Method of Undetermined Coefficients

- The method of undetermined coefficients can be used to find a particular solution Y of an n th order linear, constant coefficient, nonhomogeneous ODE

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t),$$

provided g is of an appropriate form.

- As with 2nd order equations, the method of undetermined coefficients is typically used when g is a sum or product of polynomial, exponential, and sine or cosine functions.

Example 1

Consider the differential equation

$$y''' - 6y'' + 12y' - 8y = e^{2t}$$

- For the homogeneous case,

$$y(t) = e^{rt} \quad \Rightarrow \quad r^3 - 6r^2 + 12r - 8 = 0 \Leftrightarrow \quad (r-2)^3 = 0$$

- Thus the general solution of homogeneous equation is

$$y_C(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t}$$

- For nonhomogeneous case, keep in mind the form of homogeneous solution. Thus begin with

$$Y(t) = At^3 e^{2t}$$

- It can be shown that

$$Y(t) = \frac{1}{6} t^3 e^{2t} \quad \Rightarrow \quad y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t} + \frac{1}{6} t^3 e^{2t}$$

Example 2

Consider the equation

$$y^{(4)} + 4y'' + 4y = \sin t + \cos t$$

- For the homogeneous case,

$$y(t) = e^{rt} \quad \Rightarrow \quad r^4 + 4r + 4 = 0 \Leftrightarrow \left(r^2 + 2\right)\left(r^2 + 2\right) = 0$$

- Thus the general solution of homogeneous equation is

$$y(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + c_3 t \cos\left(\sqrt{2}t\right) + c_4 t \sin\left(\sqrt{2}t\right)$$

- For the nonhomogeneous case, begin with

$$Y(t) = A \sin t + B \cos t$$

- It can be shown that

$$Y(t) = \sin t + \cos t$$

Example 3

Consider the equation

$$y^{(4)} + 4y'' + 4y = \sin\left(\sqrt{2}t\right) + \cos\left(\sqrt{2}t\right)$$

- As in Example 2, the general solution of homogeneous equation is

$$y(t) = c_1 \cos\left(\sqrt{2}t\right) + c_2 \sin\left(\sqrt{2}t\right) + c_3 t \cos\left(\sqrt{2}t\right) + c_4 t \sin\left(\sqrt{2}t\right)$$

- For the nonhomogeneous case, begin with

$$Y(t) = At^2 \sin\left(\sqrt{2}t\right) + Bt^2 \cos\left(\sqrt{2}t\right)$$

- It can be shown that

$$Y(t) = -\frac{1}{16} t^2 \sin\left(\sqrt{2}t\right) - \frac{1}{16} t^2 \cos\left(\sqrt{2}t\right)$$

Example 4

Consider the equation

$$y''' - 4y' = 3t + e^{-2t}$$

- For the homogeneous case,

$$y(t) = e^{rt} \Rightarrow r^3 - 4r = 0 \Leftrightarrow r\left(r^2 - 4\right) \Leftrightarrow r(r-2)(r+2) = 0$$

- Thus the general solution of homogeneous equation is

$$y_C(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$$

- For nonhomogeneous case, keep in mind form of homogeneous solution. Thus we have two subcases :

$$Y_1(t) = (A + Bt)t, \; Y_2(t) = Cte^{-2t},$$

- It can be shown that

$$Y_1(t) = -\frac{3}{8}t^2, \; Y_2(t) = \frac{1}{8}t\,e^{-2t}$$