Bessel's Equation of order zero

- The Bessel Equation of order zero is

$$x^2y'' + xy' + x^2y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Taking derivatives,

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n},\quad y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

- From the previous slide,

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} = 0$$

- Rewriting,

$$a_0 [r(r-1)+r] x^r + a_1 [(r+1)r + (r+1)] x^{r+1}$$

$$+ \sum_{n=2}^{\infty} \{a_n [(r+n)(r+n-1) + (r+n)] + a_{n-2}\} x^{r+n} = 0$$

- or

$$a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty} \left\{ a_n (r+n)^2 + a_{n-2} \right\} x^{r+n} = 0$$

- The indicial equation is $r^2 = 0$, and hence $r_1 = r_2 = 0$.

Recurrence Relation

- From the previous slide,

$$a_0 r^2 x^r + a_1 (r+1)^2 x^{r+1} + \sum_{n=2}^{\infty} \left\{ a_n (r+n)^2 + a_{n-2} \right\} x^{r+n} = 0$$

- Note that $a_1 = 0$; the recurrence relation is

$$a_n = -\frac{a_{n-2}}{(r+n)^2}, \ n = 2, 3, \ldots$$

- We conclude $a_1 = a_3 = a_5 = \cdots = 0$, and since $r = 0$,

$$a_{2m} = -\frac{a_{2m-2}}{(2m)^2}, \ m = 1, 2, \ldots$$

- Note : Recall dependence of a_n on r, which is indicated by $a_n(r)$. Thus we may write $a_{2m}(0)$ here instead of a_{2m}.

First Solution

- From the previous slide,

$$a_{2m} = -\frac{a_{2m-2}}{(2m)^2},\ m = 1, 2, \ldots$$

- Thus

$$a_2 = -\frac{a_0}{2^2},\ a_4 = -\frac{a_2}{4^2} = \frac{a_0}{4^2 2^2} = \frac{a_0}{2^4 (2\cdot 1)^2},\ a_6 = -\frac{a_0}{2^6 (3\cdot 2\cdot 1)^2}, \ldots$$

and in general,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} (m!)^2},\ m = 1, 2, \ldots$$

- Thus

$$y_1(x) = a_0 \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} (m!)^2} \right],\quad x > 0$$

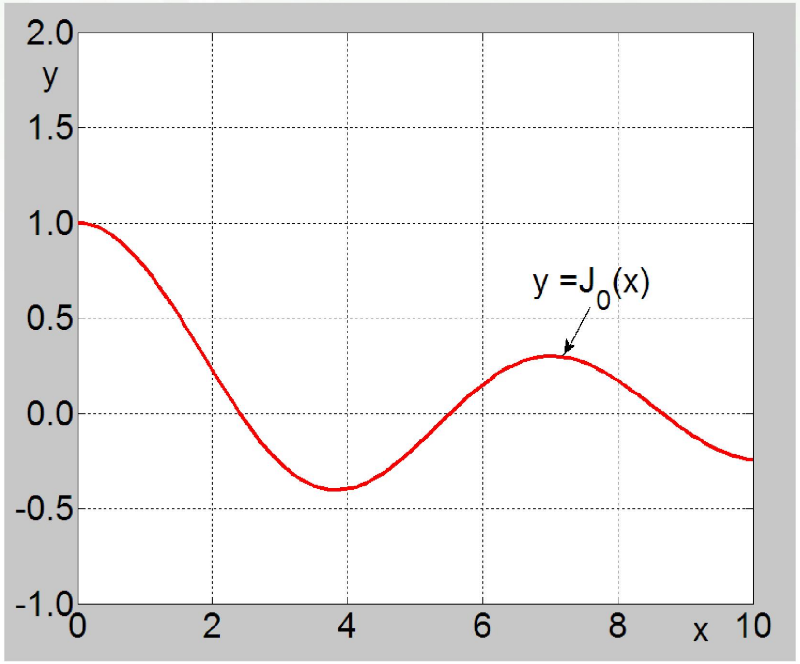
- Our first solution of Bessel's Equation of order zero is

$$y_1(x) = a_0\left[1+\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \quad x>0$$

- The series converges for all x, and is called the **Bessel function of the first kind of order zero**, denoted by

$$J_0(x) = \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}, \quad x>0$$

- The graphs of J_0 and several partial sum approximations are given here.



Second Solution

- Since indicial equation has repeated roots, recall from Section 5.7 that the coefficients in second solution can be found using

$$a_n'(r)\Big|_{r=0}$$

- Now

$$a_0(r)r^2x^r + a_1(r)(r+1)^2x^{r+1} + \sum_{n=2}^{\infty}\left\{a_n(r)(r+n)^2 + a_{n-2}(r)\right\}x^{r+n} = 0$$

- Thus

$$a_1(r) = 0 \quad \Rightarrow a_1'(0) = 0$$

- Also,

$$a_n(r) = -\frac{a_{n-2}(r)}{(r+n)^2}, \quad n = 2, 3, \ldots$$

and hence

$$a_{2m+1}'(0) = 0, \quad m = 1, 2, \ldots$$

- Thus we need only compute derivatives of the even coefficients, given by

$$a_{2m}(r) = -\frac{a_{2m-2}(r)}{(r+2m)^2} \Rightarrow a_{2m}(r) = \frac{(-1)^m a_0}{(r+2)^2 \cdots (r+2m)^2}, \quad m \geq 1$$

- It can be shown that

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = -2\left[\frac{1}{r+2} + \frac{1}{r+4} + \cdots + \frac{1}{r+2m}\right]$$

and hence

$$a'_{2m}(0) = -2\left[\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}\right]a_{2m}(0)$$

- Thus

$$a'_{2m}(0) = -H_m \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \quad m = 1, 2, \ldots$$

where

$$H_m = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2m}$$

- Taking $a_0 = 1$ and using results of Section 5.7,

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}, \quad x > 0$$

- Instead of using y_2, the second solution is often taken to be a linear combination Y_0 of J_0 and y_2, known as the **Bessel function of second kind of order zero**. Here, we take

$$Y_0(x) = \frac{2}{\pi}\left[y_2(x) + (\gamma - \ln 2)J_0(x)\right]$$

- The constant γ is the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty}\left(H_n - \ln n\right) \cong 0.5772$$

- Substituting the expression for y_2 from previous slide into equation for Y_0 above, we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}\right], \quad x > 0$$

General Solution of Bessel's Equation, Order Zero

- The general solution of Bessel's equation of order zero, $x > 0$, is given by

$$y(x) = c_1 J_0(x) + c_2 Y_0(x)$$

where

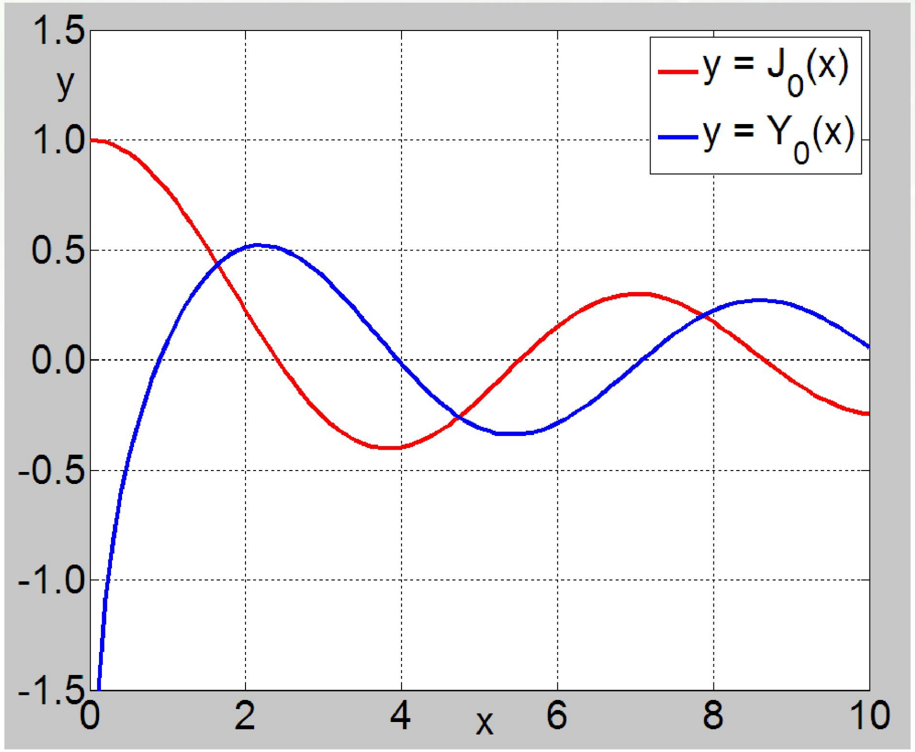
$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2},$$

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m}\right]$$

- Note that $J_0 \to 0$ as $x \to 0$ while Y_0 has a logarithmic singularity at $x = 0$. If a solution which is bounded at the origin is desired, then Y_0 must be discarded.

Graphs of Bessel's Equation, Order Zero

- The graphs of J_0 and Y_0 are given below.
- Note that the behavior of J_0 and Y_0 appear to be similar to $\sin x$ and $\cos x$ for large x, except that oscillations of J_0 and Y_0 decay to zero.



MATLAB Code

```
clear all; clc; clf; hold on

x = 0:0.1:10;

J0 = besselj(0,x);
plot(x,J0,'r-','LineWidth',2);

Y0 = bessely(0,x);
plot(x,Y0,'b-','LineWidth',2);

xlabel('x','fontsize',30)
ylabel('y','fontsize',30,'rotation',0)
grid on;

legend('y = J_0(x)','y = Y_0(x)')
axis([0 10 -1.5 1.5])
box on

set(gca,'fontsize',30)
```