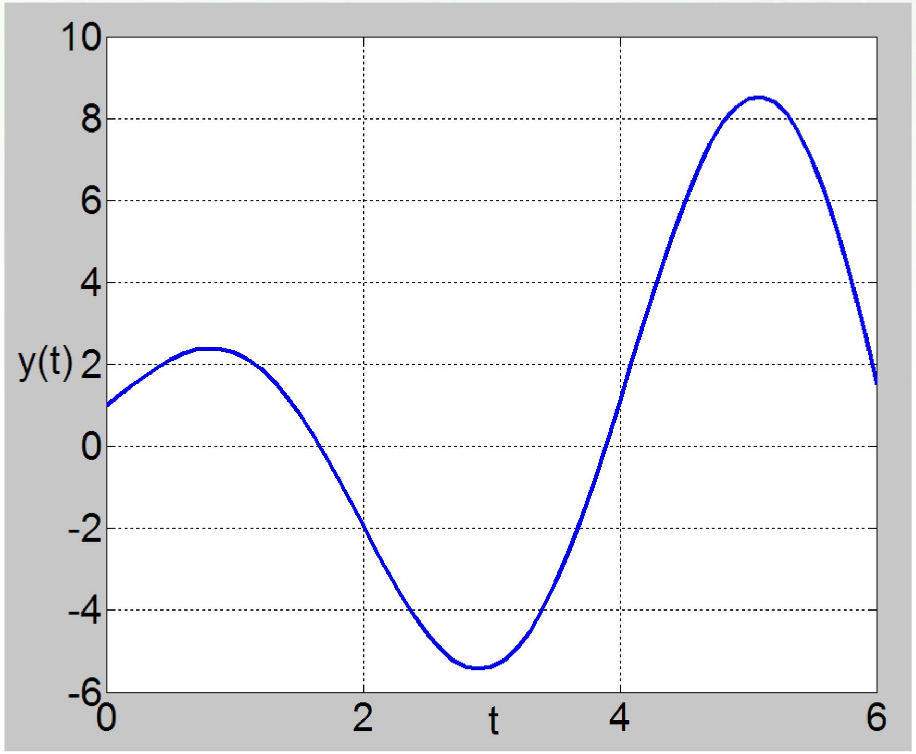
- Thus the general solution is

$$y(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + c_3 t \cos\left(\sqrt{2}t\right) + c_4 t \sin\left(\sqrt{2}t\right)$$

Graph of Solutions

- The graph of the solution with $c_1 = c_2 = c_3 = c_4 = 1$ is given below.

$$y(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + c_3 t \cos\left(\sqrt{2}t\right) + c_4 t \sin\left(\sqrt{2}t\right)$$



MATLAB Code

```
clear all; clc; clf; hold on

t = 0:0.1:6;

c1 = 1; c2 = 1; c3 = 1; c4 = 1;

y = c1*cos(sqrt(2)*t) + c2*sin(sqrt(2)*t) ...
    + c3*t.*cos(sqrt(2)*t) + c4*t.*sin(sqrt(2)*t);

plot(t,y,'b-','LineWidth',2);

xlabel('t','fontsize',30)
ylabel('y(t)','fontsize',30,'rotation',0)
grid on;

set(gca,'fontsize',30)
axis([0 6 -6 10])
box on
```