Bessel Equation of Order One-Half

- The Bessel Equation of order one-half is

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1)x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n)x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

Recurrence Relation

- Using results of previous slide, we obtain

$$\sum_{n=0}^{\infty}\left[(r+n)(r+n-1)+(r+n)-\frac{1}{4}\right]a_n x^{r+n}+\sum_{n=0}^{\infty}a_n x^{r+n+2}=0$$

or

$$\left(r^2-\frac{1}{4}\right)a_0 x^r+\left[(r+1)^2-\frac{1}{4}\right]a_1 x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-\frac{1}{4}\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- The roots of indicial equation are r_1 = 1/2, r_2 = -1/2, and note that they differ by a positive integer.
- The recurrence relation is

$$a_n(r)=-\frac{a_{n-2}(r)}{(r+n)^2-1/4},\quad n=2,3,\ldots$$

First Solution : Coefficients

- Consider first the case r_1 = 1/2. From the previous slide,

$$\left(r^2 - 1/4\right)a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[(r+n)^2 - \frac{1}{4}\right]a_n + a_{n-2}\right\}x^{r+n} = 0$$

- Since r_1 = 1/2, a_1 = 0, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$.

  For the even coefficients, we have

$$a_{2m} = -\frac{a_{2m-2}}{\left(1/2+2m\right)^2 - 1/4} = -\frac{a_{2m-2}}{2m\left(2m+1\right)}, \quad m = 1, 2, \ldots$$

- It follows that $a_2 = -\dfrac{a_0}{3!},\ a_4 = -\dfrac{a_2}{5\cdot 4} = \dfrac{a_0}{5!}, \ldots$

  and

$$a_{2m} = \frac{(-1)^m a_0}{(2m+1)!}, \quad m = 1, 2, \ldots$$

Bessel Function of First Kind, Order One-Half

- It follows that the first solution of our equation is,
  for $a_0 = 1$,

$$y_1(x) = x^{1/2}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m}\right], \quad x > 0$$

$$= x^{-1/2}\left[\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right], \quad x > 0$$

$$= x^{-1/2} \sin x, \quad x > 0$$

- The **Bessel function of the first kind of order one-half**, $J_{1/2}$, is defined as

$$J_{1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} y_1(x) \quad = \left(\frac{2}{\pi x}\right)^{1/2} \sin x, \quad x > 0$$

Second Solution : Even Coefficients

- Now consider the case r_2 = -1/2. We know that

$$\left(r^2 - 1/4\right)a_0 x^r + \left[(r+1)^2 - \frac{1}{4}\right]a_1 x^{r+1} + \sum_{n=2}^{\infty}\left\{\left[\left(r+n\right)^2 - \frac{1}{4}\right]a_n + a_{n-2}\right\}x^{r+n} = 0$$

- Since r_2 = -1/2, a_1 = arbitrary. For the even coefficients,

$$a_{2m} = -\frac{a_{2m-2}}{\left(-1/2+2m\right)^2 - 1/4} = -\frac{a_{2m-2}}{2m\left(2m-1\right)}, \quad m = 1, 2, \ldots$$

- It follows that

$$a_2 = -\frac{a_0}{2!}, \; a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!}, \ldots$$

and

$$a_{2m} = \frac{(-1)^m a_0}{(2m)!}, \quad m = 1, 2, \ldots$$

Second Solution: Odd Coefficients

- For the odd coefficients,

$$a_{2m+1} = -\frac{a_{2m-1}}{(-1/2+2m+1)^2 - 1/4} = -\frac{a_{2m-1}}{2m(2m+1)}, \quad m = 1, 2, \ldots$$

- It follows that

and $$a_3 = -\frac{a_1}{3!}, \; a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}, \ldots$$

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!}, \quad m = 1, 2, \ldots$$

Second Solution

- Therefore

$$y_2(x) = x^{-1/2}\left[a_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!}\right], \quad x > 0$$

$$= x^{-1/2}\left[a_0 \cos x + a_1 \sin x\right], \quad x > 0$$

- The second solution is usually taken to be the function

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \quad x > 0$$

where $a_0 = (2/\pi)^{1/2}$ and $a_1 = 0$.

- The general solution of Bessel's equation of order one-half is

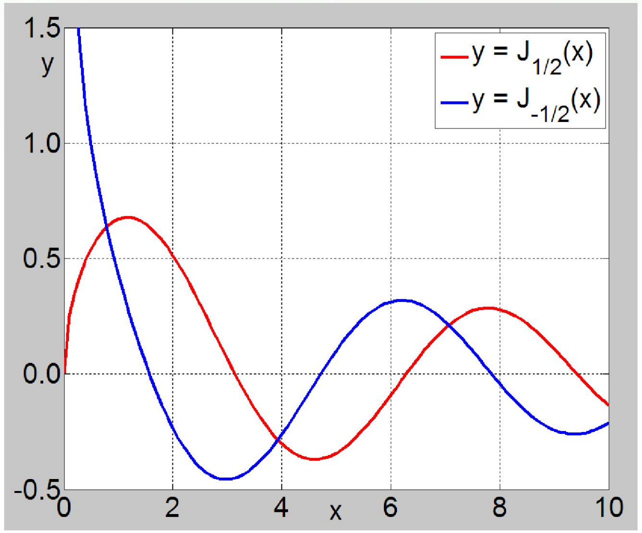
$$y(x) = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$$

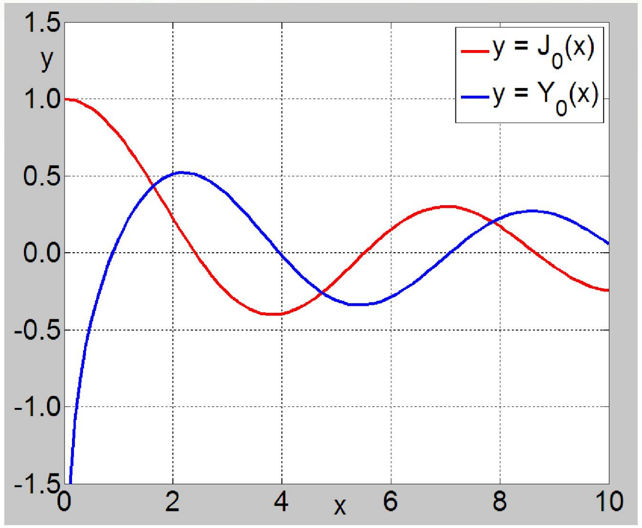
Graphs of Bessel Functions, Order One-Half

- Graphs of $J_{1/2}$, $J_{-1/2}$ are given below.
- Note behavior of $J_{1/2}$, $J_{-1/2}$ similar to J_0, Y_0 for large x, with phase shift of $\pi/4$.

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin x$$

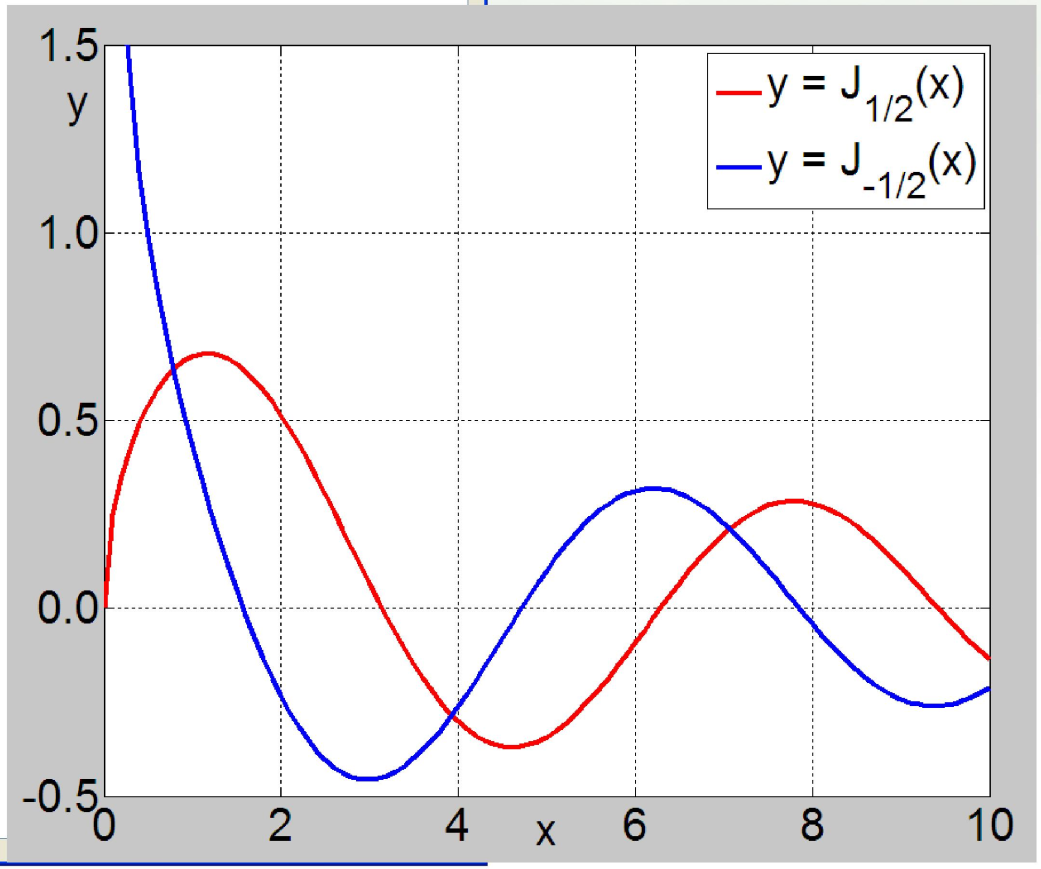
$$J_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\pi}{4}\right), \qquad Y_0(x) \cong \left(\frac{2}{\pi x}\right)^{1/2} \sin\left(x - \frac{\pi}{4}\right), \text{ as } x \to \infty$$





MATLAB Code

```
clear all; clc; clf; hold on

x = 0:0.1:10;

J1 = besselj(1/2,x);
plot(x,J1,'r-','LineWidth',2);

J2 = besselj(-1/2,x);
plot(x,J2,'b-','LineWidth',2);

xlabel('x','fontsize',30)
ylabel('y','fontsize',30,'rotation',0)
grid on;

legend('y = J_{1/2}(x)','y = J_{-1/2}(x)')
axis([0 10 -0.5 1.5])
box on
set(gca,'fontsize',30)
```



Bessel Equation of Order One

- The Bessel Equation of order one is

$$x^2 y'' + xy' + (x^2 - 1)y = 0$$

- We assume solutions have the form

$$y(x) = \phi(r, x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \text{ for } a_0 \neq 0,\ x > 0$$

- Substituting these into the differential equation, we obtain

$$\sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n} + \sum_{n=0}^{\infty} a_n (r+n) x^{r+n}$$

$$+ \sum_{n=0}^{\infty} a_n x^{r+n+2} - \sum_{n=0}^{\infty} a_n x^{r+n} = 0$$

Recurrence Relation

- Using the results of the previous slide, we obtain

$$\sum_{n=0}^{\infty}[(r+n)(r+n-1)+(r+n)-1]a_n x^{r+n}+\sum_{n=0}^{\infty}a_n x^{r+n+2}=0$$

or

$$\left(r^2-1\right)a_0 x^r+\left[(r+1)^2-1\right]a_1 x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-1\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- The roots of indicial equation are $r_1 = 1$, $r_2 = -1$, and note that they differ by a positive integer.
- The recurrence relation is

$$a_n(r)=-\frac{a_{n-2}(r)}{(r+n)^2-1},\quad n=2,3,\ldots$$

First Solution : Coefficients

- Consider first the case $r_1 = 1$. From previous slide,

$$\left(r^2-1\right)a_0x^r+\left[(r+1)^2-1\right]a_1x^{r+1}+\sum_{n=2}^{\infty}\left\{\left[(r+n)^2-1\right]a_n+a_{n-2}\right\}x^{r+n}=0$$

- Since $r_1 = 1$, $a_1 = 0$, and hence from the recurrence relation, $a_1 = a_3 = a_5 = \ldots = 0$. For the even coefficients, we have

$$a_{2m}=-\frac{a_{2m-2}}{(1+2m)^2-1}=-\frac{a_{2m-2}}{2^2(m+1)m},\quad m=1,2,\ldots$$

- It follows that

$$a_2=-\frac{a_0}{2^2\cdot 2\cdot 1},\ a_4=-\frac{a_2}{2^2\cdot 3\cdot 2}=\frac{a_0}{2^4\,3!2!},\ldots$$

and

$$a_{2m}=\frac{(-1)^m a_0}{2^{2m}(m+1)!m!},\quad m=1,2,\ldots$$

Bessel Function of First Kind, Order One

- It follows that the first solution of our differential equation is

$$y_1(x) = a_0 x \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^{2m}(m+1)!\,m!} x^{2m} \right], \quad x > 0$$

- Taking a_0 =1/2, the **Bessel function of the first kind of order one**, J_1, is defined as

$$J_1(x) = \frac{x}{2} \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m+1)!\,m!} x^{2m} \right], \quad x > 0$$

- The series converges for all x and hence J_1 is analytic everywhere.

Second Solution

- For the case r_1 = -1, a solution of the form

$$y_2(x) = a\,J_1(x)\ln x + x^{-1}\left[1+\sum_{n=1}^{\infty} c_n x^{2n}\right], \quad x>0$$

  is guaranteed by Theorem 5.7.1.
- The coefficients c_n are determined by substituting y_2 into the ODE and obtaining a recurrence relation, etc. The result is:

$$y_2(x) = -J_1(x)\ln x + x^{-1}\left[1-\sum_{m=1}^{\infty} \frac{(-1)^m\left(H_m + H_{m-1}\right)}{2^{2m}\, m!(m-1)!} x^{2n}\right], \quad x>0$$

  where H_k is as defined previously.
- Note that $J_1 \to 0$ as $x \to 0$ and is analytic at x = 0, while y_2 is unbounded at x = 0 in the same manner as $1/x$.

Bessel Function of Second Kind, Order One

- The second solution, the **Bessel function of the second kind of order one,** is usually taken to be the function

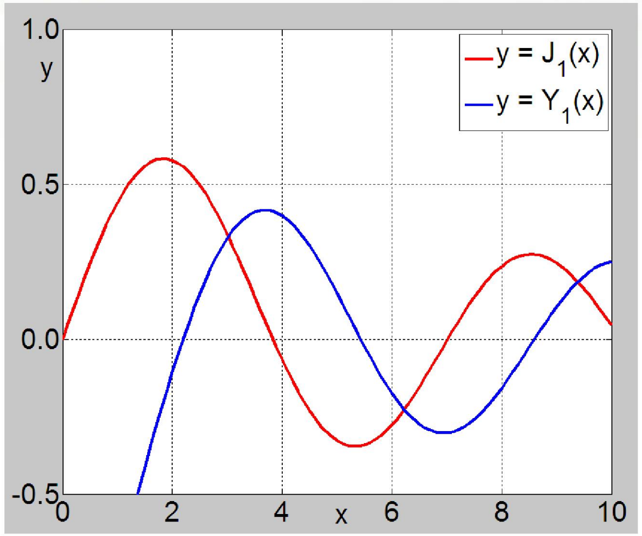
$$Y_1(x) = \frac{2}{\pi}\left[-y_2(x) + (\gamma - \ln 2)J_1(x)\right], \quad x > 0$$

where γ is the Euler-Mascheroni constant.

- The general solution of Bessel's equation of order one is

$$y(x) = c_1 J_1(x) + c_2 Y_1(x), \quad x > 0$$

- Note that J_1 , Y_1 have same behavior at $x = 0$ as observed on previous slide for J_1 and y_2.



MATLAB Code

```
clear all; clc; clf; hold on

x = 0:0.1:10;

J1 = besselj(1,x);
plot(x,J1,'r-','LineWidth',2);

Y1 = bessely(1,x);
plot(x,Y1,'b-','LineWidth',2);

xlabel('x','fontsize',30)
ylabel('y','fontsize',30,'rotation',0)
grid on;

legEnd('y = J_{1}(x)','y = Y_{1}(x)')
set(gca,'fontsize',30)
axis([0 10 -0.5 1])
box on
```