Bessel's Equation

- Bessel Equation of order n :

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad (1)$$

The general solution of this equation (1) is

$$y = AJ_n(x) + BY_n(x). \qquad (2)$$

Modified Bessel Equation 1

- Modified Bessel Equation 1

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(\lambda^2x^2-n^2)y=0 \qquad (3)$$

The general solution of this equation (3) is

$$y=AJ_n(\lambda x)+BY_n(\lambda x). \qquad (4)$$

Modified Bessel Equation 2

- Modified Bessel Equation 2

$$x^2\frac{d^2y}{dx^2} + (1-2\alpha)x\frac{dy}{dx} + \left\{\beta^2\gamma^2x^{2\gamma} + (\alpha^2 - n^2\gamma^2)\right\}y = 0 \quad (5)$$

The general solution of this equation (5) is

$$y = Ax^\alpha J_n(\beta t) + Bx^\alpha Y_n(\beta t). \quad (6)$$

Modified Bessel Equation 3

- Modified Bessel Equation 3

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + n^2)y = 0 \qquad (7)$$

Because of the same form of **[Modified Bessel Equation 1]**, the solution of Eq. (7) is

$$y(x) = AJ_n(ix) + BY_n(ix).$$

Modified Bessel Equation 4

- Modified Bessel Equation 4

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - k^2 y = 0 \qquad (8)$$

First, we rewritten Eq. (8) as

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - k^2x^2y = 0. \qquad (9)$$

Let $y = xz$. Then

$$\frac{dy}{dx} = z + x\frac{dz}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = 2\frac{dz}{dx} + x\frac{d^2z}{dx^2}. \qquad (10)$$

Modified Bessel Equation 4

Substituting the above Eq. (10) into Eq. (9), we obtain

$$x^2\left(x\frac{d^2z}{dx^2}+2\frac{dz}{dx}\right)-x\left(z+x\frac{dz}{dx}\right)-k^2x^2\cdot xz=0$$

$$x^2\frac{d^2z}{dx^2}+2x\frac{dz}{dx}-z-x\frac{dz}{dx}-k^2x^2z=0$$

$$x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}-(k^2x^2+1)z=0.$$

By substituting $t=kx$, we get

$$x^2k^2\frac{d^2z}{dt^2}+xk\frac{dz}{dt}-(t^2+1)z=0.$$

Modified Bessel Equation 4

And simplifying

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} - (t^2+1)z = 0. \qquad (11)$$

Because of the same form of **[Modified Bessel Equation 3]**, Eq. (11) has the following solution

$$\begin{aligned} z &= AI_1(t) + BK_1(t) \\ &= AI_1(kx) + BK_1(kx). \end{aligned}$$

Therefore, the solution of Eq. (8)

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - k^2y = 0$$

is

$$y = AxI_1(kx) + BxK_1(kx).$$