Research Article

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Equity-linked security pricing and Greeks at arbitrary intermediate times using Brownian bridge

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Abstract: We develop a numerical algorithm for predicting prices and Greeks of equity-linked securities (ELS) with a knock-in barrier at any time over the total time period from issue date to maturity by using Monte Carlo simulation (MCS). The ELS is one of the most important financial derivatives in Korea. In the proposed algorithm, first we calculate the probability ($0 \le p \le 1$) that underlying asset price never hits the knock-in barrier up to the intermediate evaluation date. Second, we compute two option prices V_{nk} and V_k, where V_{nk} is the option value which knock-in event does not occur and V_k is the option value which knock-in event occurs. Finally, we predict the option value with a weighted average. We apply the proposed algorithm to two- and three-asset ELS. We provide the pseudo-numerical algorithm and computational results to demonstrate the usefulness of the proposed method.

Keywords: Equity-linked securities, Monte Carlo simulation, option pricing, Brownian bridge

MSC 2010: 60J70, 65C05, 91G60, 65C20

1 Introduction

The equity-linked security (ELS) is the security whose return on investment is dependent on the performance of the underlying equities linked to the securities [10, 11]. The investment on ELS has steadily increased due to the global financial crisis. Scale of annual issuance for this derivative is over half-trillion US dollars [11]. However, ELS has encountered critical crisis since last quarter of 2015 due to a significant decrease of Hang Seng index (HSI) of Hong Kong. Therefore, more detailed studies of ELS are needed to manage risks associated with ELS.

There are many kinds of options [4, 22] such as Asian option [3, 9, 14, 16], American option [1, 15, 21], barrier option [2, 23], European option [18] and lookback option [17, 20], etc. In [2], the authors applied the Heath-Platen (HP) estimator to calculate barrier options. Compared with naive Monte Carlo simulation (MCS), HP estimator provided a variance reduction. In [9], the authors applied Brownian bridge construction to compare with Niederreiter and Sobol' sequences for the convergence rates in Asian option. In [18], the authors applied quasi-Monte Carlo methods based on lattice sequences for multidimensional numerical integration to evaluate European options. Especially, an important feature of ELS is the auto-callable knock-out condition called early redemption and knock-in condition. For this feature, Fries and Joshi mentioned conditional analytic Monte Carlo pricing method of auto-callable products in [5, 8]. Glasserman and Staum introduced conditioning on one-step survival for barrier option simulations in [7].

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The operator splitting method (OSM) was used in pricing two-asset ELS in [10] and extended to three-asset ELS in [11, 12]. A method using exit-probability was proposed in [13]. MCS was used to evaluate the price of ELS in [10, 11]. However, all the previous computations were focused on pricing ELS at only current time. In this paper, we present a numerical algorithm for predicting prices and Greeks of ELS with a knock-in barrier at intermediate times, i.e., arbitrary evaluation times between present and expiration dates, from present time using MCS and Brownian bridge. The proposed algorithm has two steps. First, by using the Brownian bridge, we compute the probability ($0 \le p \le 1$), where p is the probability that underlying asset never hits the knock-in barrier up to the intermediate evaluation date. Second, we calculate two option prices V_{nk} and V_k under the conditions without and with the knock-in event, respectively. Finally, we define the option value with the weighted average, $V_w = pV_{nk} + (1-p)V_k$.

The paper is organized as follows. In Section 2, we present general information about step-down ELS. In Section 3, we describe main idea and algorithm of Brownian bridge for Monte Carlo simulation. In Sections 4, 5 and 6, we provide numerical results for ELS prices and Greeks for one-, two-, and three-asset, respectively. In Section 7, conclusion is given.

2 Step-down equity-linked security

We consider a step-down ELS which is the most important product among the equity-linked securities. Step-down ELS is a kind of structured product consisting of knock-in and knock-out conditions. We present one-asset ELS. Two- and three-asset ELS are similarly defined.

Let $K_1 \ge K_2 \ge \cdots \ge K_N$ and $c_1, c_2, \ldots, c_N$ be strike percentages of underlying asset and coupon rates at $t_1 < t_2 < \cdots < t_N$, respectively. As time goes by, strike percentages K_i are going down. Let $S(t)$ be the value of the underlying asset at time t and $V(S,t)$ be the value of ELS at price S, time t and $X(t) = 100\frac{S(t)}{S(0)}$. Then the payoff of one-asset step-down ELS is defined as follows: If $100\frac{S(t_1)}{S(0)} \ge K_1$, then the contract is closed with $(1+c_1)F$ return. Here, F is face value. Otherwise, i.e., $X(t_1) < K_1$, we check whether $X(t_2) \ge K_2$. If it is true, the contract is closed with $(1+c_2)F$ return. Otherwise, we proceed this process until $t = t_{N-1}$ and check whether $X(t_N) \ge K_N$. If it is true, the contract is closed with $(1+c_N)F$ return. If this condition is not satisfied, we check whether the underlying asset hits the knock-in barrier D over the total period from issue date to maturity t_N. That is, if $\min_{0 \le t \le t_N} X(t) \le D$, the return is $\frac{K_N F}{S(0)}$. If $\min_{0 \le t \le t_N} X(t) > D$, the return is $(1+d)F$, where d is a dummy rate. Payoff structure of step-down ELS is shown in Figure 1.

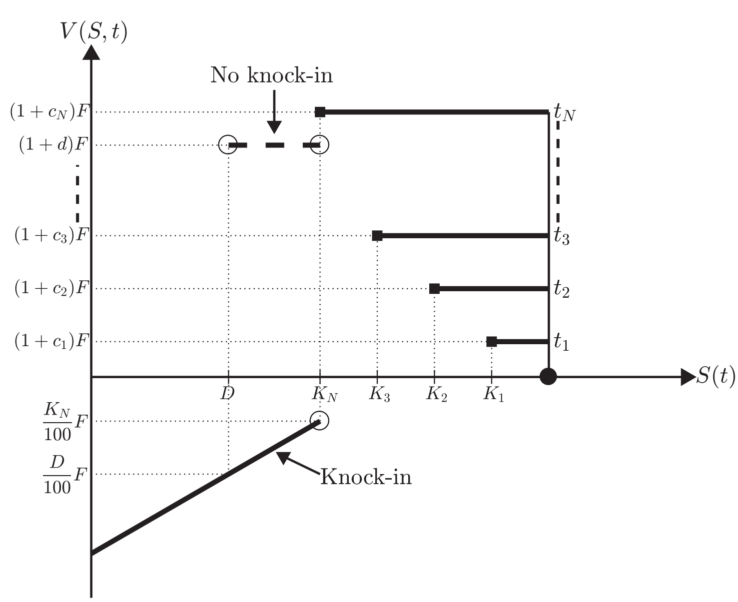

Figure 1: Payoff of step-down ELS at early redemptions and maturity.

3 Brownian bridge for Monte Carlo simulation

Now, we describe the proposed algorithm. Table 1 lists the parameter values used in this algorithm. Let $X(0) = 100$ be the reference price at $t = 0$. $F = 100$, knock-in barrier $D = 60$, dummy $d = 0.07$, risk-free interest rate $r = 0.03$, and volatility $\sigma = 0.3$ are used. The payoff structure for the one-asset step-down ELS is illustrated in Figure 2.

Early redemption date	$t_1 = 0.5$	$t_2 = 1$	$t_3 = 1.5$	$t_4 = 2$
Strike percentage	$K_1 = 95$	$K_2 = 90$	$K_3 = 85$	$K_4 = 80$
Coupon rate	$c_1 = 0.02$	$c_2 = 0.04$	$c_3 = 0.06$	$c_4 = 0.08$

Table 1: Parameter lists for the step-down ELS.

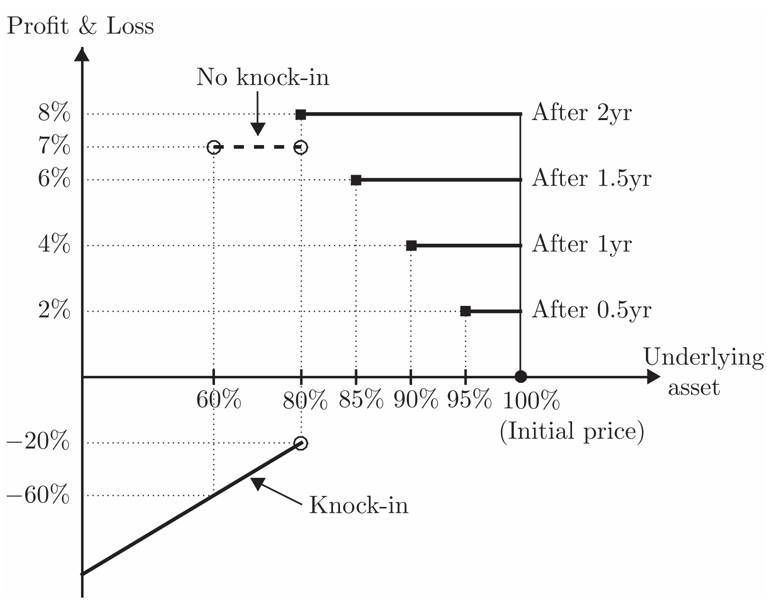

Figure 2: Payoff structure of the one-asset step-down ELS at early redemption and maturity.

Now, let us look at the seven possible cases of stock paths arising in the step-down ELS. Let $X^n := X(n\Delta t)$ denote the stock price at time $t = n\Delta t$, where Δt is time step size. Using standard Monte Carlo simulation, the stock path is defined as

$$X^{n+1} = X^n \exp((r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z^n) \quad \text{for } 0 \le n \le \frac{t_4}{\Delta t} - 1,$$

where $Z^n \in N(0, 1)$. With $r = 0.03$, $\sigma = 0.3$, and $\Delta t = \frac{1}{360}$, we can have the various path processes. We mark each case as a circled number as shown in Figure 3. Here ①–③ are early redemption cases at $t = t_1$, t_2, and t_3, respectively. Case ④ represents occurrence of obligatory redemption at the maturity. Case ⑤ illustrates the situation that the dummy is paid. The investors lose the value of their investment for the cases ⑥ and ⑦.

Algorithm 1 is the standard Monte Carlo simulation algorithm for one-asset ELS. Table 2 lists the computational results using 1. As we increase the number of MCS samples, it shows that price of ELS converges. Here, let # Sample be the number of samples.

Next, we consider the Brownian bridge procedure for Monte Carlo simulation. When we want more information between the two points, we can apply the Brownian bridge approach to generate a path connecting the specific two points. We define the standard Brownian bridge from 0 to 0 on $[0, T]$ to be the process

$$X(t) = W(t) - \frac{t}{T}W(T), \quad 0 \le t \le T,$$

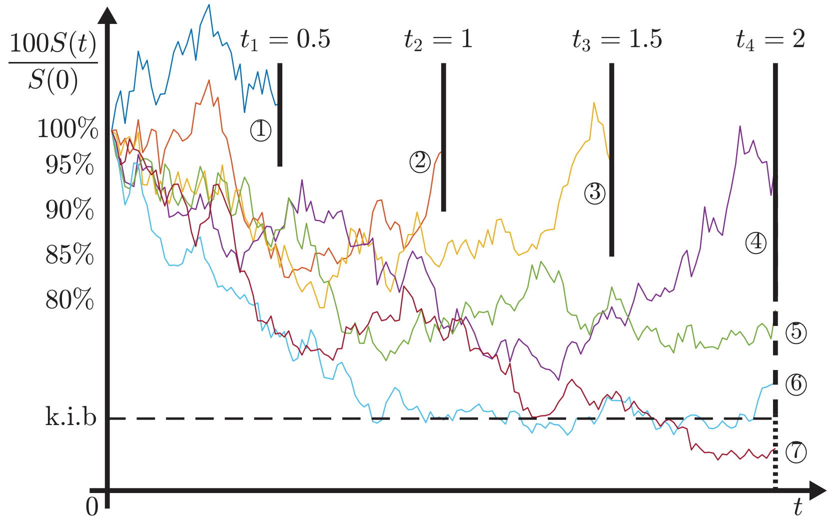

Figure 3: Seven sample random paths for the step-down ELS.

Algorithm 1. General Monte Carlo simulation algorithm for one-asset ELS.

Require: Set values for $X^0 = 100$, T, N_m, N_t, $\Delta t = T/N_t$, F, σ, r, t_j, c_j, K_j where $1 \le j \le 4$, d, and D.

Set $M_k = 0$ for $k = 1, 2, 3, 4$.
for $i = 1$ to N_m **do**
  for $n = 0$ to $N_t - 1$ **do**
    ▹ Generate process of security as
    $X^{n+1} = X^n \exp((r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z^n)$, where $Z^n \sim N(0, 1)$
  end for
  ▹ Check price of security at check dates
  if $X^{\frac{t_1}{\Delta t}} \ge K_1$ **then** $M_1 = M_1 + (1 + c_1)F$
  else if $X^{\frac{t_2}{\Delta t}} \ge K_2$ **then** $M_2 = M_2 + (1 + c_2)F$
  else if $X^{\frac{t_3}{\Delta t}} \ge K_3$ **then** $M_3 = M_3 + (1 + c_3)F$
  else if $X^{\frac{t_4}{\Delta t}} \ge K_4$ **then** $M_4 = M_4 + (1 + c_4)F$
  else if $\min\{X\} > D$ **then** $M_4 = M_4 + (1 + d)F$
  else $M_4 = M_4 + FX^{\frac{t_4}{\Delta t}}/S^0$
  end if
end for
▹ Compute the price.
$V(X^0, 0) = \sum_{k=1}^{4}(e^{-rt_k}M_k/N_m)$

#Sample	10^5	2×10^5	5×10^5	10^6
ELS Price	98.6821	98.6078	98.6666	98.6697

Table 2: ELS price with various samples.

here $W(t)$ is the Brownian motion and $W(0) = 0$. More generally, we define the Brownian bridge from a to b ($a, b \in \mathbb{R}$) on $[T_i, T_{i+1}]$ as the process

$$X^{a\to b}(t) = a + \frac{(b-a)(t-T_i)}{T_{i+1}-T_i} + W(t-T_i) - \frac{t-T_i}{T_{i+1}-T_i}W(T_{i+1}-T_i) \quad \text{for } T_i \le t \le T_{i+1}.$$

Let $X(T_i)$ and $X(T_{i+1})$ be the two given stock index values. We generate a path starting from $Y(T_i) = X(T_i)$ with the time step Δt:

$$Y(t_{j+1}) = Y(t_j)e^{w_j}, \quad j = 0, \dots, \frac{T_{i+1}-T_i}{\Delta t} - 1,$$

where $w_j = (r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z_j$ and $t_j = T_i + j\Delta t$. Let $W_j = \sum_{i=0}^{j} w_i$; then

$$Y(t_{j+1}) = Y(T_i)e^{W_j}, \quad j = 0, \ldots, \frac{T_{i+1} - T_i}{\Delta t} - 1.$$

In general, $Y(T_{i+1}) \neq X(T_{i+1})$. To construct a path connecting $X(T_i)$ and $X(T_{i+1})$, we apply the Brownian bridge to W_j. Let

$$B_j = W_j + \frac{t_j - T_i}{T_{i+1} - T_i} \log\frac{X(T_{i+1})}{Y(T_{i+1})}, \quad j = 0, \ldots, \frac{T_{i+1} - T_i}{\Delta t} - 1.$$

Then we obtain a full path connecting $X(T_i)$ and $X(T_{i+1})$ as

$$X(t_{j+1}) = X(T_i)e^{B_j}, \quad j = 0, \ldots, \frac{T_{i+1} - T_i}{\Delta t} - 1.$$

By using the Brownian bridge, the main idea of the proposed method is to calculate the ELS price with weighted average of probability weights p and $1 - p$, where p is the probability that the path never hits the knock-in barrier. Let $V_\omega(\hat{t}, \hat{X})$ be the weighted ELS time $\hat{t}$ and price at spot $\hat{X}$. The evaluation process is as follows.

Step 1: Calculate probability p. In this step, we calculate the probability p. First, we generate a sufficiently large number of paths (N_p). When the paths do not satisfy early redemption conditions, we regenerate the paths by using Brownian bridge until $t = \hat{t} = \hat{n}\Delta t$ at $X = \hat{X}$, see Figure 4. Among the paths, we choose the specific paths that passes from $(0, X^0)$ to $(\hat{t}, \hat{X})$.

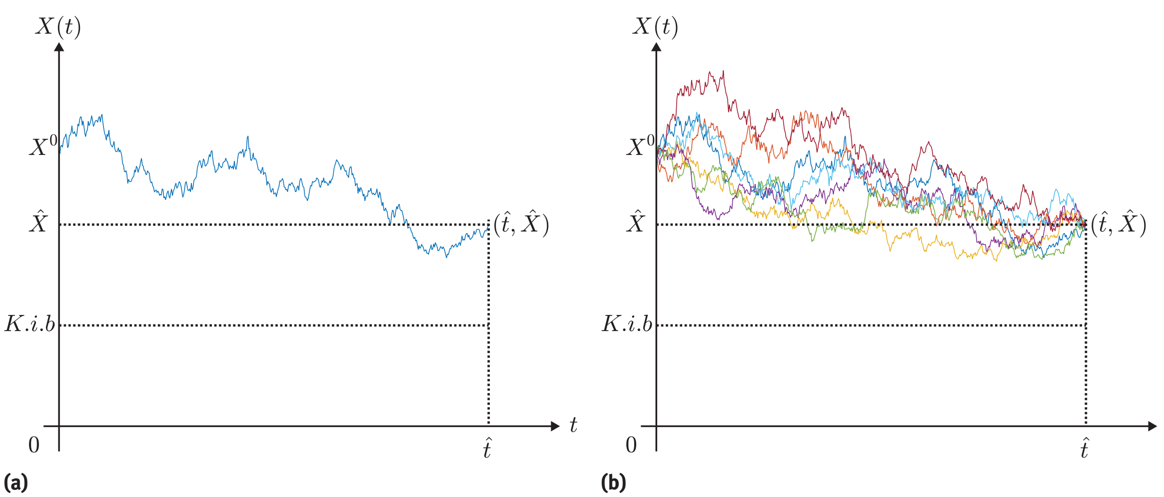

Figure 4: (a) Schematic illustration of a stock path by using Brownian bridge from $(0, X^0)$ to $(\hat{t}, \hat{X})$. (b) Generated sample paths up to time $t = \hat{t}$ of Brownian bridge.

Figure 5 shows a random path generated from $(0, X^0)$ to $(\hat{t}, \hat{X})$ and does not hit knock-in barrier. Let $\Omega_{\text{Samples}} = \{\omega_i : \omega_i^{\hat{n}} = \hat{X}\}$ be the set of the sample paths and $\Omega_{\text{not knock-in}} = \{\omega_i \in \Omega_{\text{Samples}} : D < \min_{0 \le n \le \hat{n}} \omega_i^n\}$ be the set of the paths which do not pass knock-in barrier. Next, we define the following probability:

$$p = \frac{\#\Omega_{\text{not-knock-in}}}{\#\Omega_{\text{Samples}}},$$

where # is the number of the elements. To find probability p, we perform the test that shows a effect of the number of samples on probability p. To do the test, we take $\hat{X} = 80$ unless otherwise stated in this paper.

From Figure 6, we can find that the probability converges when the number of samples is greater than 10^4. To ensure the accuracy, we set the number of samples to 10^5. Then, as shown in Figure 7, we perform the test to calculate the probability that knock-in event does not occur for each different $\hat{t}$ and $\hat{X}$.

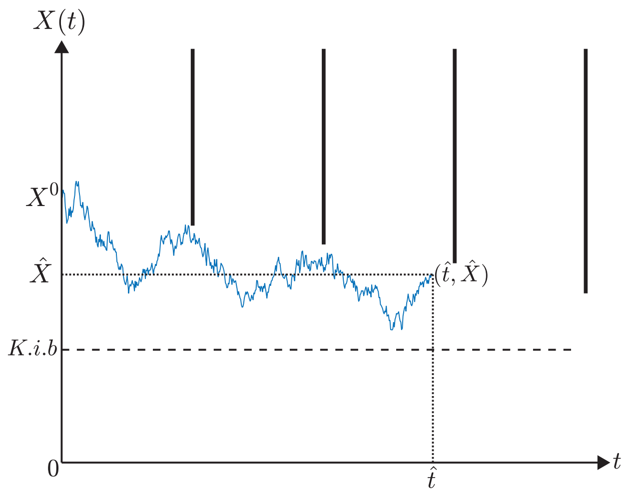

Figure 5: Schematic illustration of a Brownian bridge which does not hit the knock-in barrier.

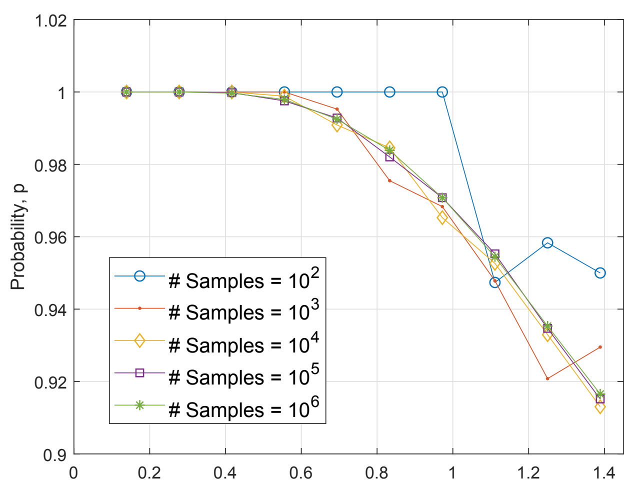

Figure 6: Probability p which does not hit knock-in barrier at different $\hat{t}$ with the different number of samples by using Brownian bridge process.

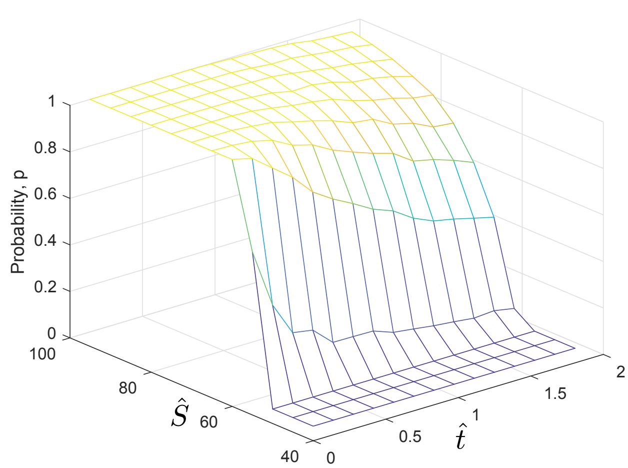

Figure 7: Probability p which does not hit knock-in barrier at different $\hat{t}$ and $\hat{X}$ by using Brownian bridge process.

Next, we compute the probability ($0 \le p \le 1$) which an underlying asset price never hits the knock-in barrier until time $t = \hat{t}$.

Step 2: Calculate V_{nk}, V_k and weighted average V_w. In this step, we calculate two option prices, V_{nk} and V_k, under the conditions without and with the knock-in event, respectively. Finally, we define the option value with a weighted average, $V_w = pV_{nk} + (1-p)V_k$. See Algorithm 2 for more detailed description.

Algorithm 2. Proposed Monte Carlo simulation algorithm for one-asset ELS.

Require: Set N_m, T, N_t, $\Delta t = T/N_t$, $\hat{t}$, $\hat{n}$, $X^0 = 100$, $\hat{X}(\hat{t})$ we will evaluate on, F, σ, r, t_j, and c_j, K_j where $1 \le j \le 4$, d, D, p.

Evaluating no knock-in price
Set $M_k = 0$ for $k = 1, 2, 3, 4$.
for $i = 1$ to N_m **do**
  for $n = \hat{n}$ to $N_t - 1$ **do**
    ▷ Generate process of security as
    $X^{n+1} = X^n \exp((r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z^n)$, where $Z^n \sim N(0,1)$
  end for
  ▷ Check price of security at check dates
  if $X^{\frac{t_j}{\Delta t}} \ge K_j$ **then** $M_j = M_j + (1 + c_j)F$ where the smallest $t_j/\Delta t > \hat{n}$
  else if $\vdots$ **then**
  else if $X^{\frac{t_4}{\Delta t}} \ge K_4$ **then** $M_4 = M_4 + (1 + c_4)F$
  else if $\min\{X\} \ge D$ **then** $M_4 = M_4 + (1 + d)F$
  else $M_4 = M_4 + FX^{\frac{t_4}{\Delta t}}/X^0$
  end if
end for
▷ Sum all results and discount to present value.
$V_{nk}(\hat{X}(\hat{t}), \hat{t}) = \sum_{k=1}^{4}(e^{-rt_k}M_k/N_m)$
Evaluating Knock-in price
Set $M_k = 0$ for $k = 1, 2, 3, 4$.
for $i = 1$ to N_m **do**
  for $n = \hat{n}$ to $N_t - 1$ **do**
    ▷ Generate process of security as
    $X^{n+1} = X^n \exp((r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z^n)$, where $Z^n \sim N(0,1)$
  end for
  ▷ Check price of security at check dates
  if $X^{\frac{t_j}{\Delta t}} \ge K_j$ **then** $M_j = M_j + (1 + c_j)F$ where the smallest $t_j/\Delta t > \hat{n}$
  else if $\vdots$ **then**
  else if $X^{\frac{t_4}{\Delta t}} \ge K_4$ **then** $M_4 = M_4 + (1 + c_4)F$
  else $M_4 = M_4 + FX^{\frac{t_4}{\Delta t}}/X^0$
  end if
end for
▷ Compute the option price.
$V_k(\hat{X}(\hat{t}), \hat{t}) = \sum_{k=1}^{4}(e^{-rt_k}M_k/N_m)$
▷ Probability
weighted price.
$V_w(\hat{X}(\hat{t}), \hat{t}) = pV_{nk}(\hat{X}(\hat{t}), \hat{t}) + (1-p)V_k(\hat{X}(\hat{t}), \hat{t})$

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4 One-asset step-down ELS

4.1 ELS pricing for one-asset step-down ELS

We consider the one-asset step-down ELS option pricing at future intermediate times. Suppose that we want to compute the option prices along a mean path which is the mean of stock processes:

$$w^n = \begin{cases} \text{mean}_{i=1}^{N_p}\{w_i^n\} & \text{if } n\Delta t \in [0, t_1), \\ \text{mean}_{i=1}^{N_p}\{w_i^n \chi_1(w_i^{t_1/\Delta t})\} & \text{if } n\Delta t \in [t_1, t_2), \\ \text{mean}_{i=1}^{N_p}\{w_i^n \chi_1(w_i^{t_1/\Delta t})\chi_2(w_i^{t_2/\Delta t})\} & \text{if } n\Delta t \in [t_2, t_3), \\ \text{mean}_{i=1}^{N_p}\{w_i^n \chi_1(w_i^{t_1/\Delta t})\chi_2(w_i^{t_2/\Delta t})\chi_3(w_i^{t_3/\Delta t})\} & \text{if } n\Delta t \in [t_3, t_4), \end{cases}$$

where

$$\chi_j(w_i^{t_j/\Delta t}) = \begin{cases} 0 & \text{if } w_i^{t_j/\Delta t} \geq K_j, \\ 1 & \text{otherwise,} \end{cases}$$

for $j = 1, 2, 3$.

Figures 8 (a), (b), and (c) represent the mean path, the probability along the mean path, and the weighted one-asset step-down ELS price, respectively. Here, we set initial price $X^0 = 100$, $\Delta t = \frac{1}{360}$, $r = 0.03$, $\sigma = 0.3$.

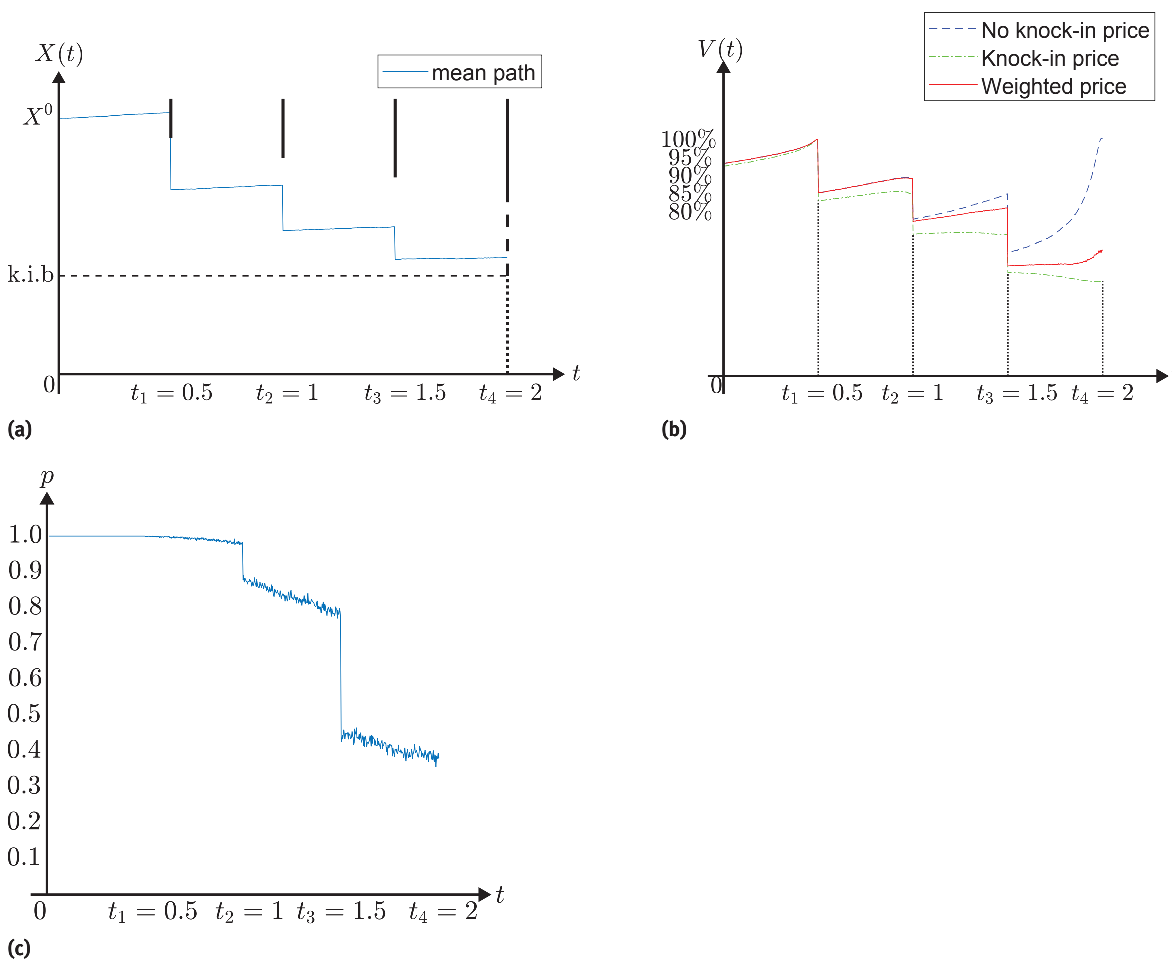

Figure 8: (a) Mean path, (b) no knock-in price (dashed line), knock-in price (dash-dotted line), weighted price, and (c) probability never hitting knock-in barrier (solid line).

4.2 Greeks for one-asset step-down ELS

In this subsection, we calculate the *delta* ($\Delta = \frac{\partial V^0}{\partial S}$) and *gamma* ($\Gamma = \frac{\partial^2 V^0}{\partial S^2}$) of the ELS. To compute Greeks, we apply the central finite difference method, i.e.,

$$\Delta \approx \frac{1}{2\Delta S}[V^0(S+\Delta S) - V^0(S-\Delta S)] \quad \text{and} \quad \Gamma \approx \frac{V^0(S-\Delta S) - 2V^0(S) + V^0(S+\Delta S)}{\Delta S^2},$$

where V^0 is the ELS price, S is the underlying asset, and $\Delta S = 1$. As stock price increases, the values of the *delta* and *gamma* converge. Figures 9 (a), (b), and (c) represent price, *delta* and *gamma* of one-asset step-down ELS, respectively.

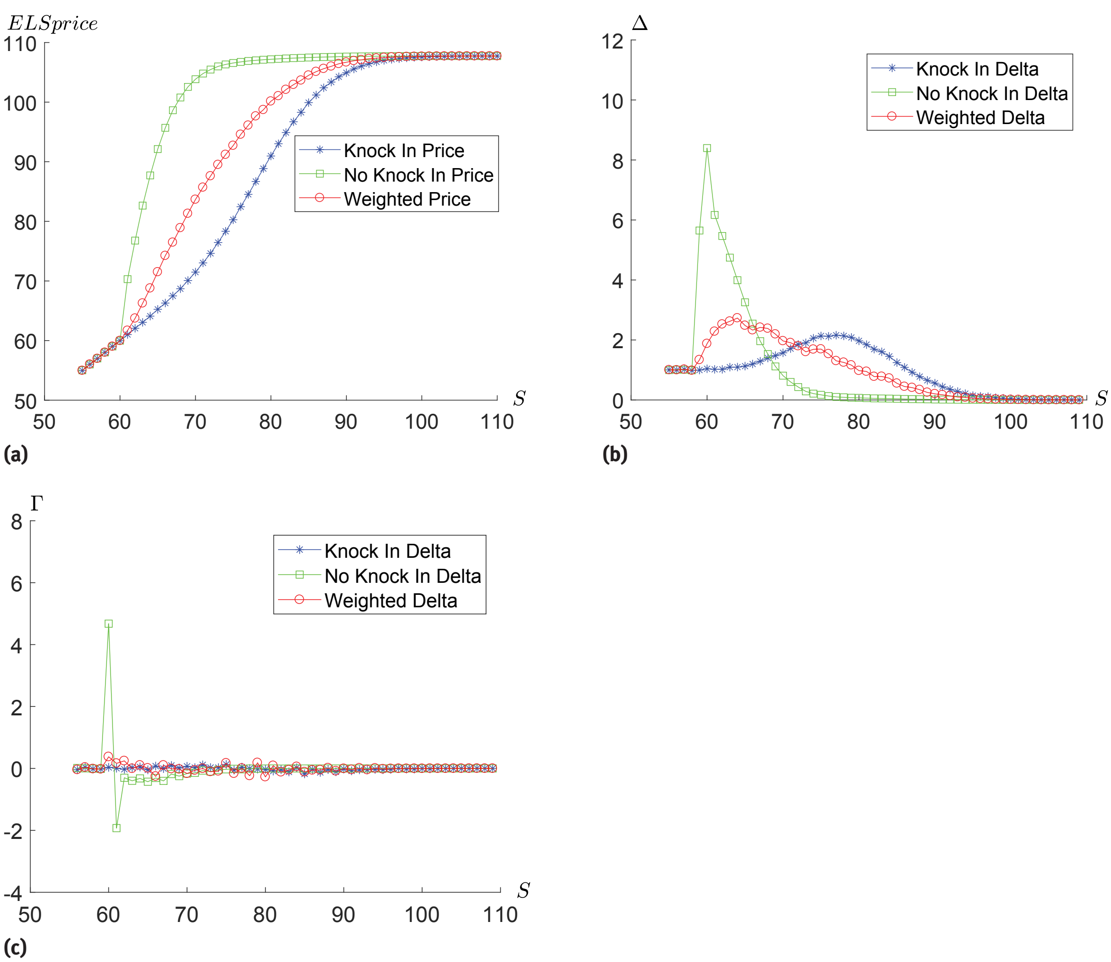

Figure 9: (a) ELS price at maturity before a month. (b) *Delta* of ELS at maturity before a month performer the stock process of process under condition and each conditioned price. (c) *Gamma* of ELS at maturity before a month performer the stock process of process under condition and each conditioned price.

5 Two-asset step-down ELS

5.1 ELS pricing for two-asset step-down ELS

We consider the pricing of a two-asset step-down ELS. Let X_1 and X_2 be the first and the second underlying stock process, respectively. σ_1 and σ_2 are the corresponding volatilities of each process. Most ELS products traded in South Korea consist of two- or three-asset. Thus, we need to generate correlated normal random numbers. Let ρ_{12} denote coefficient correlation between underlying two processes. In this paper, we use Cholesky decomposition [6, 19] to generate correlated normal random numbers:

$$X_1^{n+1} = X_1^n \exp((r - 0.5\sigma_1^2)\Delta t + \sigma_1 \sqrt{\Delta t} Z_1^n),$$
$$X_2^{n+1} = X_2^n \exp((r - 0.5\sigma_2^2)\Delta t + \sigma_2 \sqrt{\Delta t} Z_2^n),$$

where $Z_1^n = W_1^n$, $Z_2^n = \rho_{12} W_1^n + \sqrt{1 - \rho_{12}^2} W_2^n$ and W_1^n, $W_2^n \sim N(0, 1)$. The payoff of two-asset step-down ELS is determined by the worst performer of the two underlying assets. Figure 10 shows example of the worst performer of the two underlying assets. Let WP^n denote $\min\{X_1^n, X_2^n\}$. Figure 11 (a) shows two cases of paths at the first redemption time: The solid line and dash-dotted line mean that there is no early redemption and early redemption, respectively. Figure 11 (b) represents Brownian bridge path which does not hit knock-in barrier of $(\hat{X}_1, \hat{X}_2)$ at time $\hat{t}$.

Figure 12 (a) shows the mean path of two stock processes until $t = 2$. Here, we set the initial price $(X_1(0), X_2(0)) = (100, 100)$, $\Delta t = \frac{1}{360}$, $r = 0.03$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, and $\rho_{12} = 0.5$. Figure 12 (b) and (c) rep-

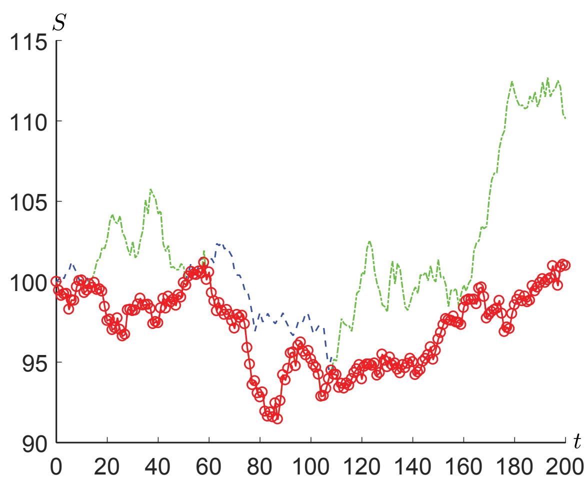

Figure 10: The circle-dash line is the worst performer of two underlying assets. Dotted line and dashed-dotted line are underlying asset 1 and 2, respectively.

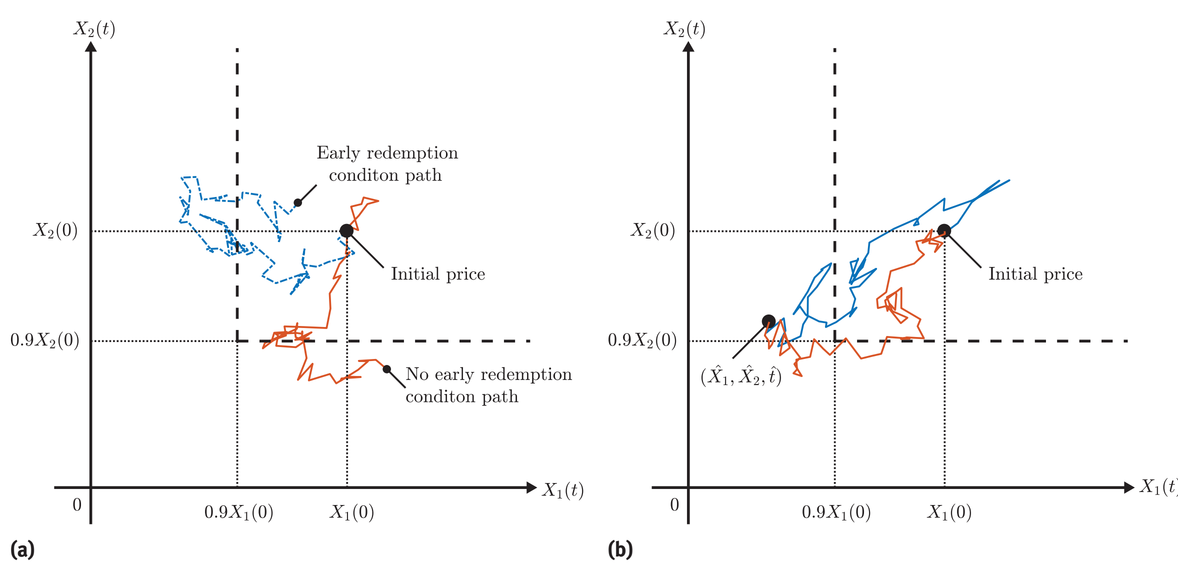

Figure 11: (a) Two cases of paths at the first redemption time: Solid line and dash-dotted line mean that there is no early redemption and early redemption, respectively. (b) Not hitting knock-in barrier Brownian bridge path of $(\hat{X}_1, \hat{X}_2)$ at time $\hat{t}$.

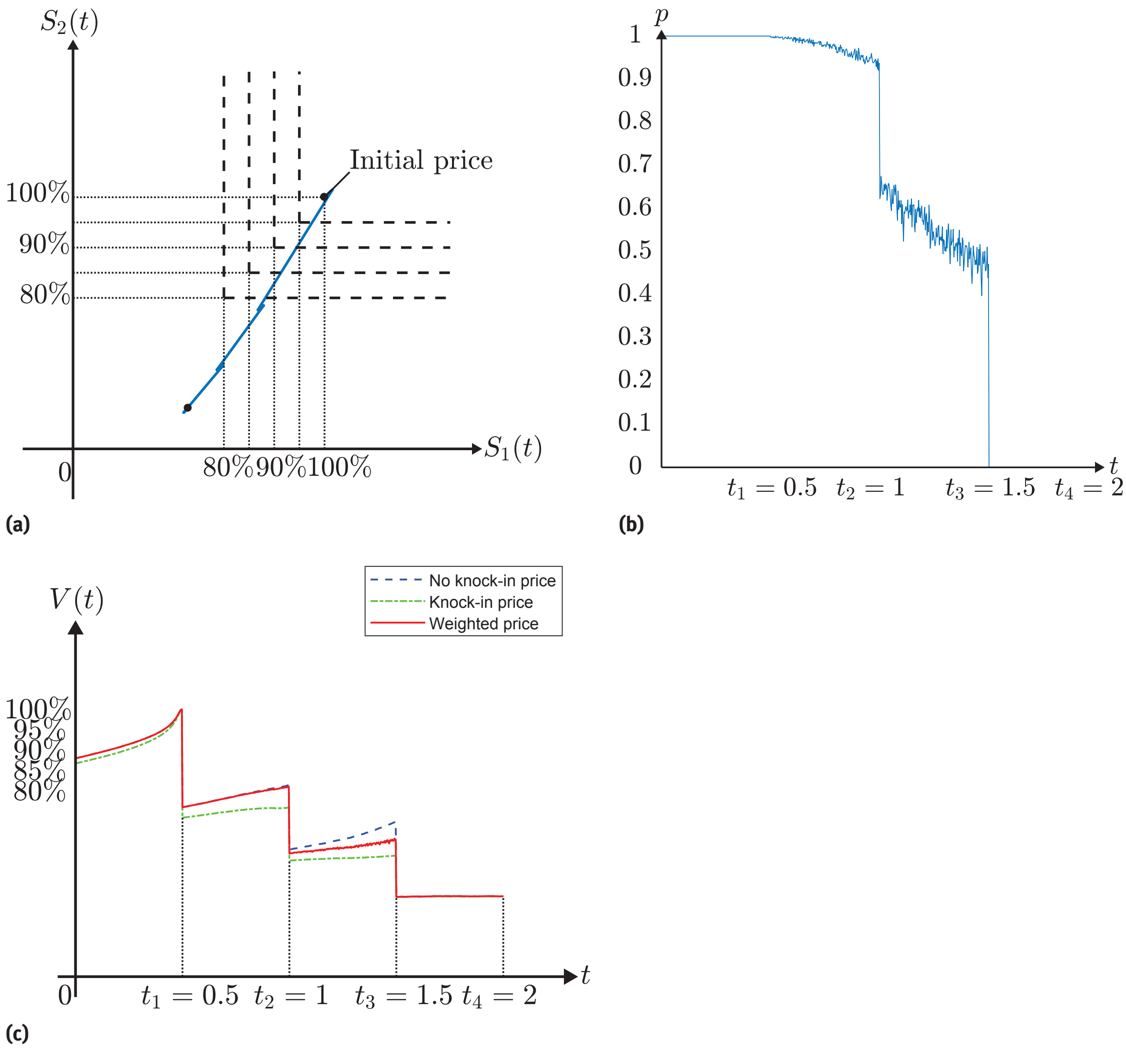

Figure 12: (a) Mean path of two-asset, (b) probability never hitting knock-in barrier of two-asset, and (c) no knock-in price of two-asset (dashed line), knock-in price of two-asset (dash-dotted line), weighted price of two-asset (solid line).

resent the probability along the mean path of two-assets and the weighted two-assets step-down ELS price, respectively. In Figure 12 (b), probability p is 0 after $t = 1.5$ because mean path of $X_2(t)$ is under knock-in barrier. Thus, Knock-in price remains only.

5.2 Greeks for two-asset step-down ELS

In this section, we calculate the *delta* ($\Delta = \frac{\partial V^0(WP)}{\partial S}$) and *gamma* ($\Gamma = \frac{\partial^2 V^0(WP)}{\partial S^2}$) of the ELS. To compute Greeks, we apply the central finite difference method, i.e.,

$$\Delta \approx \frac{V^0(WP + \Delta_{WP}) - V^0(WP - \Delta_{WP})}{2\Delta S}$$

and

$$\Gamma \approx \frac{V^0(WP - \Delta_{WP}) - 2V^0(WP) + V^0(WP + \Delta_{WP})}{\Delta S^2},$$

$\Delta S = 1$. As stock price increases, the values of the *delta* and *gamma* converge. Figures 13 (a), (b), and (c) represent price, *delta* and *gamma* of two-asset step-down ELS, respectively.

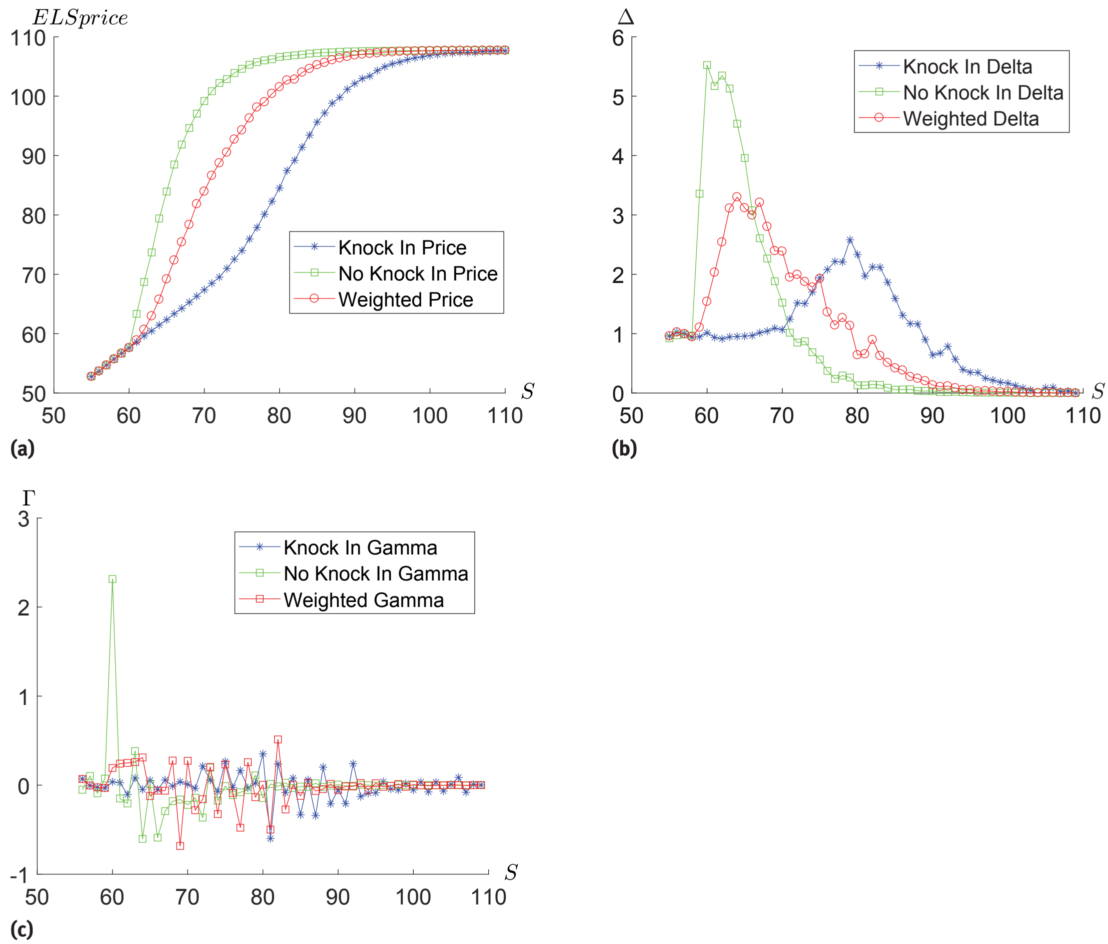

Figure 13: (a) ELS price at maturity before a month. (b) *Delta* of ELS at maturity before a month performer of two sample paths process under condition and each conditioned price. (c) *Gamma* of ELS at maturity before a month performer of two stock processes under condition and each conditioned price.

6 Three-asset step-down ELS

6.1 ELS pricing for three-asset step-down ELS

Let X_1, X_2, and X_3 be the first, second, and third underlying stock process, respectively. The payoff structure of three-asset step-down ELS is similar to that of the two-asset step-down ELS. Stock processes are as follows:

$$X_1^{n+1} = X_1^n \exp((r - 0.5\sigma_1^2)\Delta t + \sigma_1 \sqrt{\Delta t} Z_1^n),$$
$$X_2^{n+1} = X_2^n \exp((r - 0.5\sigma_2^2)\Delta t + \sigma_2 \sqrt{\Delta t} Z_2^n),$$
$$X_3^{n+1} = X_3^n \exp((r - 0.5\sigma_3^2)\Delta t + \sigma_3 \sqrt{\Delta t} Z_3^n),$$

where

$$Z_1^n = W_1^n, \quad Z_2^n = \rho_{12} W_1^n + \sqrt{1 - \rho_{12}^2} W_2^n,$$
$$Z_3^n = \rho_{13} W_1^n + \frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1 - \rho_{12}^2}} W_2^n + \sqrt{\frac{1 + 2\rho_{23}\rho_{12}\rho_{13} - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2}{1 - \rho_{12}^2}} W_3^n$$

and $W_1^n,\ W_2^n,\ W_3^n \sim N(0, 1)$. Figure 14 shows example of the worst performer of underlying three stock processes. Three-dimensional Cholesky decomposition [6, 19] is used to generate correlated normal random numbers. ρ_{ij} represents correlation coefficient of underlying for ith and jth stock processes, $1 \le i, j \le 3$.

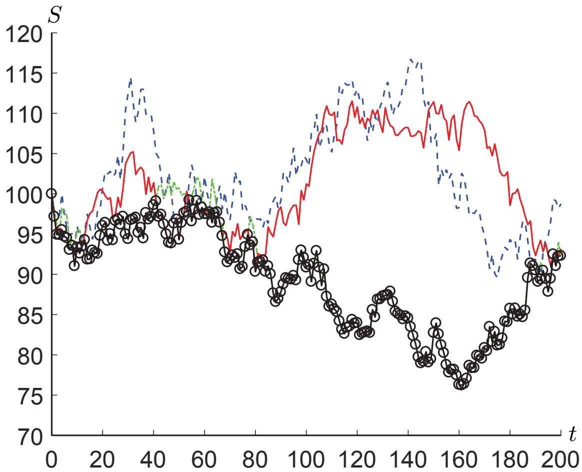

Figure 14: The circle-dash line is the worst performer of three underlying assets. Dotted line, dash-dot line and dash line are the first, second and third underlying asset respectively.

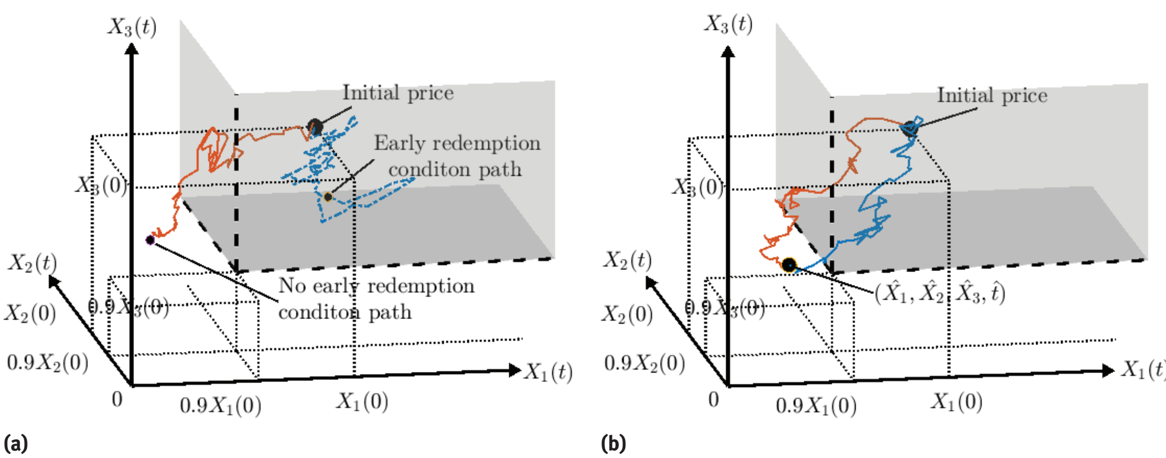

Figure 15: (a) Two cases of paths at the first redemption time: Solid line and dash-dotted line mean that there is no early redemption and early redemption, respectively. (b) Brownian bridge paths of $(\hat{X}_1, \hat{X}_2, \hat{X}_3)$ at time $\hat{t}$.

In Figure 15 (a), movement of three processes for initial price $(X_1(0), X_2(0), X_3(0)) = (100, 100, 100)$ is represented. Similar to the previous cases, we need to calculate the conditional probabilities. Figure 15 (b) represents Brownian bridge paths of $(\hat{X}_1, \hat{X}_2, \hat{X}_3)$ at time $\hat{t}$. As we did in previous section, we are going to deal with the mean of three stock processes and prices which are knock-in, no knock-in, and weighted cases onto mean of stock process.

We set $\Delta t = \frac{1}{360}$, $r = 0.03$, $\sigma_1 = 0.2$, $\sigma_2 = 0.4$, $\sigma_3 = 0.6$, $\rho_{12} = 0.5$, $\rho_{13} = 0.5$, $\rho_{23} = 0.5$. Figure 16 (a), (b) and (c) show the mean path of three stock processes until $t = 2$, the probability along the mean path of three-asset and the weighted three-asset step-down ELS price, respectively. Results are almost similar to case of two-asset ELS.

6.2 Greeks for three-asset step-down ELS

In this section, we calculate the *delta* ($\Delta = \frac{\partial V^0(WP)}{\partial S}$) and *gamma* ($\Gamma = \frac{\partial^2 V^0(WP)}{\partial S^2}$) of the ELS. To compute Greeks, we apply the central finite difference method, i.e.,

$$\Delta \approx \frac{V^0(WP + \Delta_{WP}) - V^0(WP - \Delta_{WP})}{2\Delta S} \quad \text{and} \quad \Gamma \approx \frac{V^0(WP - \Delta_{WP}) - 2V^0(WP) + V^0(WP + \Delta_{WP})}{\Delta S^2},$$

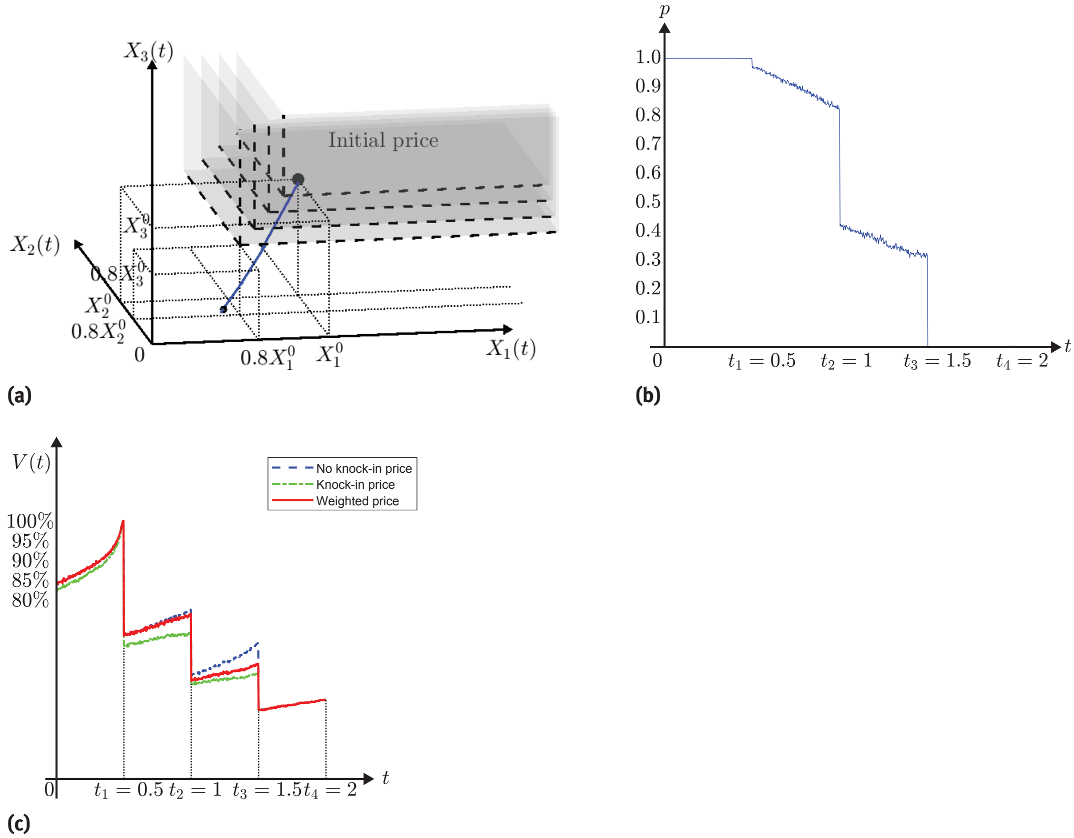

Figure 16: (a) Mean path of three-asset, (b) probability hitting knock-in barrier of three-asset, and (c) no knock-in price of three-assets (dashed line), knock-in price of three-asset (dash-dotted line), weighted price of three-asset (solid line).

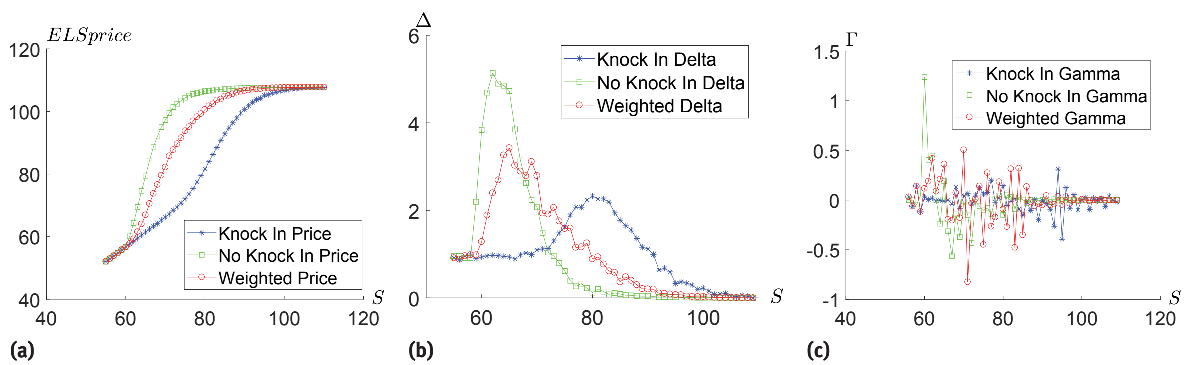

Figure 17: (a) ELS price at maturity before a month. (b) *Delta* of ELS at maturity before a month performer of three stock processes under condition and each conditioned price. (c) *Gamma* of ELS at maturity before a month performer of three stock processes under condition and each conditioned price.

$\Delta S = 1$. As stock price increases, the values of the *delta* and *gamma* converge. Figures 17 (a), (b), and (c) represent price, *delta* and *gamma* of three-asset step-down ELS, respectively.

7 Conclusion

In this article, we proposed numerical algorithm for predicting prices and Greeks of ELS with Monte Carlo simulation by using Brownian bridge at intermediate time $\hat{t}$. In the proposed algorithm, we generated the large number of sample paths and calculated probability p and two option prices V_{nk} and V_k. And we predicted the option value with weighted average $V_w = pV_{nk} + (1-p)V_k$. We presented the detailed algorithm in Section 3. Numerical experiments demonstrated that as the number of samples increases, the probability p converges. In the similar way, as stock price increases, *delta* and *gamma* converge.

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