Bull. Korean Math. Soc. **45** (2008), No. 1, pp. 169–190

# $L_q(L_p)$-THEORY OF PARABOLIC PDEs WITH VARIABLE COEFFICIENTS

KYEONG-HUN KIM

ABSTRACT. Second-order parabolic equations with variable coefficients are considered on $\mathbb{R}^d$ and $C^1$ domains. Existence and uniqueness results are given in $L_q(L_p)$-spaces, where it is allowed for the powers of summability with respect to space and time variables to be different.

## 1. Introduction

Let $\Omega$ be an open set in $\mathbb{R}^d$. In this article we are dealing with Sobolev space theory of the equations

$$u_t = a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f, \tag{1.1}$$

$$u_t = D_i(a^{ij}u_{x^j} + \bar{b}^iu + f^i) + b^iu_{x^i} + cu + f, \tag{1.2}$$

given for $x \in \Omega$ and $t \geq 0$. Throughout the article Einstein's summation convention is used. Needless to say, such equations have been extensively studied by many authors in Hölder and $L_p([0,T], L_p(\Omega))$-spaces. However, surprisingly enough, $L_q([0,T], L_p(\Omega))$-theory ($p \neq q$) of the equations had never been addressed before except in [1], [10] and [4]. In [1] and [10] the Cauchy problem with $f = 0$ was studied, and recently in [4] Krylov developed the $L_q(L_p)$-theory of the heat equation $u_t = \Delta u + f$ on $\mathbb{R}^d$ and $\mathbb{R}^d_+$.

In this article we extend the results in [4]. We give the uniqueness and existence results of equations (1.1) and (1.2) with variable coefficients on $\mathbb{R}^d$ and $C^1$ domains in $L_q([0,T], L_p)$-spaces. Since the boundary is not supposed to be regular enough we look for solutions in function spaces with weights allowing derivatives of solutions to blow up near the boundary.

Usually once one knows how to solve the heat equation in $\mathbb{R}^d$ and $\mathbb{R}^d_+$, then constructing a solvability theory of equation (1.1) with variable coefficients becomes a standard work. If $p \neq q$ then $L_q(L_p)$-theory of PDEs turns out to be one more exception to the usual situation. For instance, let $\{\zeta_n : n = 1, 2, \ldots\}$ be a standard partition of unity and each $\zeta_n$ have small support. Then based on

Received October 15, 2007.
2000 *Mathematics Subject Classification.* 35K10.
*Key words and phrases.* parabolic equations, $L_q(L_p)$-theory, Sobolev spaces with weights.
This article was supported by a Korea University grant.

some perturbation arguments, it may be possible to estimate the $L_q(L_p)$-norm of $\zeta_n u$. However relations like

$$\int_0^T \|u(t,\cdot)\|^q_{L_p(\mathbb{R}^d)}\,dt \sim \sum_{n=1}^{\infty}\int_0^T \|\zeta_n(\cdot)u(t,\cdot)\|^q_{L_p(\mathbb{R}^d)}\,dt$$

hold only if $p = q$. Thus local estimations of $u$ don't easily yield a priori estimate of $u$. Furthermore, since we are also dealing with the equations in Sobolev spaces with weights, usual perturbation arguments don't work well and require some nontrivial modifications.

In Section 2 we prove the unique solvability of equation (1.1) and equation (1.2) defined on $\mathbb{R}^d$ in $L_q([0,T],H^\gamma_p)$ and $L_q([0,T],H^1_p)$, respectively. Here $1 < p \leq q < \infty$, $\gamma \in \mathbb{R}$ and $H^\gamma_p$ is the space of Bessel potentials. In Section 3 we present the unique solvability results (Theorem 3.8 and Theorem 3.6) of the equations defined on $C^1$ domains in $L_q([0,T],H^\gamma_{p,\theta}(\Omega))$, where the weighted Sobolev space $H^\gamma_{p,\theta}(\Omega)$ is introduced in Section 3. Many advantages of $L_q(L_p)$-theory over $L_p(L_p)$-theory were investigated in [3] and we introduce some of them in Remark 3.9. In Section 4 we develop some auxiliary results, and in Section 5 and Section 6 we prove Theorem 3.8 and Theorem 3.6, respectively.

In this paper, as usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1,\dots,x^d)$, $B_r(x) = \{y \in \mathbb{R}^d : |x-y| < r\}$, $B_r = B_r(0)$ and $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$. For $i = 1,\dots,d$, multi-indices $\alpha = (\alpha_1,\dots,\alpha_d)$, $\alpha_i \in \{0,1,2,\dots\}$, and functions $u(x)$ we set

$$u_{x^i} = \partial u/\partial x^i = D_i u, \quad D^\alpha u = D_1^{\alpha_1}\cdot\dots\cdot D_d^{\alpha_d}u, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

If we write $N = N(\cdots)$, this means that the constant $N$ depends only on what are in parenthesis.

## 2. PDEs on $\mathbb{R}^d$

Here we deal with the equations on $\mathbb{R}^d$. For $n = 0,1,2,\dots$, define

$$H^n_p = H^n_p(\mathbb{R}^d) = \{u : u, Du, \dots, D^\alpha u \in L_p : |\alpha| \leq n\}.$$

In general, for $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$ define the space $H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2}L_p$ (called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the set of all distributions $u$ such that

$$\|u\|_{H^\gamma_p} := \|(1-\Delta)^{\gamma/2}u\|_{L_p} < \infty.$$

Denote

$$\mathbb{H}^{\gamma,q}_p(T) = L_q([0,T],H^\gamma_p), \quad \mathbb{L}^q_p(T) = \mathbb{H}^{0,q}_p(T), \quad U^{\gamma,q}_p = H^{\gamma-2/q}_p.$$

By $\mathcal{H}^{\gamma,q}_p(T)$ we denote the space of all functions $u \in \mathbb{H}^{\gamma,q}_p(T)$ such that $u(0,\cdot) \in U^{\gamma,q}_p$ and for some $f \in \mathbb{H}^{\gamma-2,q}_p(T)$

$$u_t = f \tag{2.1}$$

in the sense of distributions. In other words, for any $\phi \in C_0^\infty$, the equality

$$(u(t,\cdot),\phi) = (u(0),\phi) + \int_0^t (f(s,\cdot),\phi)\,ds \tag{2.2}$$

holds for all $t \le T$. Denote

$$\mathcal{H}^{\gamma,q}_{p,0}(T) = \mathcal{H}^{\gamma,q}_p(T) \cap \{u : u(0,\cdot) = 0\}.$$

The norm in $\mathcal{H}^{\gamma,q}_p(T)$ is introduced by

$$\|u\|_{\mathcal{H}^{\gamma,q}_p(T)} = \|u\|_{\mathbb{H}^{\gamma,q}_p(T)} + \|u_t\|_{\mathbb{H}^{\gamma-2,q}_p(T)} + \|u(0)\|_{U^{\gamma,q}_p}.$$

Here are the main results of this section.

**Assumption 2.1.** (i) The functions $a^{ij}, b^i, \bar{b}^i, c$ are Borel measurable in $(t,x)$, $a^{ij} = a^{ji}$ and $a$ is uniformly continuous in $x$. In other words, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|a^{ij}(t,x) - a^{ij}(t,y)| < \varepsilon$$

whenever $|x-y| < \delta$.

(ii) For any $t > 0, x \in \mathbb{R}^d$, and $\lambda \in \mathbb{R}^d$,

$$\delta_0|\lambda|^2 \le a^{ij}(t,x)\lambda^i\lambda^j \le K|\lambda|^2, \tag{2.3}$$

$$|b^i(t,x)| + |\bar{b}^i(t,x)| + |c(t,x)| \le K.$$

**Theorem 2.2.** *Let $1 < p \le q < \infty$, $\varepsilon > 0$ and Assumption 2.1 be satisfied. Then for any $f^i \in \mathbb{L}^q_p(T)$, $f \in \mathbb{H}^{-1,q}_p(T)$ and $u_0 \in H_p^{1-2/q+\varepsilon}$, equation* (1.2) *with initial data $u_0$ admits a unique solution $u$ in the class $\mathcal{H}^{1,q}_p(T)$, and for this solution*

$$\|u\|_{\mathcal{H}^{1,q}_p(T)} \le N\|f^i\|_{\mathbb{L}^q_p(T)} + N\|f\|_{\mathbb{H}^{-1,q}_p(T)} + N\|u_0\|_{H_p^{1-2/q+\varepsilon}}, \tag{2.4}$$

*where $N = N(\varepsilon, d, p, q, \delta_0, K, T)$.*

Fix $\kappa_0 > 0$, and for $\gamma \in \mathbb{R}$ define $|\gamma|_+ = |\gamma|$ if $|\gamma| = 0,1,2,\dots$ and $|\gamma|_+ = |\gamma| + \kappa_0$ otherwise. Also define

$$B^{|\gamma|_+} = \begin{cases} B(\mathbb{R}^d) & : \quad \gamma = 0 \\ C^{|\gamma|-1,1}(\mathbb{R}^d) & : \quad |\gamma| = 1,2,\dots \\ C^{|\gamma|+\kappa_0}(\mathbb{R}^d) & : \quad \text{otherwise,} \end{cases}$$

where $B(\mathbb{R}^d)$ is the space of bounded functions on $\mathbb{R}^d$, $C^{|\gamma|-1,1}(\mathbb{R}^d)$ is the space of $|\gamma|-1$ times continuously differentiable functions whose derivatives of $(|\gamma|-1)$st order are Lipschitz continuous and $C^{|\gamma|+\kappa_0}(\mathbb{R}^d)$ is the usual Hölder space.

**Assumption 2.3.** For each $t > 0$,

$$|a(t,\cdot)|_{B^{|\gamma|_+}} + |b(t,\cdot)|_{B^{|\gamma|_+}} + |c(t,\cdot)|_{B^{|\gamma|_+}} \le C.$$

**Theorem 2.4.** *Let $1 < p \le q < \infty$, $\gamma \in \mathbb{R}$, $\varepsilon > 0$ and Assumptions 2.1 and 2.3 be satisfied. Then for any $f \in \mathbb{H}_p^{\gamma,q}(T)$ and $u_0 \in H_p^{\gamma+2-2/q+\varepsilon}$, equation (1.1) with initial data $u_0$ admits a unique solution $u$ in the class $\mathcal{H}_p^{\gamma+2,q}(T)$ and for this solution*

$$\|u\|_{\mathcal{H}_p^{\gamma+2,q}(T)} \le N\|f\|_{\mathbb{H}_p^{\gamma,q}(T)} + N\|u_0\|_{H_p^{\gamma+2-2/q+\varepsilon}}, \tag{2.5}$$

*where $N = N(\gamma, \varepsilon, d, p, q, \delta_0, K, C, T)$.*

Define

$$\|u_{xx}\|_{H_p^\gamma} = \sum_{i,j=1}^{d} \|u_{x^ix^j}\|_{H_p^\gamma}.$$

We need the following lemmas to prove Theorems 2.2 and 2.4.

**Lemma 2.5.** *For $k = 1, 2, \dots, n$, let matrix $a^k = (a^{k,ij})$ be independent of $x$ and satisfy (2.3). Let $u^{(k)} \in \mathcal{H}_{p,0}^{\gamma+2,p}(T)$ be a solution of the equation*

$$u_t^{(k)} = a^{k,ij}u_{x^ix^j}^{(k)} + f^{(k)}. \tag{2.6}$$

*Then*

$$\int_0^T \prod_{k=1}^{n} \|u_{xx}^{(k)}(t)\|_{H_p^\gamma}^p dt \le N \sum_{k=1}^{n} \int_0^T \|f^{(k)}(t)\|_{H_p^\gamma}^p \prod_{\ell \ne k} \|u_{xx}^{(\ell)}(t)\|_{H_p^\gamma}^p dt, \tag{2.7}$$

*where $N = N(d, p, n, \delta_0, K)$.*

*Proof.* See Lemma 1.6 in [4]. Actually in [4] the case $a^{(k)} = a^{(\ell)}, \forall \ell, k = 1, 2, \dots, n$, is considered. But one can easily check that the proof there still holds in our case. □

**Lemma 2.6.** *For $k = 1, 2, \dots, n$, let matrix $a^k = (a^{k,ij})$ satisfy Assumptions 2.1, 2.3 and*

$$|a^{(k)}(t,x) - a^{(k)}(t,y)| \le \varepsilon, \quad \forall k, t, x, y.$$

*Let $u^{(k)} \in \mathcal{H}_{p,0}^{\gamma+2,p}(T)$ be a solution of equation (2.6). Then there exists $\varepsilon_0 \in (0, \infty)$ independent of $C$ such that if $\varepsilon \le \varepsilon_0$, then*

$$\begin{aligned} &\int_0^T \prod_{k=1}^{n} \|u^{(k)}(t)\|_{H_p^{\gamma+2}}^p dt \\ &\le N \sum_{k=1}^{n} \int_0^T \|f^{(k)}(t)\|_{H_p^\gamma}^p \prod_{\ell \ne k} \|u^{(\ell)}(t)\|_{H_p^{\gamma+2}}^p dt \\ &\quad + N \sum_{J \in \Gamma} \int_0^T \Big(\prod_{k \in J} \|u^{(k)}(t)\|_{H_p^{\gamma+2}}^p\Big)\Big(\prod_{\ell \notin J} \|u^{(\ell)}(t)\|_{H_p^\gamma}^p\Big) dt, \end{aligned}$$

*where $\Gamma$ is the collection of all subsets $A$ of $\{1, 2, \dots, n\}$ such that $A \ne \{1, 2, \dots, n\}$ and $N = N(d, p, n, \gamma, \delta_0, K, C)$.*

*Proof.* Denote $a_0^{k,ij}(t,x) = a^{k,ij}(t,0)$ and $f_0^{(k)} = (a^{k,ij} - a_0^{k,ij})u^{(k)}_{x^ix^j} + f^{(k)}$. Then by Lemma 2.5, (2.7) holds with $f_0^{(k)}$ instead of $f^{(k)}$. Here we use

$$\|u\|_{H_p^{\gamma+2}} \sim (\|u_{xx}\|_{H_p^\gamma} + \|u\|_{H_p^\gamma})$$
$$\|(a^{(k)} - a_0^{(k)})u^{(k)}_{xx}\|_{H_p^\gamma} \le N|a^{(k)} - a_0^{(k)}|_{B^{|\gamma|+}} \|u^{(k)}\|_{H_p^{\gamma+2}},$$

and get

$$\begin{aligned}
&\int_0^T \prod_{k=1}^n \|u^{(k)}(t)\|^p_{H_p^{\gamma+2}} dt \\
&\le N\int_0^T \prod_{k=1}^n (\|u^{(k)}_{xx}(t)\|^p_{H_p^\gamma} + \|u^{(k)}(t)\|^p_{H_p^\gamma}) dt \\
&\le N\int_0^T \prod_{k=1}^n \|u^{(k)}_{xx}(t)\|^p_{H_p^\gamma} dt \\
&\quad + N\sum_{J\in\Gamma}\int_0^T (\prod_{k\in J}\|u^{(k)}_{xx}(t)\|^p_{H_p^\gamma})(\prod_{\ell\notin J}\|u^{(\ell)}(t)\|^p_{H_p^\gamma})\, dt \\
&\le N\sup_{k,t}|a^{(k)} - a_0^{(k)}|_{B^{|\gamma|+}} \int_0^T \prod_{k=1}^n \|u^{(k)}(t)\|^p_{H_p^{\gamma+2}} dt \\
&\quad + N\sum_{k=1}^n \int_0^T \|f^{(k)}(t)\|^p_{H_p^\gamma} \prod_{\ell\neq k}\|u^{(\ell)}(t)\|^p_{H_p^{\gamma+2}} dt \\
&\quad + N\sum_{J\in\Gamma}\int_0^T (\prod_{k\in J}\|u^{(k)}(t)\|^p_{H_p^{\gamma+2}})(\prod_{\ell\notin J}\|u^{(\ell)}(t)\|^p_{H_p^\gamma})\, dt.
\end{aligned}$$

Thus our lemma holds true if $N\sup_{k,t}|a^{(k)} - a_0^{(k)}|_{B^{|\gamma|+}} < 1/2$. Observe that for $a_m^{(k)}(t,x) := a^{(k)}(t/m^2, x/m)$ we have

$$|a_m^{(k)}(t,\cdot) - a_m^{(k)}(t,\cdot)|_{B^{|\gamma|+}} \le \varepsilon + m^{-(|\gamma|+\wedge 1)}C,$$

and the second on the right can be dropped if $\gamma = 0$. Denote $\varepsilon_0 = (4N)^{-1}$ and fix $m$ such that $Nm^{-(|\gamma|+\wedge 1)}C \le 1/4$. Then it follows that the lemma holds if we replace $a^{(k)}, u^{(k)}, f^{(k)}$ and $T$ by $a_m^{(k)}$, $u^{(k)}(t/m^2, x/m)$, $m^{-2}f^{(k)}(t/m^2, x/m)$ and $m^2T$, respectively. Finally it suffices to observe that $\|\cdot\|_{H_p^\gamma}$ norms of $u^{(k)}(t/m^2, x/m)$ and $u^{(k)}(t,x)$ are comparable. The lemma is proved. □

**Lemma 2.7.** *For $k = 1, 2, \ldots, n$, let matrix $a^k = (a^{k,ij})$ satisfy Assumption 2.1 and*

$$|a^{(k)}(t,x) - a^{(k)}(t,y)| \le \varepsilon, \quad \forall k, t, x, y.$$

*Let $u^{(k)} \in \mathcal{H}^{1,p}_{p,0}(T)$ be a solution of the equation*

$$u_t^{(k)} = D_i(a^{k,ij}u^{(k)}_{x^j} + f^{i(k)}) + f^{(k)}.$$

*Then there exists* $\varepsilon_1 \in (0,\infty)$ *such that if* $\varepsilon \le \varepsilon_1$, *then*

$$\int_0^T \prod_{k=1}^n \|u^{(k)}(t)\|^p_{H^1_p}\,dt \le N\sum_{k=1}^n \int_0^T \|f^{i(k)}(t)\|^p_{L_p} \prod_{\ell\neq k} \|u^{(\ell)}(t)\|^p_{H^1_p}\,dt$$
$$+\sum_{k=1}^n \int_0^T \|f^{(k)}(t)\|^p_{H^{-1}_p} \prod_{\ell\neq k} \|u^{(\ell)}(t)\|^p_{H^1_p}\,dt$$
$$+N\sum_{J\in\Gamma}\int_0^T (\prod_{k\in J} \|u^{(k)}(t)\|^p_{H^1_p})(\prod_{\ell\notin J}\|u^{(\ell)}(t)\|^p_{L_p})\,dt,$$

*where* $N = N(d,p,n,\gamma,\delta_0,K)$.

*Proof.* Denote $\bar{f}_0^{(k)} = D_i((a^{k,ij}-a_0^{k,ij})u^{(k)}_{x^j}+f^{i(k)})+f^{(k)}$. Then by Lemma 2.5, (2.7) holds with $\gamma=-1$ and $\bar{f}_0^{(k)}$ instead of $f^{(k)}$. Here we use

$$\begin{aligned}&\|\bar{f}_0^{(k)}\|_{H^{-1}_p}\\ &\le N\|(a^{(k)}-a_0^{(k)})u^{(k)}_x + f^{i(k)}\|_{L_p} + \|f^{(k)}\|_{H^{-1}_p}\\ &\le N\varepsilon\|u^{(k)}_x\|_{L_p} + N\|f^{i(k)}\|_{L_p} + \|f^{(k)}\|_{H^{-1}_p}\\ &\le N\varepsilon\|u^{(k)}\|_{H^1_p} + N\|f^{i(k)}\|_{L_p} + \|f^{(k)}\|_{H^{-1}_p}\end{aligned}$$

and get

$$\int_0^T \prod_{k=1}^n \|u^{(k)}(t)\|^p_{H^1_p}dt \le N\int_0^T \prod_{k=1}^n (\|u^{(k)}_{xx}(t)\|^p_{H^{-1}_p} + \|u^{(k)}(t)\|^p_{L_p})dt$$
$$\le N\varepsilon \int_0^T \prod_{k=1}^n \|u^{(k)}(t)\|^p_{H^1_p}dt + \sum_{k=1}^n\int_0^T \|f^{i(k)}(t)\|^p_{L_p}\prod_{\ell\neq k}\|u^{(\ell)}(t)\|^p_{H^1_p}\,dt$$
$$+N\sum_{k=1}^n\int_0^T \|f^{(k)}(t)\|^p_{H^{-1}_p}\prod_{\ell\neq k}\|u^{(\ell)}(t)\|^p_{H^1_p}\,dt$$
$$+N\sum_{J\in\Gamma}\int_0^T (\prod_{k\in J}\|u^{(k)}(t)\|^p_{H^1_p})(\prod_{\ell\notin J}\|u^{(\ell)}(t)\|^p_{L_p})\,dt.$$

Obviously this proves the lemma. □

*Proof of Theorem 2.4.* The theorem is already known ([4]) if $a=(a^{ij})$ is independent of $x$ and $b^i=c=0$. Thus due to the method of continuity we only need to show that estimate (2.5) holds given that a solution $u\in\mathcal{H}^{\gamma+2,q}_p(T)$ already exists. We know from [4] that there is a unique solution $v\in\mathcal{H}^{\gamma+2,q}_p(T)$ of the equation

$$v_t=\Delta v,\quad v(0,\cdot)=u_0$$

and

$$\|v\|_{\mathcal{H}^{\gamma+2,q}_p(T)}\le N\|u_0\|_{H^{\gamma+2-2/q+\varepsilon}_p}.$$

If we replace the function $u$ by $u-v$, we come to the situation with zero initial data. Also we may assume that $q=np$ for some positive integer $n$. The case $q \neq np$ is covered by Marcinkiewicz interpolation theorem.

Choose $r>0$ such that

$$|a(t,x)-a(t,y)|<\varepsilon_0/2$$

whenever $|x-y|<r$. Let $\{\zeta_m : m=1,2,\dots\}$ be a standard partition of unity such that for any $m$ the support of $\zeta_m$ lies in a ball $B_{r/4}(x_m)$. Also for each $m$, choose a function $\eta_k \in C_0^\infty(B_{r/2}(x_m))$ such that $0 \le \eta_m \le 1$ and $\eta_m=1$ on the support of $\zeta_m$. Denote

$$a^{(m)}(t,x)=(a^{m,ij}):=a(t,x)\eta_m(x)+(1-\eta_m(x))a(t,x_m).$$

Then one can easily check that

$$\sup_{t,x,y}|a^{(m)}(t,x)-a^{(m)}(t,y)|<\varepsilon_0.$$

By Lemma 6.7 in [5] for any $\gamma \in \mathbb{R}$ and $v \in H_p^\gamma$,

$$\sum_m \|\zeta_m v\|^p_{H_p^\gamma} \sim \|v\|^p_{H_p^\gamma}, \tag{2.8}$$

$$\sum_m \|\zeta_{mx} v\|^p_{H_p^\gamma} \le N\|v\|^p_{H_p^\gamma}, \quad \sum_m \|\zeta_{mxx} v\|^p_{H_p^\gamma} \le N\|v\|^p_{H_p^\gamma}. \tag{2.9}$$

Therefore,

$$\begin{aligned} \int_0^T \|u\|^{np}_{H_p^{\gamma+2}}dt &\le N\int_0^T \Big(\sum_m \|\zeta_m u\|^p_{H_p^{\gamma+2}}\Big)^n dt \\ &= N\sum_{m_1,m_2,\dots,m_n}\int_0^T \prod_{k=1}^n \|\zeta_{m_k}u\|^p_{H_p^{\gamma+2}}dt. \end{aligned} \tag{2.10}$$

Note that $\zeta_{m_k}u$ satisfies

$$(\zeta_{m_k}u)_t = a^{m_k,ij}(\zeta_{m_k}u)_{x^ix^j}+f^{(m_k)},$$

where

$$f^{(m_k)} := -2a^{ij}u_{x^i}\zeta_{m_k x^i} - a^{ij}u\zeta_{m_k x^i x^j} + b^i u_{x^i}\zeta_{m_k} + cu\zeta_{m_k} + \zeta_{m_k} f$$

and (see Lemma 5.2 in [5]),

$$\begin{aligned} \|f^{(m_k)}\|_{H_p^\gamma} \le N\|u_{x^i}\zeta_{m_k x^j}\|_{H_p^\gamma} + N\|u\zeta_{m_k x^i x^j}\|_{H_p^\gamma} + N\|u_{x^i}\zeta_{m_k}\|_{H_p^\gamma} \\ + N\|u\zeta_{m_k}\|_{H_p^\gamma} + \|\zeta_{m_k} f\|_{H_p^\gamma}. \end{aligned}$$

By Lemma 2.6, for any $t \le T$,

$$\int_0^t \prod_{k=1}^n \|\zeta_{m_k}u\|^p_{H_p^{\gamma+2}}\,ds$$

$$\leq N\sum_{k=1}^{n}\sum_{m_1,m_2,\dots,m_n}\int_0^t \|f^{(m_k)}\|^p_{H^\gamma_p}\prod_{\ell\neq k}\|u\zeta_{m_\ell}\|^p_{H^{\gamma+2}_p}\,ds$$
$$+N\sum_{J\in\Gamma}\sum_{m_1,m_2,\dots,m_n}\int_0^t \Big(\prod_{k\in J}\|u\zeta_{m_k}\|^p_{H^{\gamma+2}_p}\Big)\Big(\prod_{\ell\notin J}\|u\zeta_{m_\ell}\|^p_{H^\gamma_p}\Big)\,ds.$$

Here

$$\sum_{J\in\Gamma}\sum_{m_1,m_2,\dots,m_n}\int_0^t \Big(\prod_{k\in J}\|u\zeta_{m_k}\|^p_{H^{\gamma+2}_p}\Big)\Big(\prod_{\ell\notin J}\|u\zeta_{m_\ell}\|^p_{H^\gamma_p}\Big)\,ds$$
$$=\sum_{J\in\Gamma}\int_0^t \prod_{k\in J}\Big(\sum_{m_k}\|u\zeta_{m_k}\|^p_{H^{\gamma+2}_p}\Big)\prod_{\ell\notin J}\Big(\sum_{m_\ell}\|u\zeta_{m_\ell}\|^p_{H^\gamma_p}\Big)\,ds$$
$$\leq N\sum_{\ell=0}^{n-1}\int_0^t \|u\|^{\ell p}_{H^{\gamma+2}_p}\|u\|^{(n-\ell)p}_{H^\gamma_p}\,ds \leq \varepsilon\int_0^t \|u\|^{np}_{H^{\gamma+2}_p}dt + N\int_0^t \|u\|^{np}_{H^\gamma_p}\,ds$$

and

$$\sum_{k=1}^{n}\sum_{m_1,m_2,\dots,m_n}\int_0^t \|u_{x^i}\zeta_{m_k x^j}\|^p_{H^\gamma_p}\prod_{\ell\neq k}\|u\zeta_{m_\ell}\|^p_{H^{\gamma+2}_p}\,ds$$
$$=\sum_{k=1}^{n}\int_0^t \Big(\sum_{m_k}\|u_{x^i}\zeta_{m_k x^j}\|^p_{H^\gamma_p}\Big)\prod_{\ell\neq k}\Big(\sum_{m_\ell}\|u\zeta_{m_\ell}\|^p_{H^{\gamma+2}_p}\Big)\,ds$$
$$\leq N\int_0^t \|u_x\|^p_{H^\gamma_p}\|u\|^{(n-1)p}_{H^{\gamma+2}_p}\,ds.$$

Similar computation based on (2.8) and (2.9) shows

$$\sum_{k=1}^{n}\sum_{m_1,m_2,\dots,m_n}\int_0^t \|f^{(m_k)}\|^p_{H^\gamma_p}\prod_{\ell\neq k}\|u\zeta_{m_\ell}\|^p_{H^{\gamma+2}_p}\,ds$$
$$\leq N\int_0^t \big(\|u_x\|^p_{H^\gamma_p}+\|u\|^p_{H^\gamma_p}+\|f\|^p_{H^\gamma_p}\big)\|u\|^{(n-1)p}_{H^{\gamma+2}_p}\,ds$$
$$\leq \varepsilon\|u\|^{np}_{\mathbb{H}^{\gamma+2,np}_p(t)}+N\|u_x\|^{np}_{\mathbb{H}^{\gamma,np}_p(t)}+N\|u\|^{np}_{\mathbb{H}^{\gamma,np}_p(t)}+N\|f\|^{np}_{\mathbb{H}^{\gamma,np}_p(t)}.$$

Now take $\varepsilon>0$ sufficiently small then for each $t\leq T$,

$$\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(t)}=\|u\|^{np}_{\mathbb{H}^{\gamma+2,np}_p(t)}+\|a^{ij}u_{x^ix^j}+b^iu_{x^i}+cu+f\|^{np}_{\mathbb{H}^{\gamma,np}_p(t)}$$
$$\leq N\|u\|^{np}_{\mathbb{H}^{\gamma+1,nq}_p(t)}+N\|f\|^{nq}_{\mathbb{H}^{\gamma,nq}_p(t)}.$$

Here we use (see Theorem 4.2 in [3])

$$\sup_{s\leq t}\|u(s)\|^p_{H^{\gamma+1}_p}\leq N(\gamma,p,d,T)\|u\|^p_{\mathcal{H}^{\gamma+2,p}_p(t)} \tag{2.11}$$

and get

$$\sup_{s\leq t}\|u(s)\|^{np}_{H^{\gamma+1}_p}\leq N\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(t)},$$

$$(2.12)\qquad \|u\|^{np}_{\mathbb{H}^{\gamma+1,np}_p(t)} \le \int_0^t \sup_{r\le s}\|u(r)\|^{np}_{H^{\gamma+1,np}_p}\,ds \le N\int_0^t \|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(s)}\,ds,$$

$$\|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(t)} \le N\int_0^t \|u\|^{np}_{\mathcal{H}^{\gamma+2,np}_p(s)}\,ds + N\|f\|^{nq}_{\mathbb{H}^{\gamma,nq}_p(T)}.$$

Actually there is a restriction that $p \ge 2$ in Theorem 4.2 of [5], but by inspecting the proofs of Theorem 4.2 and Theorem 7.1 in [5] one can easily check that in our (deterministic) case the result holds for all $p > 1$. Finally Gronwall's inequality leads to (2.5). The theorem is proved. □

*Proof of Theorem 2.2.* We prove that the estimate (2.4) holds assuming that a solution $u \in \mathbb{H}^{1,q}_p(T)$ of equation (1.2) already exists. As before, we also assume that $u_0 = 0$ and $q = np$.

**Step 1**. Assume that $b^i \equiv 0$. Define a partition of unity $\{\zeta_m : m = 1,2,\dots\}$ and $a^{(m)}$ as in the proof of Theorem 2.4 such that

$$\sup_{t,x,y}|a^{(m)}(t,x) - a^{(m)}(t,y)| \le \varepsilon_1.$$

As in (2.11),

$$\int_0^t \|u\|^{np}_{H^1_p}\,ds \le N\sum_{m_1,\dots,m_n}\int_0^t\prod_{k=1}^n \|\zeta_{m_k}u\|^p_{H^1_p}\,ds.$$

Note that $v^{(m_k)} := \zeta_{m_k}u$ satisfies

$$v^{(m_k)}_t = D_i(a^{m_k,ij}v^{(m_k)}_{x^j} + f^{i(m_k)}) + f^{(m_k)},$$

where

$$\begin{aligned} f^{i(m_k)} &:= \bar{b}^i\zeta_{m_k}u + f^i\zeta_{m_k} - a^{ij}u\zeta_{m_kx^j},\\ f^{(m_k)} &:= -a^{ij}u_{x^j}\zeta_{m_kx^i} - \bar{b}^iu\zeta_{m_kx^i} - f^i\zeta_{m_kx^i} + c\zeta_{m_k}u + f\zeta_{m_k}.\end{aligned}$$

Obviously,

$$\|f^{i(m_k)}\|_{L_p} \le N\|\zeta_{m_k}u\|_{L_p} + \|f^i\zeta_{m_k}\|_{L_p} + \|u\zeta_{m_kx^j}\|_{L_p}$$

and (remember that $\|u\|_{H^{\gamma-1}_p} \le \|u\|_{H^\gamma_p}$)

$$\begin{aligned}\|f^{(m_k)}\|_{H^{-1}_p} \le \|a^{ij}u_{x^j}\zeta_{m_kx^i}\|_{H^{-1}_p} + N\|u\zeta_{m_kx^i}\|_{L_p} + \|f^i\zeta_{m_kx^i}\|_{L_p}\\ + N\|\zeta_{m_k}u\|_{L_p} + \|f\zeta_{m_k}\|_{H^{-1}_p}.\end{aligned}$$

Also, we claim that for any $\varepsilon > 0$,

$$(2.13)\qquad \|a^{ij}u_{x^j}\zeta_{m_kx^i}\|_{H^{-1}_p} \le \varepsilon\|u_{x^j}\zeta_{m_kx^i}\|_{L_p} + N(\varepsilon)\|u_{x^j}\zeta_{m_kx^i}\|_{H^{-1}_p}.$$

Indeed, take a sequence of smooth functions $a_n$ satisfying

$$\sup_x|a - a_n| \le 1/n.$$

Then

$$\|a^{ij}u_{x^j}\zeta_{m_kx^i}\|_{H_p^{-1}} \le \|a_n^{ij}u_{x^j}\zeta_{m_kx^i}\|_{H_p^{-1}} + \|(a^{ij}-a_n^{ij})u_{x^j}\zeta_{m_kx^i}\|_{L_p}$$
$$\le N|a_n|_{C^1}\|u_{x^j}\zeta_{m_kx^i}\|_{H_p^{-1}} + 1/n\|u_{x^j}\zeta_{m_kx^i}\|_{L_p}.$$

Now we use Lemma 2.7 to estimate

$$\sum_{m_1,\dots,m_n}\int_0^T\prod_{k=1}^n\|\zeta_{m_k}u\|_{H_p^1}^p\,dt.$$

Similar computations as in the proof of Theorem 2.4 shows that

$$\|u\|^{np}_{\mathbb{H}_p^{1,np}(t)} \le N\varepsilon\|u\|^{np}_{\mathbb{H}_p^{1,np}(t)} + N(\varepsilon)\|u\|^{np}_{\mathbb{L}_p^{np}(t)} + N\|f^i\|^{np}_{\mathbb{L}_p^{np}(T)} + N\|f\|^{np}_{\mathbb{H}_p^{-1,np}(T)}.$$

This, (2.12) and Gronwall's inequality certainly yield (2.4).

**Step 2**. Drop the condition $b^i\equiv 0$ in Step 1. We prove that there exists $\varepsilon\in(0,1)$ such that (2.4) holds if $T\le\varepsilon$. Assume that $T\le 1$. Since $b^iu_{x^i}\in\mathbb{L}_p^{np}(T)$, by Theorem 2.4, the equation

$$v_t = \Delta v + b^iu_{x^i}$$

has a unique solution $v\in\mathcal{H}_{p,0}^{2,np}(T)$, and furthermore

$$\|v\|_{\mathcal{H}_p^{2,np}(T)} \le N\|bu_x\|_{\mathbb{L}_p^{np}(T)} \le N\|u_x\|_{\mathbb{L}_p^{np}(T)}, \tag{2.14}$$

where $N$ is independent of $T$, since $T\le 1$. Thus by (2.12),

$$\|v\|_{\mathbb{H}_p^{1,np}(T)} \le NT\|v\|_{\mathcal{H}_p^{2,np}(T)} \le NT\|u_x\|_{\mathbb{L}_p^{np}(T)}.$$

Observe that $\bar u = u - v$ satisfies

$$\bar u_t = D_i(a^{ij}\bar u_{x^j} + \bar b^i\bar u + \bar f^i) + c\bar u + \bar f,$$

with

$$\bar f^i := f^i + \bar b^iv + (a^{ij}-\delta^{ij})v_{x^j}, \quad \bar f := f + cv.$$

By the result of step 1 and (2.14)

$$\|u\|^{np}_{\mathbb{H}_p^{1,np}(T)} \le \|\bar u\|^{np}_{\mathbb{H}_p^{1,np}(T)} + \|v\|^{np}_{\mathbb{H}_p^{1,np}(T)}$$
$$\le NT\|u\|^{np}_{\mathbb{H}_p^{1,np}(T)} + N\|f^i\|^{np}_{\mathbb{L}_p^{np}(T)} + N\|f\|^{np}_{\mathbb{H}_p^{-1,np}(T)}.$$

Thus if $NT\le 1$, then

$$\|u\|^{np}_{\mathbb{H}_p^{1,np}(T)} \le N\|f^i\|^{np}_{\mathbb{L}_p^{np}(T)} + N\|f\|^{np}_{\mathbb{H}_p^{-1,np}(T)}.$$

This yields (2.4).

**Step 3**. General case. First we prove the following lemma.

**Lemma 2.8.** *Let $\tau\le T$, $u\in\mathcal{H}_{p,0}^{\gamma+2,q}(\tau)$ and $u_t = f$. Then there exists a unique $\tilde u\in\mathcal{H}_{p,0}^{\gamma+2,q}(T)$ such that $\tilde u(t) = u(t)$ for $t\le\tau$ and, on $(0,T)$,*

$$u_t = \Delta\tilde u + \tilde f(t), \tag{2.15}$$

*where $\tilde f = (f(t) - \Delta u(t))I_{t\le\tau}$. Furthermore,*

$$\|\tilde u\|_{\mathcal{H}_p^{\gamma+2,q}(T)} \le N\|u\|_{\mathcal{H}_p^{\gamma+2,q}(\tau)}, \tag{2.16}$$

*where $N$ is independent of $u$ and $\tau$.*

*Proof.* Note $\tilde{f} \in \mathbb{H}_p^{\gamma,q}(T)$, so that, by Theorem 2.4, equation (2.15) has a unique solution $\tilde{u} \in \mathcal{H}_{p,0}^{\gamma+2,q}(T)$ and (2.16) follows. To show that $\tilde{u}(t) = u(t)$ for $t \leq \tau$, notice that, for $t \leq \tau$, the function $v(t) = \tilde{u}(t) - u(t)$ satisfies the equation

$$v_t = \Delta v, \quad v(0, \cdot) = 0.$$

It follows from Theorem 2.4 that $v(t) = 0$ for $t \leq \tau$. □

Now, take an integer $M \geq 2$ such that $T/M \leq \varepsilon$, and denote $t_m = Tm/M$. Assume that, for $m = 1, 2, \ldots, M-1$, we have the estimate (2.4) with $t_m$ in place of $T$ (and $N$ depending only on $d, p, \delta_0, K, T$). We are going to use the induction on $m$. Let $u^{(m)}$ be the continuation of $u$ on $[t_m, T]$, which exists by Lemma 2.8 with $\gamma = -1$. Then

$$\|u_m\|_{\mathcal{H}_p^{1,q}(T)} \leq N\|u\|_{\mathcal{H}_p^{1,q}(t_m)}. \tag{2.17}$$

Denote $v^{(m)} := u - u^{(m)}$, then $v^{(m)}(t_m, \cdot) = 0$, and for $t \in [t_m, T]$

$$v_t^{(m)} = D_i(a^{ij}v_{x^j}^{(m)} + \bar{b}^i v^{(m)} + f_m^i) + b^i v_{x^i}^{(m)} + cv^{(m)} + f_m,$$

where

$$f_m^i = (a^{ij} - \delta^{ij})u_{x^j}^{(m)} + \bar{b}^i u^{(m)} + f^i, \quad f_m = b^i u_{x^i}^{(m)} + cu^{(m)} + f.$$

Next, in a natural way, introduce spaces $L_q([t_m, t], H_p^\gamma)$. Then we get a counterpart of the result of step 2 and conclude that

$$\int_{t_m}^{t_{m+1}} \|(u - u_m)(s)\|_{H_p^1}^q \, ds$$
$$\leq N \int_{t_m}^{t_{m+1}} (\|f_m^i(s)\|_{L_p}^q + \|f_m(s)\|_{H_p^{-1}}^q) \, ds.$$

Thus by (2.17) and the induction hypothesis we get

$$\int_0^{t_{m+1}} \|u(s)\|_{H_p^1}^q \, ds$$
$$\leq N \int_0^T \|u_m(s)\|_{H_p^1}^p \, ds + N \int_{t_m}^{t_{m+1}} \|(u - u_m)(s)\|_{H_p^1}^q \, ds$$
$$\leq N\|f^i\|_{\mathbb{L}_p^q(t_{m+1})}^q + N\|f\|_{\mathbb{H}_p^{-1,q}(t_{m+1})}^p.$$

We see that the induction goes through and thus the theorem is proved. □

## 3. PDEs on bounded $C^1$ domains

Here we deal with the equations on bounded domains.

**Assumption 3.1.** The domain $\Omega$ is of class $C_u^1$. In other words, for any $x_0 \in \partial\Omega$, there exist constants $r_0, K_0 \in (0, \infty)$ and a one-to-one continuously differentiable mapping $\Psi$ of $B_{r_0}(x_0)$ onto a domain $G \subset \mathbb{R}^d$ such that

(i) $G_+ := \Psi(B_{r_0}(x_0) \cap \Omega) \subset \mathbb{R}^d_+$ and $\Psi(x_0) = 0$;
(ii) $\Psi(B_{r_0}(x_0) \cap \partial\Omega) = G \cap \{y \in \mathbb{R}^d : y^1 = 0\}$;
(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))} \le K_0$ and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \le K_0|y_1 - y_2|$ for any $y_i \in G$;
(iv) $\Psi_x$ is uniformly continuous in for $B_{r_0}(x_0)$.

Here is a well known property of $C^1$ domains (see, for instance, [2]). Denote $\rho(x) := \mathrm{dist}(x, \partial\Omega)$.

**Lemma 3.2.** *There is a real-valued bounded function $\psi$ defined on $\bar{\Omega}$ such that for any multi-index $\alpha$,*

$$(3.1) \qquad \lim_{\rho(x)\to 0} \rho(x)\psi_{xx}(x) = 0, \quad \sup_{\Omega} \rho^{|\alpha|}(x)|D^\alpha \psi_x(x)| < \infty$$

*and for some constant $N = N(\Omega) > 0$,*

$$N^{-1}\rho(x) \le \psi(x) \le N\rho(x).$$

We use the Banach spaces introduced in [3], [6], and [9]. If $\theta \in \mathbb{R}$ and $n$ is a nonnegative integer, then

$$L_{p,\theta}(\Omega) := H^0_{p,\theta}(\Omega) = L_p(\Omega, \rho^{\theta-d}dx),$$

$$(3.2) \qquad H^n_{p,\theta}(\Omega) := \{u : u, \rho u_x, \dots, \rho^{|\alpha|}D^\alpha u \in L_{p,\theta}(\Omega) : |\alpha| \le n\}.$$

In general, for $\gamma \in \mathbb{R}$, the weighted Sobolev space $H^\gamma_{p,\theta}(\Omega)$ is defined as set of all distributions $u$ on $\Omega$ such that

$$(3.3) \qquad \|u\|^p_{H^\gamma_{p,\theta}(\Omega)} := \sum_{n=-\infty}^{\infty} e^{n\theta}\|\zeta_{-n}(e^n\cdot)u(e^n\cdot)\|^p_{H^\gamma_p} < \infty,$$

where $\{\zeta_n : n \in \mathbb{Z}\}$ is a sequence of functions $\zeta_n \in C_0^\infty(\Omega)$ such that

$$\sum_n \zeta_n > c > 0, \quad |D^m\zeta_n(x)| \le N(m)e^{mn}.$$

We also introduce Banach spaces defined on $\mathbb{R}^d_+$. Fix a function $\zeta \in C_0^\infty(\mathbb{R}_+)$ such that

$$(3.4) \qquad \sum_{n\in\mathbb{Z}} \zeta(e^{n+x}) > c > 0, \quad \forall x \in \mathbb{R},$$

and define $\zeta_n(x) = \zeta(e^n x)$, so that (3.3) becomes

$$(3.5) \qquad \|u\|^p_{H^\gamma_{p,\theta}} := \sum_{n=-\infty}^{\infty} e^{n\theta}\|\zeta(\cdot)u(e^n\cdot)\|^p_{H^\gamma_p} < \infty.$$

It is known that the set $H^\gamma_{p,\theta}(\Omega)$ is independent of the choice of $\zeta_n$, and the norms generated by different choices of $\zeta_n$ are all equivalent. In particular, if $\gamma$ is a nonnegative integer then as in (3.2)

$$(3.6) \qquad \|u\|^p_{H^\gamma_{p,\theta}(\Omega)} \sim \sum_{|\alpha|\le\gamma} \int_\Omega |\rho^{|\alpha|}D^\alpha u|^p \rho^{\theta-d}dx.$$

Denote $\rho(x,y)=\rho(x)\wedge\rho(y)$. For $\sigma\in\mathbb{R}$, $\nu\in(0,1]$ and $k=0,1,2,\dots$, define

$$|u|_{C(\Omega)}=\sup_{\Omega}|u(x)|,\quad [u]_{C^\nu(\Omega)}=\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\nu},$$

$$[u]_k^{(\sigma)}=[u]_{k,\Omega}^{(\sigma)}=\sup_{\substack{x\in\Omega\\|\beta|=k}}\rho^{k+\sigma}(x)|D^\beta u(x)|,$$

$$[u]_{k+\nu}^{(\sigma)}=[u]_{k+\nu,\Omega}^{(\sigma)}=\sup_{\substack{x,y\in\Omega\\|\beta|=k}}\rho^{k+\nu+\sigma}(x,y)\frac{|D^\beta u(x)-D^\beta u(y)|}{|x-y|^\nu},$$

$$|u|_k^{(\sigma)}=|u|_{k,\Omega}^{(\sigma)}=\sum_{j=0}^k[u]_{j,\Omega}^{(\sigma)},\quad |u|_{k+\nu}^{(\sigma)}=|u|_{k+\nu,\Omega}^{(\sigma)}=|u|_{k,\Omega}^{(\sigma)}+[u]_{k+\nu,\Omega}^{(\sigma)}.$$

We collect some properties of the space $H^\gamma_{p,\theta}(\Omega)$. The constants $N$'s in the following lemma depend only $d,p,\gamma,\theta$ and $\Omega$.

**Lemma 3.3** ([9]). (i) *Assume that $\gamma-d/p=m+\nu$ for some $m=0,1,\dots$ and $\nu\in(0,1]$. Let $i,j$ be multi-indices such that $|i|\le m$, $|j|=m$. Then for any $u\in H^\gamma_{p,\theta}(\Omega)$, we have*

$$\psi^{|i|+\theta/p}D^iu\in C(\Omega),\quad \psi^{m+\nu+\theta/p}D^ju\in C^\nu(\Omega),$$
$$|\psi^{|i|+\theta/p}D^iu|_{C(\Omega)}+[\psi^{m+\nu+\theta/p}D^ju]_{C^\nu(\Omega)}\le N\|u\|_{H^\gamma_{p,\theta}(\Omega)}.$$

(ii) $\psi D, D\psi : H^\gamma_{p,\theta}(\Omega)\to H^{\gamma-1}_{p,\theta}(\Omega)$ *are bounded linear operators, and for any $u\in H^\gamma_{p,\theta}(\Omega)$*

$$\|u\|_{H^\gamma_{p,\theta}(\Omega)}\le N\|\psi u_x\|_{H^{\gamma-1}_{p,\theta}(\Omega)}+N\|u\|_{H^{\gamma-1}_{p,\theta}(\Omega)}\le N\|u\|_{H^\gamma_{p,\theta}(\Omega)}, \tag{3.7}$$

$$\|u\|_{H^\gamma_{p,\theta}(\Omega)}\le N\|(\psi u)_x\|_{H^{\gamma-1}_{p,\theta}(\Omega)}+N\|u\|_{H^{\gamma-1}_{p,\theta}(\Omega)}\le N\|u\|_{H^\gamma_{p,\theta}(\Omega)}. \tag{3.8}$$

(iii) *For any $\nu,\gamma\in\mathbb{R}$, $\psi^\nu H^\gamma_{p,\theta}(\Omega)=H^\gamma_{p,\theta-p\nu}(\Omega)$, and*

$$\|u\|_{H^\gamma_{p,\theta-p\nu}(\Omega)}\le N\|\psi^{-\nu}u\|_{H^\gamma_{p,\theta}(\Omega)}\le N\|u\|_{H^\gamma_{p,\theta-p\nu}(\Omega)}. \tag{3.9}$$

(iv) *There exists a constant $N>0$ such that*

$$\|au\|_{H^\gamma_{p,\theta}(\Omega)}\le N|a|^{(0)}_{|\gamma|_+}\|u\|_{H^\gamma_{p,\theta}(\Omega)}.$$

Denote

$$\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,T)=L_q([0,T],H^\gamma_{p,\theta}(\Omega)),\quad \mathbb{H}^{\gamma,q}_{p,\theta}(T)=L_q([0,T],H^\gamma_{p,\theta}),$$

$$\mathbb{L}^q_{p,\theta}(\Omega,T)=\mathbb{H}^{0,q}_{p,\theta}(\Omega,T),\quad U^{\gamma,q}_{p,\theta}=\psi^{1-2/q}H^{\gamma-2/q}_{p,\theta}(\Omega),$$

where

$$\|u\|_{U^{\gamma,q}_{p,\theta}}:=\|\psi^{2/q-1}u\|_{H^{\gamma-2/q}_{p,\theta}(\Omega)}.$$

By $\mathfrak{H}^{\gamma,q}_{p,\theta}(\Omega,T)$ we denote the space of all functions $u \in \psi\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,T)$ such that $u(0,\cdot) \in U^{\gamma,q}_{p,\theta}$ and for some $f \in \psi^{-1}\mathbb{H}^{\gamma-2,q}_{p,\theta}(\Omega,T)$,

$$u_t = f \tag{3.10}$$

in the sense of distributions. We define $\mathfrak{H}^{\gamma,q}_{p,\theta,0}(\Omega,T) = \mathfrak{H}^{\gamma,q}_{p,\theta}(\Omega,T)\cap\{u : u(0,\cdot) = 0\}$. The norm in $\mathfrak{H}^{\gamma,q}_{p,\theta}(\Omega,T)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\Omega,T)} = \|\psi^{-1}u\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,T)} + \|\psi u_t\|_{\mathbb{H}^{\gamma-2,q}_{p,\theta}(\Omega,T)} + \|u_0\|_{U^{\gamma,q}_{p,\theta}}.$$

By Lemma 3.3(iii), the norm $\|u\|_{\mathfrak{H}^{\gamma,q}_{p,\theta}(\Omega,T)}$ is equivalent to

$$\|u\|_{\mathbb{H}^{\gamma,q}_{p,\theta-p}(\Omega,T)} + \|u_t\|_{\mathbb{H}^{\gamma-2,q}_{p,\theta+p}(\Omega,T)} + \|u_0\|_{U^{\gamma,q}_{p,\theta}}.$$

From this point on, we assume that

$$d-1 < \theta < d-1+p.$$

Here are the main results of this section.

**Assumption 3.4.** (i) The functions $a, b, \bar{b}, c$ are Borel measurable in $(t,x)$, $a^{ij} = a^{ji}$ and $a$ is uniformly continuous in $x$.

(ii) For any $t > 0, x \in \Omega$, and $\lambda \in \mathbb{R}^d$,

$$\delta_0|\lambda|^2 \le a^{ij}(t,x)\lambda^i\lambda^j \le K|\lambda|^2. \tag{3.11}$$

(iii) For any $t, x$,

$$\rho(x)|\bar{b}^i(t,x)| + \rho(x)|b^i(t,x)| + \rho^2(x)|c(t,x)| \le K \tag{3.12}$$

and there is a control on the behavior of $\bar{b}, b$ and $c$ near $\partial\Omega$, namely,

$$\lim_{\substack{\rho(x)\to 0\\ x\in\Omega}} \sup_t(\rho(x)|\bar{b}(t,x)| + \rho(x)|b(t,x)| + \rho^2(x)|c(t,x)|) = 0. \tag{3.13}$$

*Remark* 3.5. (3.12) and (3.13) allow the coefficients $\bar{b}^i, b^i$ and $c$ to be unbounded and to blow up near the boundary of $\Omega$. For instance, those conditions are satisfied if for some $\varepsilon, N > 0$

$$|\bar{b}(t,x)| + |b(t,x)| \le N\rho^{\varepsilon-1}(x), \quad |c(t,x)| \le N\rho^{\varepsilon-2}(x).$$

**Theorem 3.6.** *Let $\varepsilon > 0$ and $1 < p \le q < \infty$ and Assumption 3.4 be satisfied. Then for any $f^i \in \mathbb{L}^q_{p,\theta}(\Omega,T)$, $f \in \psi^{-1}\mathbb{H}^{-1,q}_{p,\theta}(\Omega,T)$ and*

$$u_0 \in \psi^{-2/q+1+\varepsilon}H^{1-2/q+\varepsilon}_{p,\theta}(\Omega),$$

*equation (1.2) with initial data $u_0$ has a unique solution $u \in \mathfrak{H}^{1,q}_{p,\theta}(\Omega,T)$, and for this solution*

$$\begin{aligned} &\|u\|_{\mathfrak{H}^{1,q}_{p,\theta}(\Omega,T)} \\ \le\; & N(\|\psi^{2/q-1-\varepsilon}u_0\|_{H^{1-2/q+\varepsilon}_{p,\theta}(\Omega)} + \|f^i\|_{\mathbb{L}^q_{p,\theta}(\Omega,T)} + \|\psi f\|_{\mathbb{H}^{-1,q}_{p,\theta}(\Omega,T)}), \end{aligned} \tag{3.14}$$

*where the constant $N$ is independent of $u$, $f$, and $u_0$.*

**Assumption 3.7.** For each $t > 0$,

$$|a(t,\cdot)|^{(0)}_{|\gamma|_+} + |b(t,\cdot)|^{(1)}_{|\gamma|_+} + |c(t,\cdot)|^{(2)}_{|\gamma|_+} \leq K. \tag{3.15}$$

**Theorem 3.8.** *Let $\gamma \in \mathbb{R}, \varepsilon > 0$, $1 < p \leq q < \infty$ and Assumptions* 3.4 *and* 3.7 *be satisfied. Then for any $f \in \psi^{-1}\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,T)$ and*

$$u_0 \in \psi^{-2/q+1+\varepsilon} H^{\gamma+2-2/q+\varepsilon}_{p,\theta}(\Omega),$$

*equation* (1.1) *with initial data $u_0$ admits a unique solution $u$ in the class $\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,T)$, and for this solution*

$$\|u\|_{\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,T)} \leq N(\|\psi^{2/q-1-\varepsilon}u_0\|_{H^{\gamma+2-2/q+\varepsilon}_{p,\theta}(\Omega)} + \|\psi f\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,T)}), \tag{3.16}$$

*where the constant $N$ is independent of $u$, $f$, and $u_0$.*

*Remark* 3.9. Various Hölder estimates of the solution $u \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,T)$ are investigated in [3]. For instance (see (4.10) and (4.17) there), if

$$2/q < \alpha < \beta \leq 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon$$

where $k = 0, 1, \ldots, \varepsilon \in (0,1]$. Then for $\nu := \beta - 1 + \theta/p$ and multi-indices $i$ and $j$ such that $|i| \leq k$ and $|j| = k$, we have

$$\sup_{t\neq s} |t-s|^{-(\alpha/2-1/q)}(|\psi^{\nu+|i|}D^i(u(t)-u(s))|_{C(\Omega)}$$
$$+ [\psi^{\nu+|j|+\varepsilon}D^j(u(t)-u(s))]_{C^\varepsilon(\Omega)}) < \infty.$$

In particular, if $\gamma = -1$, $\theta = d$ and $\kappa_0 := 1 - 2/q - d/p > 0$, then for any $\kappa \in (0, \kappa_0)$

$$\sup_{t\leq T}\sup_{x,y\in\Omega} \frac{|u(t,x)-u(t,y)|}{|x-y|^\kappa} + \sup_{x\in\Omega}\sup_{t,s\leq T} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\kappa/2}} < \infty.$$

Indeed, to estimate the second term, take $\beta = \kappa_0 - \kappa + 2/q$, then we have $\varepsilon = 1 - \beta - d/p = \kappa = -\nu$. For the first term in the above, take $\alpha = \kappa + 2/q$, and note that $2/q < \alpha < 1 - d/p < 1$ and $q\alpha/2 - 1 = q\kappa/2$.

## 4. Auxiliary results

In this section we develop some estimates for PDEs defined on $\mathbb{R}^d_+$. By $M^\alpha$ we denote the operator of multiplying $(x^1)^\alpha$ and $M = M^1$.

**Theorem 4.1.** *Let $1 < p \leq q < \infty, \gamma \in \mathbb{R}$ and $\varepsilon > 0$. Also let Assumption 3.4*(i)-(ii)*, Assumption 3.7 hold and*

$$|a^{ij}(t,x) - a^{ij}(t,y)| + x^1|b^i(t,x)| + (x^1)^2|c(t,x)| < \beta, \quad \forall t,x,y.$$

*Then there exists $\beta_0 > 0$ such that if $\beta \leq \beta_0$ then for any $f \in M^{-1}\mathbb{H}^{\gamma,q}_{p,\theta}(T)$ and $u_0 \in M^{-2/q+1+\varepsilon}H^{\gamma+2-2/q+\varepsilon}_{p,\theta}$, equation* (1.1) *with initial data $u_0$ has a unique solution $u \in M\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)$, and for this solution*

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \leq N\|Mf\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} + N\|M^{2/q-1-\varepsilon}u_0\|_{H^{\gamma+2-2/q+\varepsilon}_{p,\theta}}, \tag{4.1}$$

*where* $N = N(d,p,q,\gamma,\varepsilon,\delta_0,K)$.

*Proof.* By Theorem 3.2 in [4], the theorem is true if $a = (a^{ij})$ is independent of $x$ and $b^i = c = 0$. Thus we only need to show that there exists $\beta_0$ such that the estimate (4.1) holds given that a solution $u$ already exists and $\beta \le \beta_0$. As before, we also assume that $u_0 = 0$.

**Case 1.** $|\gamma| \notin \{1,2,\dots\}$. Fix $x_0 \in \mathbb{R}^d_+$ and denote $a_0(t,x) = a(t,x_0)$. Then $u$ satisfies

$$u_t = a_0^{ij} u_{x^ix^j} + \bar{f}, \quad \bar{f} := (a^{ij} - a_0^{ij})u_{x^ix^j} + b^i u_{x^i} + cu + f.$$

By Theorem 3.2 in [4],

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \le N\|M\bar{f}\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}. \tag{4.2}$$

Here we use (Lemma 3.3(ii) or Theorem 2.1 in [6])

$$\|Mu_{xx}\|_{H^\gamma_{p,\theta}} + \|u_x\|_{H^\gamma_{p,\theta}} \le N\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}} \tag{4.3}$$

and (Lemma 3.6 in [2])

$$\|au\|_{H^\gamma_{p,\theta}} \le N \sup_x |a|^s \|u\|_{H^\gamma_{p,\theta}}, \tag{4.4}$$

where $s = 1$ if $\gamma = 0$ and $s = 1 - |\gamma|/|\gamma|_+$ $(> 0)$, otherwise. Thus

$$\|M(a-a_0)u_{xx}\|_{H^\gamma_{p,\theta}} \le N\beta^s\|Mu_{xx}\|_{H^\gamma_{p,\theta}} \le N\beta^s\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}},$$

$$\|Mbu_x\|_{H^\gamma_{p,\theta}} + \|Mcu\|_{H^\gamma_{p,\theta}} \le N\beta^s\|u_x\|_{H^\gamma_{p,\theta}} + N\beta^s\|M^{-1}u\|_{H^\gamma_{p,\theta}}.$$

Consequently,

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \le N\beta^s\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} + N\|Mf\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}.$$

Now it suffices to take $\beta_0$ sufficiently small such that $N\beta_0^s \le 1/2$.

**Case 2.** $\gamma \in \{1,2,\dots\}$. Proceed as before, but instead of (4.4) we use (Lemma 3.6 in [2])

$$\|au\|_{H^\gamma_{p,\theta}} \le N\sup_x |a|\|u\|_{H^\gamma_{p,\theta}} + N|a|^{(0)}_\gamma\|u\|_{H^{\gamma-1}_{p,\theta}}.$$

By Theorem 2.6 in [7] for any $\varepsilon > 0$,

$$\|u\|_{H^{\gamma+1}_{p,\theta}} \le \varepsilon\|u\|_{H^{\gamma+2}_{p,\theta}} + N(\varepsilon)\|u\|_{H^2_{p,\theta}}.$$

Thus

$$\begin{aligned}\|M\bar{f}\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} &\le N\beta\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} + N\|M^{-1}u\|_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(T)} \\ &\quad + N\|Mf\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)} \\ &\le N(\beta+\varepsilon)\|M^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} + N\|M^{-1}u\|_{\mathbb{H}^{2,q}_{p,\theta}(T)} + \|Mf\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}.\end{aligned}$$

This and the result of Case 1 (when $\gamma = 0$) easily yield (4.1).

**Case 3.** $\gamma \in \{-1,-2,\dots\}$. We proceed by induction on $\gamma$ and assume that there exists $\beta_0 > 0$ such that the theorem holds for $\gamma + 1$ in place of $\gamma$. The possibility to start the induction from $\gamma = -1$ is justified in case 1.

Note that if (3.15) holds for $\gamma$ then it also holds for $\gamma+1$. Let $\beta \le \beta_0$. Then the operator $\mathcal{R}$ which maps $f \in M^{-1}\mathbb{H}^{\gamma+1,q}_{p,\theta}(T)$ into the solution $u \in M\mathbb{H}^{\gamma+3,q}_{p,\theta}(T)$ of equation (1.1) with zero initial data is well defined and bounded.

Take $f \in M^{-1}\mathbb{H}^{\gamma,q}_{p,\theta}(T)$, then by Corollary 2.12 of [6] we have the representation

$$f = \sum_{k=1}^{d} MD_k f^k,$$

where $f^k \in M^{-1}\mathbb{H}^{\gamma+1}_{p,\theta}$ and

$$\sum_{k=1}^{d} \|Mf^k\|_{\mathbb{H}^{\gamma+1}_{p,\theta}} \le N\|Mf\|_{\mathbb{H}^{\gamma}_{p,\theta}}. \tag{4.5}$$

Now define

$$w^k = \mathcal{R}f^k, k = 1,2,\dots,d, \quad v = \sum_{k=1}^{d} MD_k w^k.$$

Owing to the induction hypothesis, (4.3) and (4.5) we have

$$\begin{aligned}\|M^{-1}v\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} &\le \sum_k \|w^k_x\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)} \\ &\le N\sum_k \|M^{-1}w^k\|_{\mathbb{H}^{\gamma+3,q}_{p,\theta}(T)} \le N\|Mf\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(T)}.\end{aligned}$$

Furthermore,

$$v_t = a^{ij}v_{x^ix^j} + b^i v_{x^i} + cv + f + \bar{f},$$

with

$$\begin{aligned}M\bar{f} = &\, Mw^k_{x^ix^j}MD_k a^{ij} + w^k_{x^i}M^2D_k b^i + M^{-1}w^kM^3D_k c \\ &- 2a^{i1}Mw^k_{x^kx^i} - w^k_{x^k}Mb^1.\end{aligned}$$

In addition,

$$\begin{aligned}|MD_k a|_{|\gamma+1|} &= |MD_k a|_{|\gamma|-1} \le N|a|_{|\gamma|} \le NK, \\ |M^2D_k b|_{|\gamma+1|} &= |M^2D_k b|_{|\gamma|-1} \le N|b|^{(1)}_{|\gamma|},\end{aligned}$$

and similar estimates hold for $M^3D_k c$, $a$, and $Mb$. Hence from the construction of $w^k$, we infer that

$$\|M\bar{f}\|_{\mathbb{H}^{\gamma+1}_{p,\theta}} \le N\|Mf\|_{\mathbb{H}^{\gamma}_{p,\theta}}.$$

Now we define $\bar{u} = \mathcal{R}(\bar{f})$ and $\tilde{u} = v - \bar{u}$. Then $\tilde{u}$ belongs to $M\mathbb{H}^{\gamma+2,q}_{p,\theta}(T)$ and satisfies equation (1.1) and (4.1) follows from the above estimates. Finally by reducing $\beta_0$ if necessary (we are free to do this) we show that $u = \tilde{u}$. Since $p \le q$,

$$u, \tilde{u} \in M\mathbb{H}^{\gamma+2,p}_{p,\theta}(T).$$

It follows from Theorem 2.14 in [2] that $u = \tilde{u}$ (if $\beta_0$ sufficiently small). The theorem is proved. □

**Theorem 4.2.** *Let $1 < p \leq q < \infty$ and $\varepsilon > 0$. Also let Assumption 3.4*(i)-(ii) *hold and*

$$|a^{ij}(t,x) - a^{ij}(t,y)| + x^1|\bar{b}^i(t,x)| + x^1|b^i(t,x)| + (x^1)^2|c(t,x)| < \beta, \quad \forall t,x,y.$$

*Then there exists $\beta_1 > 0$ such that if $\beta \leq \beta_1$ then for any $f^i \in \mathbb{L}^q_{p,\theta}(T)$, $f \in M^{-1}\mathbb{H}^{-1,q}_{p,\theta}(T)$ and $u_0 \in M^{-2/q+1+\varepsilon}H^{1-2/q+\varepsilon}_{p,\theta}$, equation* (1.2) *with initial data $u_0$ has a unique solution $u \in M\mathbb{H}^{1,q}_{p,\theta}(T)$, and for this solution*

$$\begin{aligned} \|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)} &\leq N\|f^i\|_{\mathbb{L}^q_{p,\theta}(T)} + N\|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)} \\ &\quad + \|M^{2/q-1-\varepsilon}u_0\|_{H^{1-2/q+\varepsilon}_{p,\theta}}, \end{aligned} \tag{4.6}$$

*where $N = N(d,p,q,\gamma,\varepsilon,\delta_0,K)$.*

*Proof.* Fix $x_0 \in \mathbb{R}^d_+$, denote $a_0(t,x) = a(t,x_0)$ and

$$\bar{f} = D_i((a^{ij} - a^{ij}_0)u_{x^j} + \bar{b}^i u + f^i) + b^i u_{x^i} + cu + f.$$

Then by Theorem 3.2 in [4],

$$\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(T)} \leq M\|M\bar{f}\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)}.$$

Since $MD : H^\gamma_{p,\theta} \to H^{\gamma-1}_{p,\theta}$ is a bounded operator,

$$\|MD_i((a^{ij} - a^{ij}_0)u_{x^j})\|_{H^{-1}_{p,\theta}} \leq N\|(a^{ij} - a^{ij}_0)u_{x^j}\|_{L_{p,\theta}} \leq N\beta\|u_x\|_{L_{p,\theta}},$$

$$\|MD_i(\bar{b}^i u)\|_{H^{-1}_{p,\theta}} \leq N\|M\bar{b}^i M^{-1}u\|_{L_{p,\theta}} \leq N\beta\|M^{-1}u\|_{L_{p,\theta}},$$

$$\|Mb^i u_{x^i} + Mcu\|_{H^{-1}_{p,\theta}} \leq \|Mb^i u_{x^i}\|_{L_{p,\theta}} + \|M^2cM^{-1}u\|_{L_{p,\theta}} \leq N\beta\|M^{-1}u\|_{H^1_{p,\theta}}.$$

Thus,

$$\|M\bar{f}\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)} \leq N\beta\|M^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}} + N\|f^i\|_{\mathbb{L}^q_{p,\theta}(T)} + \|Mf\|_{\mathbb{H}^{-1,q}_{p,\theta}(T)}.$$

Obviously, this proves the theorem. □

## 5. Proof of Theorem 3.8

We repeat arguments in the proof of Theorem 2.10 in [2], where the theorem is proved when $p = q$. As in the proof of Theorem 2.10 in [2] we may assume that $\partial\Omega$ is infinitely differentiable. Indeed, there is $\varepsilon > 0$ and a $C^\infty$ diffeomorphism $\mu : \Omega_\varepsilon := \{x \in \Omega : \psi(x) > \varepsilon\} \to \Omega$ such that the mappings $\mu$ and $\mu^{-1}$ induce one-to-one linear bounded mappings from $H^\gamma_{p,\theta}(\Omega)$ onto $H^\gamma_{p,\theta}(\Omega_\varepsilon)$ and vice versa, and proving that the function $u \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,T)$ satisfies (1.1) and admits estimate (3.16) is equivalent to proving the function $\tilde{u} = u(\mu) \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega_\varepsilon,T)$ satisfies the corresponding equation in $\Omega_\varepsilon$, and admits the natural modification of estimate (3.16). Furthermore, the mappings $\mu$ and $\mu^{-1}$ preserve all the assumptions on the coefficients. Remember that $\psi$ is bounded and infinitely differentiable, and therefore $\Omega_\varepsilon \in C^\infty$.

Next we establish a priori estimate (3.16) given that a solution $u$ already exists. Let $x_0 \in \partial\Omega$ and $\Psi$ be the function in Assumption 3.1. As mentioned above, we may assume that $\Psi$ is infinitely differentiable with bounded derivatives.

Define $r = r_0/K_0$ and fix a smooth function $\eta \in C_0^\infty(B_r)$ such that $0 \le \eta \le 1$ and $\eta = 1$ in $B_{r/2}$. Observe that $\Psi(B_{r_0}(x_0))$ contains $B_r$. For $m = 1, 2, \ldots$, $t > 0$, $x \in \mathbb{R}^d_+$, introduce $\eta_m(x) = \eta(mx)$,

$$\hat{a}_m(t,x) := \tilde{a}\eta_m + (1-\eta_m)\tilde{a}(t,0), \quad \hat{b}_m := \tilde{b}\eta_m, \quad \hat{c}_m := \tilde{c}\eta_m,$$

where

$$\tilde{a}^{ij}(t,x) = \check{a}^{ij}(t,\Psi^{-1}(x)), \quad \check{a}^{ij} = a^{rs}\Psi^i_{x^r}\Psi^j_{x^s},$$
$$\tilde{b}^i(t,x) = \check{b}^i(t,\Psi^{-1}(x)), \quad \check{b}^i = a^{rs}\Psi^i_{x^r x^s} + b^r\Psi^i_{x^r}, \quad \tilde{c}(t,x) = c(t,\Psi^{-1}(x)).$$

One can easily check that there is a constant $\bar{K}$ independent of $x_0$ such that

$$\sup_{m\ge 1}\sup_t (|\hat{a}_m|^{(0)}_{|\gamma|_+} + |\hat{b}_m|^{(1)}_{|\gamma|_+} + |\hat{c}_m|^{(2)}_{|\gamma|_+}) \le \bar{K}.$$

Take $\beta_0$ from Theorem 4.1 corresponding to $\delta, p, q, K, \gamma$ and $\bar{K}$. By Assumption 3.4 one can easily find $m$ such that

$$|\hat{a}_m(t,x) - \hat{a}_m(t,y)| + x^1|\hat{b}_m(t,x)| + (x^1)^2|\hat{c}_m(t,x)| \le \beta_0, \forall t,x,y.$$

Now we fix $m$ and $\rho_0 < r_0$ such that

$$\Psi(B_{\rho_0}(x_0)) \subset B_{r/(2m)}.$$

Let $\xi$ be a smooth function with support in $B_{\rho_0}(x_0)$ and denote $v := (u\xi)(\Psi^{-1})$ and continue $v$ as zero in $\mathbb{R}^d_+ \setminus \Psi(B_{\rho_0}(x_0))$. Since $\eta_m = 1$ on $\Psi(B_{\rho_0}(x_0))$, the function $v$ satisfies

$$v_t = \hat{a}^{ij}_m v_{x^i x^j} + \hat{b}^i_m v_{x^i} + \hat{c}_m v + \hat{f},$$

where

$$\hat{f} = \tilde{f}(\Psi^{-1}), \quad \tilde{f} = -2a^{ij}u_{x^i}\xi_{x^j} - a^{ij}u\xi_{x^i x^j} - b^i u\xi_{x^i} + \xi f.$$

Next we observe that by Theorem 3.2 in [9] for any $\nu, \alpha \in \mathbb{R}$ and $h \in \psi^{-\alpha}H^\nu_{p,\theta}(\Omega)$ with support in $B_{\rho_0}(x_0)$

$$\|\psi^\alpha h\|_{H^\nu_{p,\theta}(\Omega)} \sim \|M^\alpha h(\Psi^{-1})\|_{H^\nu_{p,\theta}}. \tag{5.1}$$

Therefore by Theorem 4.1 we have, for any $t \le T$,

$$\|M^{-1}v\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(t)} \le N\|M\hat{f}\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(t)} + N\|M^{2/q-1-\varepsilon}u_0(\Psi^{-1})\xi\|_{H_p^{\gamma+2-2/q+\varepsilon}}.$$

By using (5.1) again we obtain

$$\begin{aligned}\|\psi^{-1}u\xi\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(\Omega,t)} \leq{}& N\|a\xi_x\psi u_x\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}\\ &+N\|a\xi_{xx}\psi u\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}\\ &+N\|\xi_x\psi bu\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}+N\|\xi\psi f\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}\\ &+N\|\psi^{2/q-1-\varepsilon}u_0\|_{H^{\gamma+2-2/q+\varepsilon}_p(\Omega)}.\end{aligned}$$

Also since

$$\sup_t(|a\xi_x|^{(0)}_{|\gamma|_+}+|\xi_x\psi b|^{(0)}_{|\gamma|_+}+|a\xi_{xx}\psi|^{(0)}_{|\gamma|_+})<\infty,$$

we conclude

$$\begin{aligned}\|\psi^{-1}u\xi\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(\Omega,t)} \leq{}& N\|\psi u_x\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}+N\|u\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}\\ &+N\|\psi f\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}+\|\psi^{2/q-1-\varepsilon}u_0\|_{H^{\gamma+2-2/q+\varepsilon}_p(\Omega)}.\end{aligned}$$

Observe that $\rho_0, m, K', N$ are independent of $x_0$. To estimate the norm $\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(G,t)}$, one introduces a partition of unity $\zeta_{(i)}$, $i=0,1,2,\dots,N_0$ such that $\zeta_{(0)}\in C_0^\infty(\Omega)$ and $\zeta_{(i)}\in C_0^\infty(B_{\rho_0}(x_i))$ for $i=1,2,\dots,N_0$, where $x_i\in\partial\Omega$.

Then one estimates $\|\psi^{-1}u\zeta_{(0)}\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(\Omega,t)}$ using Theorem 2.4 and the other norms as above. By summing up those estimates (this is possible since $\Omega$ is bounded and thus $N_0<\infty$) one gets

$$\begin{aligned}&\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2,q}_{p,\theta}(\Omega,t)}\\ \leq{}& N\|\psi u_x\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}+N\|u\|_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\Omega,t)}\\ &+N\|\psi f\|_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,t)}+N\|\psi^{2/q-1-\varepsilon}u_0\|_{H^{\gamma+2-2/q+\varepsilon}_p}.\end{aligned}\tag{5.2}$$

Furthermore, we know that

$$\|\psi u_x\|_{H^{\gamma}_{p,\theta}(\Omega)}\leq N\|u\|_{H^{\gamma+1}_{p,\theta}(\Omega)}.$$

Therefore (5.2) yields

$$\begin{aligned}&\|u\|^q_{\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,t)}\\ \leq{}& N\|u\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\Omega,t)}+N\|\psi f\|^q_{\mathbb{H}^{\gamma,q}_{p,\theta}(\Omega,T)}\\ &+N\|\psi^{2/q-1-\varepsilon}u_0\|_{H^{\gamma+2-2/q+\varepsilon}_p(\Omega)}.\end{aligned}\tag{5.3}$$

Now we use (Theorem 2.7 in [8])

$$\sup_{s\leq t}\|u(s)\|^p_{H^{\gamma+1}_{p,\theta}(\Omega)}\leq N\|u\|^p_{\mathfrak{H}^{\gamma+2,p}_{p,\theta}(\Omega,t)}$$

to get

$$\|u\|^q_{\mathbb{H}^{\gamma+1,q}_{p,\theta}(\Omega,t)}\leq N\int_0^t\|u\|^q_{\mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,s)}\,dt.$$

Actually in [8] there is a restriction that $p \geq 2$. But as mentioned before, in our (deterministic) case the inequality holds for all $p > 1$. Now (5.3) and Gronwall's inequality lead to (3.16).

The a priori estimate combined with the method of continuity show that it only remains to prove solvability in the case of the heat equation

$$u_t = \Delta u + f, \quad u(0,\cdot) = u_0.$$

Since $C_0^\infty(\Omega)$ is dense in $H^\nu_{p,\theta}(\Omega)$ for any $\nu$ and $\theta$, it suffices to concentrate on $u_0 \in C_0^\infty(\Omega)$. Then passing from $u$ to $u - u_0$ we see that we may assume that $u_0 = 0$. Again using the fact that $C_0^\infty(\Omega)$ is dense in the spaces $H^\gamma_{p,\theta}(\Omega)$ we easily convince ourselves that it suffices to only consider $f$'s which are bounded on $\Omega \times [0,T]$ along with each derivative in $(t,x)$ and vanish if $x$ is in a neighborhood of the boundary of $\Omega$. In that case it is well known that there exists a classical solution $u$ of the heat equation with zero boundary and initial data. It turns out (see [2]) that $\psi^{-1}u$ is infinitely differentiable and has bounded derivatives. It easily follows that $u \in \mathfrak{H}^{\gamma+2,q}_{p,\theta}(\Omega,T)$. The theorem is proved.

## 6. Proof of Theorem 3.6

Due to Theorem 3.8 we only need to establish estimate (3.14) assuming that a solution $u$ already exists and $u(0) = 0$. Also note that $D_i f^i \in \rho^{-1}\mathbb{H}^{-1,q}_{p,\theta}(\Omega,T)$ since

$$\|\rho D_i f^i\|_{H^{-1}_{p,\theta}(\Omega)} \leq N\|f^i\|_{L_{p,\theta}(\Omega)}.$$

If we replace $f$ with $f^i_{x^i} + f$ then we come to the situation with $f^i = 0$. Thus without loss of generality we assume that $f^i = 0$.

Let $x_0 \in \partial\Omega$. Take $\eta_m$ and $\hat{a}_m$ from the proof of Theorem 3.8. Define $\Phi^i_r := D_i(\Psi^i_{x^r}(\Psi^{-1}))(\Psi)$,

$$\hat{\bar{b}}_m = \tilde{\bar{b}}\eta_m, \quad \hat{b}_m = \tilde{b}\eta_m, \quad \hat{c}_m = \tilde{c}\eta_m$$

where

$$\tilde{\bar{b}}^i(t,x) = \bar{b}^r(t,\Psi^{-1}(x))\Psi^i_{x^r}(\Psi^{-1}(x)), \quad \tilde{b}(t,x) = \check{b}(t,\Psi^{-1}),$$
$$\check{b}^i = -a^{rs}\Psi^i_r\Phi^j_s + b^r\Phi^i_r, \quad \tilde{c}(t,x) = \check{c}(t,\Psi^{-1}(x)), \quad \check{c} = c - \bar{b}^r\Phi^i_r.$$

Then, one can easily find $m$ such that for each $t,x,y$,

$$|\hat{a}_m(t,x) - \hat{a}_m(t,y)| + x^1|\hat{\bar{b}}_m(t,x)| + x^1|\hat{b}_m(t,x)| + (x^1)^2|\hat{c}_m(t,x)| \leq \beta_1.$$

Now we fix $m$ and $\rho_1 < r_0$ such that $\Psi(B_{\rho_1}(x_0)) \subset B_{r/(2m)}$. Let $\xi$ be a smooth function with support in $B_{\rho_1}(x_0)$ and denote $v := (u\xi)(\Psi^{-1})$ and continue $v$ as zero in $\mathbb{R}^d_+ \setminus \Psi(B_{\rho_1}(x_0))$. Then,

$$v_t = D_i(\hat{a}^{ij}_m v_{x^j} + \hat{\bar{b}}^i_m v + \hat{f}^i) + \hat{b}^i_m v_{x^i} + \hat{c}_m v + \hat{f},$$

where

$$\hat{f}^i = \tilde{f}^i(\Psi^{-1}), \quad \tilde{f}^i = a^{ij}u\xi_{x^j},$$

$$\hat{f} = \tilde{f}(\Psi^{-1}), \quad \tilde{f} = f\xi + b^i u\xi_{x^i} - a^{ij}u_{x^j}\xi_{x^i} - \bar{b}^i u\xi_{x^i}.$$

By Theorem 4.2 and (5.1),

$$\|\rho^{-1}u\xi\|_{\mathbb{H}^{1,q}_{p,\theta}(\Omega,T)} \le N\|\tilde{f}^i\|_{\mathbb{L}^q_{p,\theta}(\Omega,T)} + N\|\psi\tilde{f}\|_{\mathbb{H}^{-1,q}_{p,\theta}(\Omega,T)}.$$

As in (2.13), one can show that for any $\varepsilon > 0$,

$$\|\psi a^{ij}u_{x^j}\xi_{x^i}\|_{H^{-1}_{p,\theta}(\Omega)} \le \varepsilon\|u_{x^j}\xi_{x^i}\|_{L_{p,\theta}(\Omega)} + N(\varepsilon)\|\psi u_{x^j}\xi_{x^i}\|_{H^{-1}_{p,\theta}(\Omega)}.$$

To estimate the norm $\|\psi^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(\Omega,t)}$, one introduces a partition of unity $\zeta_{(i)}$, $i = 0,1,2,\dots,N_1$ such that $\zeta_{(0)} \in C_0^\infty(\Omega)$ and $\zeta_{(i)} \in C_0^\infty(B_{\rho_1}(x_i))$ for $i = 1,2,\dots,N_1$, where $x_i \in \partial\Omega$. Then one estimates $\|\psi^{-1}u\zeta_{(0)}\|_{\mathbb{H}^{1,q}_{p,\theta}(\Omega,t)}$ using Theorem 2.2 and the other norms as above. By summing up those estimates one gets, for each $t \le T$,

$$\|\psi^{-1}u\|_{\mathbb{H}^{1,q}_{p,\theta}(\Omega,t)} \le N\varepsilon\|u_x\|_{\mathbb{L}^q_{p,\theta}(\Omega,t)} + N\|\psi u_x\|_{\mathbb{H}^{-1,q}_{p,\theta}(\Omega,t)} + N\|\psi f\|_{\mathbb{H}^{-1,q}_{p,\theta}(\Omega,t)}.$$

This yields (5.2) with $\gamma = -1$ and finishes the proof.

## References


[1] A. Benedek, A. P. Calderoń, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365.

[2] K. Kim and N. V. Krylov, *On the Sobolev space theory of parabolic and elliptic equations in $C^1$ domains*, SIAM J. Math. Anal. **36** (2004), no. 2, 618–642.

[3] N. V. Krylov, *Some properties of traces for stochastic and deterministic parabolic weighted Sobolev spaces*, J. Funct. Anal. **183** (2001), no. 1, 1–41.

[4] ———, *The heat equation in $L_q((0,T), L_p)$-spaces with weights*, SIAM J. Math. Anal. **32** (2001), no. 5, 1117–1141.

[5] ———, *An Analytic Approach to SPDEs*, Stochastic partial differential equations: six perspectives, 185–242, Math. Surveys Monogr., 64, Amer. Math. Soc., Providence, RI, 1999.

[6] ———, *Weighted Sobolev spaces and Laplace's equation and the heat equations in a half space*, Comm. Partial Differential Equations **24** (1999), no. 9-10, 1611–1653.

[7] ———, *Some properties of weighted Sobolev spaces in $R^d_+$*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **28** (1999), no. 4, 675–693.

[8] S. V. Lototksy, *Dirichlet problem for stochastic parabolic equations in smooth domains*, Stochastics and Stochastics Rep. **68** (1999), no. 1-2, 145–175.

[9] ———, *Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations*, Methods Appl. Anal. **7** (2000), no. 1, 195–204.

[10] P. Weidemaier, *On the sharp initial trace of functions with derivatives in $L_q(0,T;L_p(\Omega))$*, Boll. Un. Mat. Ital. B (7) **9** (1995), no. 2, 321–338.



Department of Mathematics
Korea University
Seoul 136-701, Korea
*E-mail address*: kyeonghun@korea.ac.kr