Definition of the Laplace Transform

The **Laplace Transform of *f*** is defined as

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

- Let $f(t)=1$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt = -\lim_{b\to\infty}\left.\frac{e^{-st}}{s}\right|_0^b = \frac{1}{s}, \quad s > 0$$

- Let $f(t)=e^{at}$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,dt = \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\,dt$$

$$= -\lim_{b\to\infty}\left.\frac{e^{-(s-a)t}}{s-a}\right|_0^b = \frac{1}{s-a}, \quad s > a$$

- Let $f(t) = \sin(at)$ for $t \geq 0$. Using integration by parts twice, the Laplace transform $F(s)$ of f is found as follows:

$$F(s) = L\{\sin(at)\} = \int_0^\infty e^{-st} \sin atdt = \lim_{b\to\infty} \int_0^b e^{-st} \sin atdt$$

$$= \lim_{b\to\infty}\left[-(e^{-st} \cos at)/a\Big|_0^b - \frac{s}{a}\int_0^b e^{-st} \cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a}\lim_{b\to\infty}\left[\int_0^b e^{-st} \cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a}\lim_{b\to\infty}\left[(e^{-st} \sin at)/a\Big|_0^b + \frac{s}{a}\int_0^b e^{-st} \sin at \right]$$

$$= \frac{1}{a} - \frac{s^2}{a^2}F(s) \Rightarrow F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

- Let $f(t) = \cos(t)$ for $t \geq 0$. Using integration by parts twice, the Laplace transform $F(s)$ of f is found as follows:

$$F(s) = L\{\cos(t)\} = \int_0^\infty e^{-st}\cos(t)dt = \lim_{b\to\infty}\int_0^b e^{-st}\cos(t)dt$$

Now, we write

$$\int_0^\infty e^{-st}\cos(t)dt = \lim_{b\to\infty}\left[-\frac{1}{s}\cos(t)e^{-st}\Big|_0^b - \frac{1}{s}\int_0^b e^{-st}\sin(t)dt\right]$$

$$= \frac{1}{s} - \frac{1}{s^2}\int_0^\infty e^{-st}\cos(t)dt$$

Therefore, $F(s) = \dfrac{s}{s^2+1}$

Linearity of the Laplace Transform

Linearity of the Laplace transform

If c_1 and c_2 are the constants, then

$$L\{c_1 f(t) + c_2 g(t)\} = c_1 \int_0^\infty e^{-st} f(t)dt + c_2 \int_0^\infty e^{-st} g(t)dt$$

$$= c_1 L\{f(t)\} + c_2 L\{g(t)\}$$

for all s such tat the Laplace transforms of the function f and g both exist.

- Let $f(t) = 4e^{-3t} - 2\sin(5t)$ for $t \geq 0$.

  Then by linearity of the Laplace transform, and using results of previous examples, the Laplace transform $F(s)$ of f is:

$$
\begin{aligned}
F(s) &= L\{f(t)\} \\
&= L\left\{4e^{-3t} - 2\sin(5t)\right\} \\
&= 4L\left\{e^{-3t}\right\} - 2L\left\{\sin(5t)\right\} \\
&= \frac{4}{s+3} - \frac{10}{s^2+25}, \quad s > 0.
\end{aligned}
$$

The Laplace Transform of *f ′* and *f ″*

- We have $L\{f'(t)\} = sL\{f(t)\} - f(0)$

  Proof : $\lim_{b\to\infty} \int_0^b e^{-st} f'(t)dt = \lim_{b\to\infty} \left[e^{-st} f(t)\Big|_0^b - \int_0^b (-s)e^{-st} f(t)dt \right]$

$$= \lim_{b\to\infty} \left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)dt \right]$$

$$= \lim_{b\to\infty} \left[s\int_0^b e^{-st} f(t)dt \right] - f(0) = sL\{f(t)\} - f(0).$$

- We have $L\{f''(t)\} = sL\{f'(t)\} - f'(0)$

  Proof :

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s[sL\{f(t)\} - f(0)] - f'(0) \\ &= s^2 L\{f(t)\} - sf(0) - f'(0). \end{aligned}$$

Example 1 : Using Chapter 3 Method

Consider the initial value problem

$$y'' + 5y' + 4y = 0, \quad y(0) = 1, \ y'(0) = 2$$

- Recall from Section 3.1:

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 5r + 4 = 0 \ \Leftrightarrow \ (r+1)(r+4) = 0$$

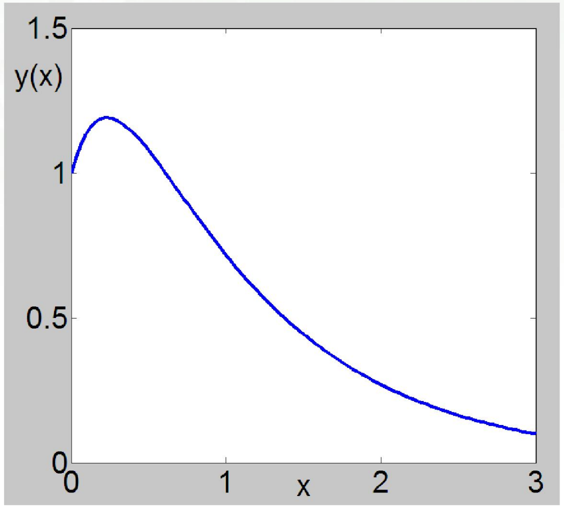
- Thus r_1 = -1 and r_2 = -4, and general solution has the form $y(t) = c_1 e^{-t} + c_2 e^{-4t}$

- Using initial conditions:

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 - 4c_2 &= 2 \end{aligned}\right\} \Rightarrow c_1 = 2, \ c_2 = -1$$

- Thus

$$y(t) = 2e^{-t} - e^{-4t}$$



- We now solve this problem using Laplace Transforms.

Example 1 : Laplace Transform Method

Consider the initial value problem

$$y'' + 5y' + 4y = 0, \quad y(0) = 1, \; y'(0) = 2$$

- By Laplace transform method,

$$L\{y'' + 5y' + 4y\} = L\{y''\} + 5L\{y'\} + 4L\{y\} = L\{0\} = 0$$

and hence

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 5\left[sL\{y\} - y(0)\right] + 4L\{y\} = 0$$

- Letting $Y(s) = L\{y\}$, we have

$$\left(s^2 + 5s + 4\right)Y(s) - \left(s + 5\right)y(0) - y'(0) = 0$$

- Substituting in the initial conditions, we obtain

$$\left(s^2 + 5s + 6\right)Y(s) - \left(s + 5\right) - 2 = 0$$

- Thus

$$L\{y\} = Y(s) = \frac{s+7}{(s+1)(s+4)}.$$

- Using partial fraction decomposition, $Y(s)$ can be rewritten:

$$\frac{s+7}{(s+1)(s+4)} = \frac{A}{(s+1)} + \frac{B}{(s+4)}$$

$$s+7 = A(s+4) + B(s+1)$$

$$s+7 = (A+B)s + (4A+B)$$

$$A+B=1,\ \ 4A+B=7$$

$$A=2,\ B=-1$$

- Therefore,

$$L\{y\} = Y(s) = \frac{2}{(s+1)} - \frac{1}{(s+4)}.$$

- Recall from Section 6.1:

$$L\{e^{at}\} = F(s) = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a$$

- Thus

$$Y(s) = \frac{2}{(s+1)} - \frac{1}{(s+4)} = 2L\{e^{-t}\} - L\{e^{-4t}\}, \ s > -1,$$

- Recalling $Y(s) = L\{y\}$, we have

$$L\{y\} = L\{2e^{-t} - e^{-4t}\}$$

and hence

$$y(t) = 2e^{-t} - e^{-4t}.$$

Example 2 : Non-homogeneous Problem

Consider the initial value problem

$$y'' + 2y = -\sin 3t, \quad y(0) = 1, \ y'(0) = 2$$

- Taking the Laplace transform of the differential equation, we have

$$s^2 L\{y\} - sy(0) - y'(0) + 2L\{y\} = -3/(s^2+9)$$

- Letting $Y(s) = L\{y\}$, we have

$$(s^2+2)Y(s) - sy(0) - y'(0) = -3/(s^2+9)$$

- Substituting in the initial conditions, we obtain

$$(s^2+2)Y(s) - s - 2 = -3/(s^2+9)$$

- Thus

$$Y(s) = \frac{s^3 + 2s^2 + 9s + 15}{(s^2+2)(s^2+9)}$$

- Using partial fractions,

$$Y(s) = \frac{s^3 + 2s^2 + 9s + 15}{(s^2 + 2)(s^2 + 9)} = \frac{As + B}{s^2 + 2} + \frac{Cs + D}{s^2 + 9}$$

Then

$$s^3 + 2s^2 + 9s + 15 = (As + B)(s^2 + 9) + (Cs + D)(s^2 + 2)$$

$$= (A + C)s^3 + (B + D)s^2 + (9A + 2C)s + (9B + 2D)$$

- Solving, we obtain *A* = 1, *B* = 0, *C* = 11/7, and *D* = 3/7. Thus

$$Y(s) = \frac{s}{s^2 + 2} + \frac{11/7}{s^2 + 2} + \frac{3/7}{s^2 + 9}$$

Hence

$$y(t) = 2\cos t + \frac{11}{7}\sin 2t + \frac{3}{7}\sin 3t$$