Revival of Chinese theory of equations in Joseon

—Around the Tianyuanshu—

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August 19–20, 2014
ICM 2014, COEX (Seoul)

Goal of this Presentation

- The history of the theory of equations in Joseon.
- The development of Tianyuanshu in later Joseon.
- Modern concepts and structural viewpoint of Park Yul.
- Completion and Extension by Hong Jeongha.

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2. Park Yul
3. Hong Jeongha and his Guiljib
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Background of Mathematics of Joseon

The JiuZhang SuanShu(九章算術) was not available in Joseon.
Other books were not available either except ⋯

Chinese Mathematics Books in Joseon

- Yang Hui SuanFa (1274–1275)
- SuanXue QiMeng (1299)
- XiangMing SuanFa (1373)
- Suanfa Tongzong (1592)

There was a revolution in the theory of equations in Song-Qin-Yuan Period, namely···

The 11–13$^{\text{th}}$ c. Theory of Equations in China

- Apparently divided into two steps.
- The contruction/representation method: Tianyuanshu
- The solution method: Kaifangshu
- Kaifangshu evolved from Shisuo Kaifangfa to Zengcheng Kaifangfa in 11–13$^{\text{th}}$ c.

But not much details had been brought into Joseon.

Mathematics of 15^{th} c. Joseon

We have The Annals of the Joseon Dynasty (UNESCO's Memory of the World Programme) which has the records that $\cdots$

- The Great King Sejong studied Suanxue Qimeng himself.
- The mathematicians of early 15^{th} c. could manipulate and solve the polynomial equations of degree 10.
- King Sejong gave order to create a calendar with Seoul as the primary meridian. (Took about 20 years. A probable reason for the intensive study of mathematics.)

No other records are left $\cdots$.

History of Later Joseon

There were two devastating foreign invasions in 15–16th c.

- Invasion of Japan (1592–1598) and of Qing (1636–1637).
- New calendar from China: Shixianli (時憲曆, 1645) by Johann Adam Schall von Bell.
- Reprinting of Suanxue Qimeng in 1660.
- 100 years of adoptation process of Shixianli. (on and off)
- Gradual reconstruction process in 17–18th c.

RE-development of mathmatics in Joseon

- Struggle to restore mathematics: Gyeong Seonjing.
- Publication of first Tianyuanshu book (1770) since 13^{th} c.: Park Yul (1621–1668).
- Early 18^{th} c. mathematicians: Im Jun, Choe Seokjeong.
- Wide and fluent use of Tianyuanshu and the completion of the theory of equation by Hong Jeongha (1684–?): Guiljib (1713, 1724).
- Hong Jeongha met delegates from Qing and discussed mathematics: The essence of Tianyuanshu and Kaifangshu of 13^{th} c. were not used in Qing.
- Hong Jeongha's influence in 19^{th} c. Joseon: Lee Sanghyeok and Nam Byeonggil.

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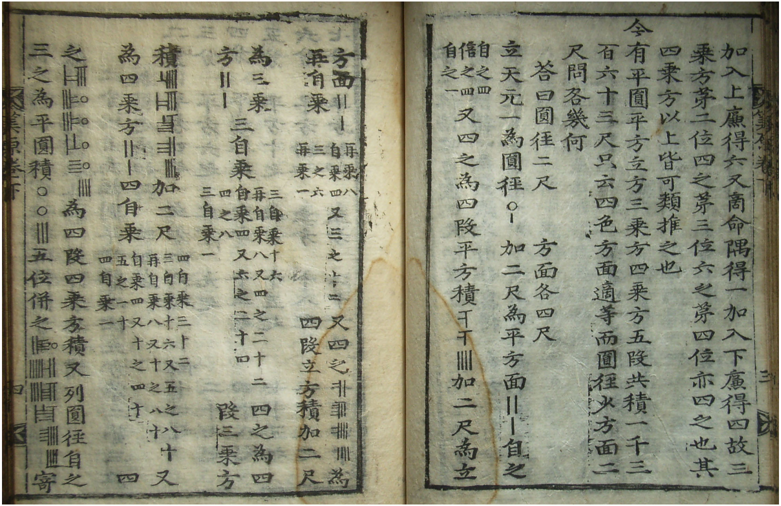
Park Yul's Work: Sanhak Wonbon (1700; mid 17th c.)

- He begins the first page with the quadratic equation and its linear interpolation formula from Biangu Tongyuan and its reverse formula: He considers this as the core of Kaifangshu.
- He recognized the Kaifangshu as the approximation to a precise solution: Compares the interpolation solution to the precise one.
- He clarifies the error: Corrects and gives a "possible proof" or ground to the statement of Yang Hui about this approximation.

- In the Tianyuanshu theory, he begins directly with polynomials of degree 5 without explaining degree 2 or 3 using squares or cubes: He was comfortable with the higher dimensional hypercubes or the abstract concept of x^5 in Tianyuanshu.
- He even interpreted the Cuifen problems as the problem of linear equations: a unified approach to mathematics.

1st Problem in Book of Tianyuanshu (vol. 3)



加入上廉得六又商命隅得一加入下廉得四故三
乘方第二位四之第三位六之第四位亦四之也其
四乘方以上皆可類推之也
今有平圓平方立方三乘方四乘方五段共積一千三
百六十三尺只云四色方面適等而圓徑少方面二
尺問各幾何
荅曰圓徑二尺 方面各四尺

Method of Biangu Tongyuan

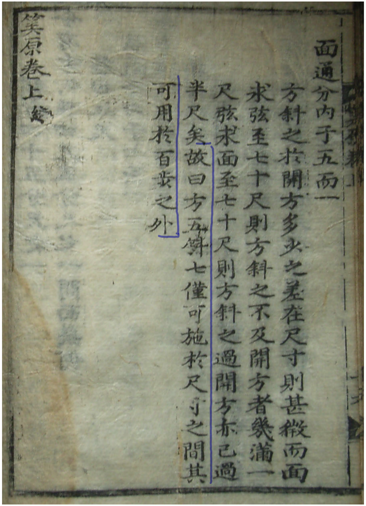
$x^2 - a = 0$ has interpolation formula $\sqrt{a} = \alpha + \frac{a-\alpha^2}{1+2\alpha}$. ($\alpha = [\sqrt{a}]$)

If $\sqrt{a} \approx \frac{n}{m}$, then the reverse formula says

$$a = \frac{n^2 + (m - (a - \alpha^2))(a - \alpha^2)}{m^2}.$$

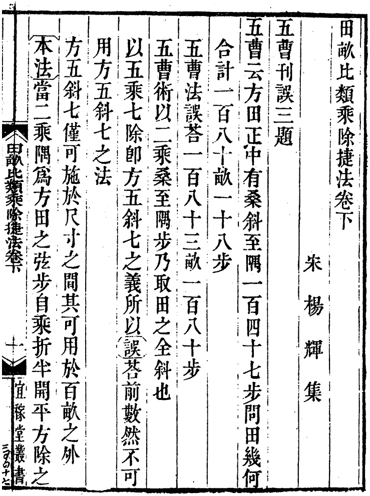
Park Yul's Error Analysis

- "Fangwu Xieqi (方五斜七) or approximation by $\sqrt{2} = 1.4$ has quite small error when dealing small sizes like a few feet or less. Therefore when computing the diagonal from the side of a square, one can use this method [with error less than 1 ft] as long as the side is no longer than 70 ft. Also when computing the side from the diagonal $\cdots$."
- He quotes the following from Yang Hui's *Tianmu Bilei Chengchu Jiefa*:
  "Therefore it is said [in Yang Hui] that Fangwu Xieqi method can be used when dealing with problems of feet or less, and can be used if it does not exceed 100 mu."
- He had the correct solution for $\sqrt{2}x - 1.4x \leq 1$ or equivalently $x^2 - 70x - 25 \leq 0$ as $x \leq 70$: Such computations are hardly seen in the E. Asia before the 18^{th} c.



筭原卷上

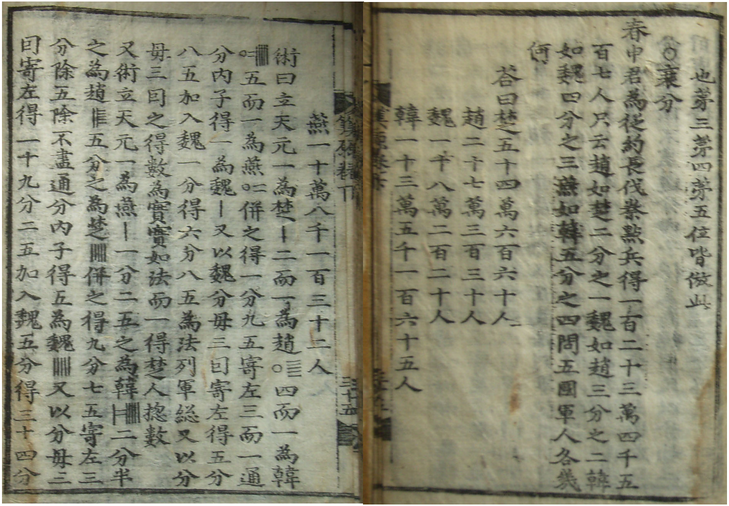
面通分內子五而一
方斜之於開方多少之差在尺寸則甚微而面
求弦至七十尺則方斜之不及開方者幾滿一
尺弦求面至七十尺則方斜之過開方亦已過
半尺矣故曰方五斜七僅可施於尺寸之間其
可用於百畝之外



田畝比類乘除捷法卷下
宋 楊輝 集
五曹刊誤三題
五曹云方田正中有桑斜至隅一百四十七步問田幾何
合計一百八十畝一十八步
五曹法誤荅一百八十三畝一百八十步
五曹術以二乘桑至隅步乃取田之全斜也
以五乘七除即方五斜七之義所以誤荅前數然不可
用方五斜七之法
方五斜七僅可施於尺寸之間其可用於百畝之外
本法當二乘隅為方田之弦步自乘折半開平方除之

田畝比類乘除捷法卷下 十一 宜稼堂叢書

Cuifen problem using Tianyuanshu



也第三第四第五位皆倣此
○衰分
春中君為從約長伐楚點兵得一百二十三萬四千五百七人只云趙如楚二分之一魏如趙三分之二韓如魏四分之三燕如韓五分之四問五國軍人各幾何
答曰楚五十四萬六百六十人
趙二十七萬三百三十人
魏一十八萬二百二十人
韓一十三萬五千一百六十五人
燕一十萬八千一百三十二人
術曰立天元一為楚二而一為趙○四而一為韓五而一為燕○併之得一分九五寄左三而一通分內子得一為魏又以魏分母三因寄左得五分八五加入魏一分得六分八五為法列軍総又以分母三因之得數為實實如法而一得楚人摠數
又術立天元一為燕一分二五之為韓二分半之為趙五分之為楚併之得九分七五寄左三分除五除不盡通分內子得五為魏又以分母三因寄左得一十九分二五加入魏五分得三十四分

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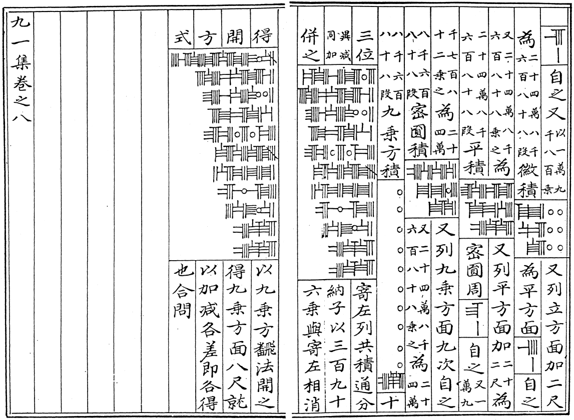
Hong Jeongha's Guiljib

- Hong used Tianyuanshu extensively and described Kaifangshu completely.
- Guiljib consists of 9 volumes and is divided into 21 chapters and an appendix.
- Guiljib deals with all the mathematics developed in China.
- It contains 166 problems which uses Tianyuanshu. (Suanxue Qimeng has 27.)
- Tianyuanshu problems are spread in 7 chapters, and among them one is about Gougushu (pythagorean methods) and three is about Kaifangshu.

Hong Jeongha's Tianyuanshu

- Tianyuanshu is used in the chapters GucheokHaeeunMun, BubyeongToetaMun, ChangdonJeoksokMun, GugoHoeunMun, GaebangGaksulMun (I, II, III).
- Starting from the first problem of of BubyeongToetaMun, the Shi (the constant term) has the negative sign: This means he used Tianyuanshu to manipulate polynomials.
- In several problems of ChangdonJeoksokMun, he chose one variable and then another as the Tianyuan and compared the two equations.
- He explained in several (complicated) problems to show how to manipulate Tianyuanshu.
- In Gougu problems he systematically used tianyuanshu to do rational polynomials. (involving terms of powers of $1/x$)

A Gougu Problem (No. 28)



The Tianyuanshu of Hong JeongHa from *Guiljib* (1724). The Tianyuan diagrams show how to get an equation of degree 10.

A Gougu Problem (No. 28)

Right △ with sides a, b, the hypothenuse c and $ab = 1080$, $c = a + 27$

Gou is $\boxed{\begin{array}{c}0(\text{太})\\1\end{array}}$, Gu is $\boxed{\begin{array}{c}1080\\0(\text{太})\end{array}}$, Xian is $\boxed{\begin{array}{c}27(\text{太})\\1\end{array}}$

And the Pythagorean theorem says

$$\boxed{\begin{array}{c}0(\text{太})\\1\end{array}}\ \text{squared} + \boxed{\begin{array}{c}1080\\0(\text{太})\end{array}}\ \text{squared} = \boxed{\begin{array}{c}27(\text{太})\\1\end{array}}\ \text{squared.}$$

$$(太)\begin{array}{|c|}\hline \\ \\ 0 \\ 0 \\ 1 \\ \hline\end{array} + \begin{array}{|c|}\hline (1080)^2 \\ 0 \\ 0 \\ \\ \\ \hline\end{array} = \begin{array}{|c|}\hline \\ 27^2 \\ 2\times 27 \\ 1 \\ \hline\end{array}$$

which becomes

$$\begin{array}{|c|}\hline -(1080)^2 \\ 0 \\ 27^2 \\ 2\times 27 \\ 0 \\ \hline\end{array}$$

Notice no base (太) position.

Associative law, Distributive law, Shenwai Jiafa

In the expansion of $(x+7)(x+8)(x+7.5)$
he first computed

$$(x+8)(x+7.5) = x^2 + (8+7.5)x + 8 \times 7.5 = x^2 + 15.5x + 60$$

Then used Shenwai Jiafa:

1	15.5	60	0
	7	108.5	420
1	22.5	168.5	420

Combined Multiplication

In the expansion of $(-x+32)(-x+33)(-x+32.5)$

Hong introduces the terminology YuSeung (維乘; Combined Multiplication) of the array 32, 33, 32.5 as:

$$32 \times 33, \quad 32 \times 32.5, \quad 33 \times 32.5$$

And their SUM.

Also the sum of the three numbers, etc. and found the expansion.

He realized the "elementary symmetric polynomials" when expanding product of up to 3 linear polynomials.

His method to understand the expansion was the formula:

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} = \frac{b_1(a_2a_3) + b_2(a_1a_3) + b_3(a_1a_2)}{a_1a_2a_3}$$

Especially, when $b_1 = b_2 = b_3 = 1$ the sum of combined multiplication appears in the numerator.
(This method does not apply to 4 factored expansion, and the synthetic expansion was devised.)

Hong's Synthetic Expansion Method

In his book *A History of Chinese Mathematics*, J.-C. Martzloff asked: *"Qin Jiushao was not the only one to use this (Horner's) method. ⋯ Until now no-one has been able to give a satisfactory explanation of this, ⋯"* (pp. 246–247)

- This question was answered by a hypothetical theory by Hong and Kim. (Liu Yi and Hong Jung Ha's Kaifangshu. HPM 2012 (Daejeon, Korea))
- Hong Jeongha's synthetic division method is discovered and gives a firm ground to our theory.

What did Hong Jeongha achieve?

- Hong Jeongha used the method of Tianyuanshu fluently and explained them in detail.
- Hong devised a clever method of expanding product of linear polynomials, which is not very well recognized even today.
- This method is an essential ingredient in both the Tianyuanshu (polynomial expansion) and the Kaifangshu (synthetic expansion).
- This viewpoint suggests that the development of 11–13$^{\text{th}}$ c. Chinese theory of equations might had been a single process: Hong shows the possibility of this explanation, and he himself achieved it.

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Conclusions

- In the field of equations, the 17^{th}–early 18^{th} c. Joseon mathematics shows both the hint of modern/structural thought and the completion of traditional methods.
- Park Yul's book is free from the tradition.
- Hong Jeongha formulated and explained the step-by-step reasoning of the construction and solution.
- They showed the possibilities lying in the east asian mathematics. (Hong's expansion method is unique in explaining the synthetic division process.)

What does this suggest?

- In the Chinese theory of equations, we understand its development of Tianyuanshu and Kaifangshu as a twofold process. But according to Hong Jeongha, Zengcheng Kaifangshu is nothing but a systematic use of (generalized) Shenwai Jiafa (身外加法).
- Shenwai Jiafa can be considered as a technic in Tianyuanshu and the essence of Kaifangshu is an Tianyuanshu.
- This suggests that possibly the original development in the 11–13$^{\text{th}}$ c. Song-Yuan theory was a single process: i.e. the Tianyuanshu and Kaifangshu were no two different things in their development.
- Even if there is almost no way to find this out, Hong showed this possibility.