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FAST PRICING OF FOUR ASSET EQUITY-LINKED SECURITIES USING BROWNIAN BRIDGE

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ABSTRACT. In this study, we present a fast option pricing method for four asset equity-linked securities (ELS) using Brownian bridge. The proposed method is based on Monte Carlo simulation (MCS) and a Brownian bridge approach. Currently, three asset ELS is the most popular ELS among multi-asset ELSs. However, four asset ELS emerged as an alternative to three asset ELS under low interest rate environment to give higher coupon rate to investors. We describe in detail the computational solution algorithm for the four underlying asset step-down ELS. The numerical tests confirm the accuracy and speed of the method.

1. INTRODUCTION

In this article, we present a fast option pricing method for four asset equity-linked securities (ELS) using Brownian bridge technique. The proposed method is based on Monte Carlo simulation (MCS) and a Brownian bridge approach. MCS is one of the most important techniques in quantitative finance and this method can be utilized as a simple and powerful alternative tool for option pricing. Fu and Hu [1] introduced techniques for the sensitivity analysis of option pricing, which provided an estimate of the option value and estimates of the sensitivities of the option value to various parameters. Duffie and Glynn [2] provided an asymptotically efficient algorithm for the allocation of computing resources to the problem of continuous-time security prices. Chidambaran [3] used MCS to generate stock and option price data needed to develop a genetic option pricing program. Ballotta and Kyriakou [4] presented a joint Monte Carlo-Fourier transform sampling scheme for pricing derivative products under a Carr–Geman–Madan–Yor model exhibiting jumps of infinite activity and finite or infinite variation. Abbas–Turki et al. [5] researched the pricing for European and American options on

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graphics processing units using MCS. They reduced the computational energy consumed and showed faster performance compared to CPU operation. In addition, there are various option pricing studies using MCS [6, 7, 8, 9, 10, 11].

Since the advent of ELS in 2002, the types of ELS have evolved in various ways. For example, knock-out type (either single-directional and bi-directional), bull-spread type, digital type, reverse-convertible type and step-down type with knock-in and no knock-in have emerged as investment circumstance changes over time. Because step-down ELS is the most frequently issued type whether knock-in barrier is embedded or not, we are focusing on step-down type. One of the key features of step-down type of ELS is that it is path dependent option like structured derivatives. Early redemption condition is Bermudan like option and knock-in barrier condition is American like option in terms of frequency of checking days. Early redemption condition requires to check only number of days already designated at issuing date, whereas knock-in barrier condition requires to check daily basis. We used Brownian bridge algorithm to reduce computational cost by reducing cases that is needed to be checked daily basis.

Currently, three underlying asset ELS is the most popular ELS among multi-asset ELSs. However, four underlying asset ELS emerged as an alternative to three asset ELS under low interest rate environment to give higher coupon rate to investors. Intuitively, not only more complex structure of ELS gives higher return to investors but also it gives more risk to capital market as well as investors. In March 2020, foreign currency market experienced huge shock that never had for a decade. Security firms issued multi-underlying asset ELS faced huge amount of margin call as foreign stock market fell steeply because of COVID-19. The effect of margin calls from ELS immediately had a severe impact on the foreign exchange market as large security firms needed to exchange currency to make-up margin calls. The severity of shock of foreign currency market can be checked by looking swap cost fluctuation from the mid of March to early of April. Changes in condition of capital market and foreign exchange market can adversely affect the general industry and must be managed.

Because ELS has become an important investment tool in capital market of republic of Korea, its hedging strategy and risk management are becoming more important than ever. We believe that one of the key factors in risk management is how accurately profit and loss of ELS can be quantified. Therefore, we propose a numerical solution algorithm that makes it feasible. We describe the computational solution algorithm for the four underlying asset step-down ELS. The numerical tests confirm the accuracy and speed of the method.

The contents of this paper are as follows. In Section 2, four underlying asset ELS is described. In Section 3, the computational solution algorithm is given. In Section 4, we present computational experiments with the proposed method and conclusion is given in Section 5.

2. Four asset ELS

Figure 1 shows the schematic illustration of the four asset ELS option payoff. Let $K_1 \geq K_2 \geq K_3 \geq K_4 \geq K_5 \geq K_6$ be strike prices and $c_1 < c_2 < c_3 < c_4 < c_5 < c_6$ be coupon rates at times $T_1 < T_2 < T_3 < T_4 < T_5 < T_6$. Let us introduce the following notation for $k = 1, 2, 3, 4$: $X_k(t) = 100S_k(t)/S_k(0)$, where $S_k(t)$ is the k-th underlying asset value at time

t. Let us define the worst performer ($WP(t)$) among four asset paths:

$$WP(t) = \min(X_1(t), X_2(t), X_3(t), X_4(t)).$$

At $t = T_1$, if $WP(T_1) \geq K_1$, then $(1 + c_1)F$ is paid, where F is the face value. Otherwise, the contract will be continued. At time $t = T_2$, if $WP(T_2) \geq K_2$, then $(1 + c_2)F$ is paid. At time $t = T_3$, if $WP(T_3) \geq K_3$, then $(1 + c_3)F$ is paid. At time $t = T_4$, if $WP(T_4) \geq K_4$, then $(1 + c_4)F$ is paid. At time $t = T_5$, if $WP(T_5) \geq K_5$, then $(1 + c_5)F$ is paid. At $t = T_6$, we first check whether $WP(T_6) \geq K_6$ or not. If it is true, then $(1 + c_6)F$ is paid. Otherwise, if $\min_{0 \leq t \leq T_6} WP(t) \leq D$, then $WP(T_6)F/100$ is paid. Otherwise, it is $(1 + d)F$, where d is a dummy rate. Figure 1 shows the payoff structure.

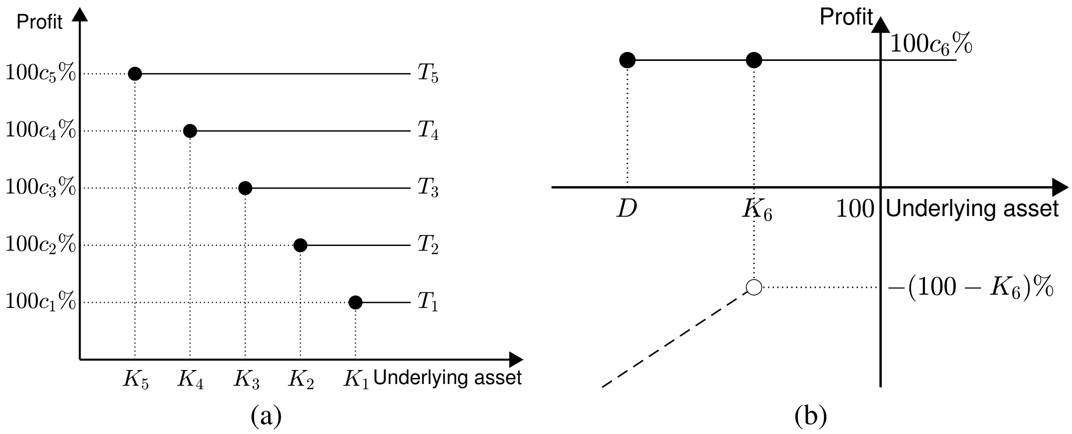

FIGURE 1. Schematic illustration of the payoff structures of the four-asset step-down ELS at times (a) $t = T_1$, T_2, T_3, T_4, T_5 and (b) $t = T_6$. Here, $d = c_6$ is used.

3. NUMERICAL METHOD

Now, let us describe the numerical solution algorithm in detail. Let us consider the following coefficient matrix

$$A = \begin{pmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} \\ \rho_{13} & \rho_{23} & 1 & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & 1 \end{pmatrix}.$$

Here, ρ_{ij} is the correlation coefficient between i and j underlying assets. We can decompose the matrix A using the Cholesky factorization [13] as follows:

$$A = LL^T,$$

where

$$L = \begin{pmatrix} L_{1,1} & 0 & 0 & 0 \\ L_{2,1} & L_{2,2} & 0 & 0 \\ L_{3,1} & L_{3,2} & L_{3,3} & 0 \\ L_{4,1} & L_{4,2} & L_{4,3} & L_{4,4} \end{pmatrix}.$$

Here,

$$L_{j,j} = \sqrt{A_{j,j} - \sum_{k=1}^{j-1} L_{j,k}^2},\ L_{i,j} = \frac{1}{L_{j,j}}\left(A_{i,j} - \sum_{k=1}^{j-1} L_{i,k}L_{j,k}\right) \text{ for } i > j.$$

We generate correlated random numbers Z_1^*, Z_2^*, Z_3^*, and Z_4^* from a standard multivariate normal distribution Z_1, Z_2, Z_3, and Z_4 using

$$\begin{pmatrix} Z_1^* \\ Z_2^* \\ Z_3^* \\ Z_4^* \end{pmatrix} = L \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix} = \begin{pmatrix} L_{1,1} & 0 & 0 & 0 \\ L_{2,1} & L_{2,2} & 0 & 0 \\ L_{3,1} & L_{3,2} & L_{3,3} & 0 \\ L_{4,1} & L_{4,2} & L_{4,3} & L_{4,4} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix}.$$

That is

$$\begin{aligned} Z_1^* &= L_{1,1}Z_1,\ Z_2^* = L_{2,1}Z_1 + L_{2,2}Z_2, \\ Z_3^* &= L_{3,1}Z_1 + L_{3,2}Z_2 + L_{3,3}Z_3, \\ Z_4^* &= L_{4,1}Z_1 + L_{4,2}Z_2 + L_{4,3}Z_3 + L_{4,4}Z_4. \end{aligned}$$

We make the following four correlated asset paths:

$$\begin{aligned} X_1(t_{i+1}) &= X_1(t_i)e^{(r-0.5\sigma_1^2)\Delta t_i + \sigma_1\sqrt{\Delta t_i}Z_{1i}^*}, \\ X_2(t_{i+1}) &= X_2(t_i)e^{(r-0.5\sigma_2^2)\Delta t_i + \sigma_2\sqrt{\Delta t_i}Z_{2i}^*}, \\ X_3(t_{i+1}) &= X_3(t_i)e^{(r-0.5\sigma_3^2)\Delta t_i + \sigma_3\sqrt{\Delta t_i}Z_{3i}^*}, \\ X_4(t_{i+1}) &= X_4(t_i)e^{(r-0.5\sigma_4^2)\Delta t_i + \sigma_4\sqrt{\Delta t_i}Z_{4i}^*}, \end{aligned}$$

where $\Delta t_i = t_{i+1} - t_i$. Let $WP(t_i)$ be the worst performer among four asset paths:

$$WP(t_i) = \min(X_1(t_i), X_2(t_i), X_3(t_i), X_4(t_i)).$$

We generate random samples at $T_1,\ T_2,\ T_3,\ T_4,\ T_5,\ T_6$. That is,

$$WP(T_i),\ i = 0, \ldots, 6,$$

where $WP(T_0) = 100$ and $T_0 = 0$. If an early redemption condition is satisfied, then the contract is ended with an appropriate payoff as shown in Fig. 2(a). Even if the knock-in barrier has been touched, if an early redemption or the maturity condition is satisfied as shown in Fig. 2(b), then the given payoff is paid.

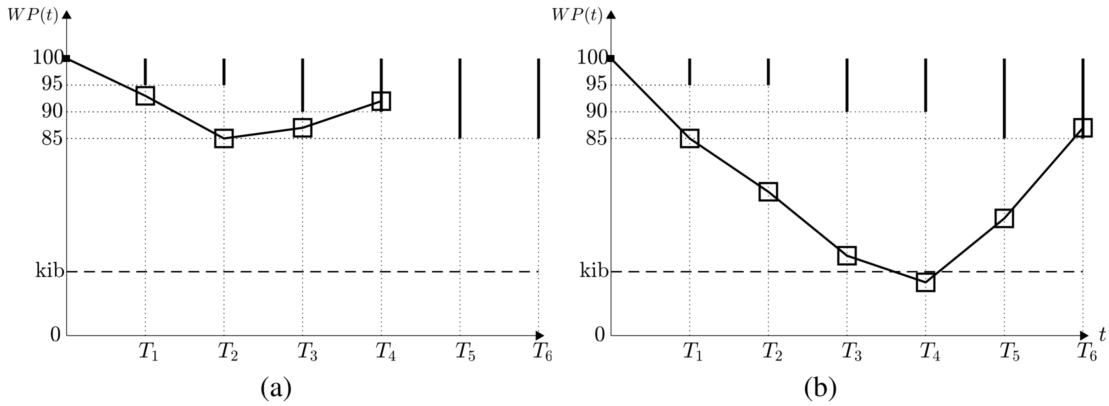

FIGURE 2. (a) Early redemption condition is satisfied and (b) maturity condition is satisfied even if the knock-in barrier has been touched.

If the early redemption and maturity conditions are not satisfied and the knock-in barrier has been touched, i.e., $\min\{WP(T_1), WP(T_2), WP(T_3), WP(T_4), WP(T_5), WP(T_6)\} \leq D$ as shown in Fig. 3, then the payoff is $WP(T_6)F/100$.

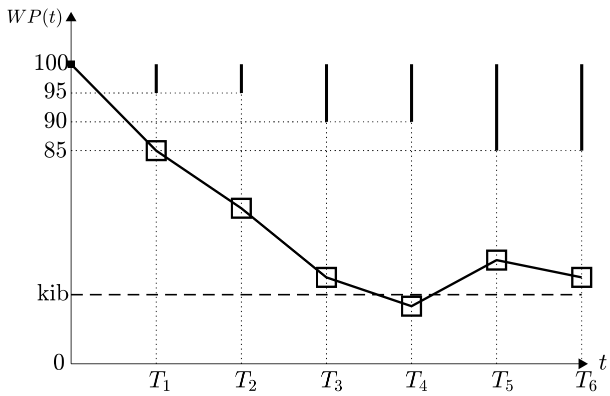

FIGURE 3. Early redemption and maturity conditions are not satisfied and the knock-in barrier has been touched.

Otherwise, if $\min\{WP(T_1), WP(T_2), WP(T_3), WP(T_4), WP(T_5), WP(T_6)\} > D$ as shown in Fig. 4, then we make a full path connecting the prices at the check days using the Brownian bridge technique. In Fig. 4, the dashed line is the regenerated full path. Using the regenerated full path, if $\min_{1\leq i\leq T_6/\Delta t} WP(t_i) > D$, see Fig. 4 (a), then the return is $(1+d)F$, where d is a dummy rate. If $\min_{1\leq i\leq T_6/\Delta t} WP(t_i) \leq D$, refer to Fig. 4(b), then it is $WP(T_6)F/100$.

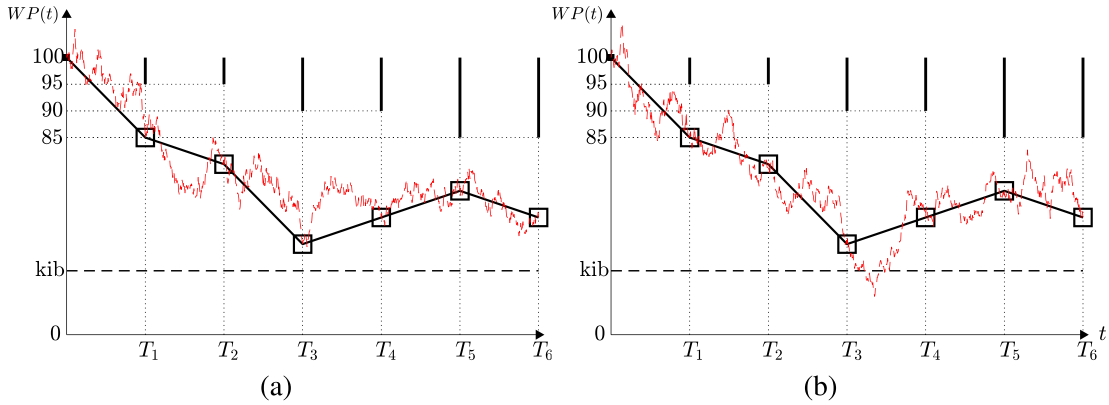

FIGURE 4. Reconstruction of a full path connecting the prices at the check days: (a) no knock-in case and (b) knock-in case.

When we want additional values between the two given values, we can use the Brownian bridge technique [14, 15] to make a path passing the two given values. For $k = 1, 2, 3, 4$, let $X_k(T_i)$ and $X_k(T_{i+1})$ be the two given values at $t = T_i$ and $t = T_{i+1}$, respectively, then we generate a path starting from $Y_k(T_i) = X_k(T_i)$ with the time step $\Delta t = 1/365$.

$$Y_k(t_{j+1}) = Y_k(t_j)e^{w_j},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1,$$

where $w_j = (r - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}Z^*_{kj}$ and $t_j = T_i + j\Delta t$. Let $W_j = \sum_{i=0}^{j} w_i$, then $Y_k(t_{j+1}) = Y_k(T_i)e^{W_j},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1$. In general, $Y_k(T_{i+1}) \neq X_k(T_{i+1})$. To make a path passing $X_k(T_i)$ and $X_k(T_{i+1})$, we use the Brownian bridge technique to W_j. Let

$$B_j = W_j + \frac{t_j - T_i}{T_{i+1} - T_i} \log \frac{X_k(T_{i+1})}{Y_k(T_{i+1})},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1.$$

Then, we get a full path passing $X_k(T_i)$ and $X_k(T_{i+1})$ as

$$X_k(t_{j+1}) = X_k(T_i)e^{B_j},\ j = 0, \ldots, (T_{i+1} - T_i)/\Delta t - 1.$$

Finally, we define the worst performer among four full asset paths:

$$WP(t_i) = \min(X_1(t_i), X_2(t_i), X_3(t_i), X_4(t_i)).$$

Using this worst performer, we apply the standard MCS. More details about generating new path using the Brownian bridge technique can be found in [12]. We compute the prices of ELS products with four assets using **Algorithms 1** and **2**.

4. NUMERICAL EXPERIMENT

Now, we perform characteristic computational tests. All computations are run in MATLAB version R2020a on a Intel(R) Core(TM) i5-7400 CPU @ 3.00GHz 3.00 GHz PC with 12.0 GB RAM.

Algorithm 1 Initial setting for the Brownian bridge MCS algorithm for four asset ELS

Require:

$K_1 \geq K_2 \geq K_3 \geq K_4 \geq K_5 \geq K_6$: Strike percentages
$c_1 < c_2 < c_3 < c_4 < c_5 < c_6$: Coupon rates
$T_1 < T_2 < T_3 < T_4 < T_5 < T_6$: Automatic redemption times
N_s: The number of sample paths
F: Face value
σ_1, σ_2, σ_3, and σ_4: Volatilities
ρ_{12}, ρ_{13}, ρ_{14}, ρ_{23}, ρ_{24}, ρ_{34}: Correlation coefficients
Z_1^*, Z_2^*, Z_3^*, Z_4^*: Correlated random numbers
r: Risk-neutral interest rate
d: Dummy rate
D: knock-in barrier
$S_1(t), S_2(t), S_3(t), S_4(t)$: Underlying asset prices
$X_1(t) = 100S_1(t)/S_1(0)$, $X_2(t) = 100S_2(t)/S_2(0)$, $X_3(t) = 100S_3(t)/S_3(0)$,
$X_4(t) = 100S_4(t)/S_4(0)$: Scaled underlying asset prices
$WP(t) = \min(X_1(t), X_2(t), X_3(t), X_4(t))$: Worst performer
$M_1 = M_2 = M_3 = M_4 = M_5 = M_6 = 0$

4.1. **Parameter setting.** We have conducted convergence test and comparison of CPU time with strike prices $K_1 = 85$, $K_2 = 80$, $K_3 = 75$, $K_4 = 70$, $K_5 = 65$, $K_6 = 60$, $T_i = 0.5i$, $i = 1, \ldots, 6$, knock-in barrier $D = 50$, volatilities $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\sigma_3 = 0.25$, $\sigma_4 = 0.24$, the correlation coefficients $\rho_{12} = 0.7$, $\rho_{13} = 0.48$, $\rho_{14} = 0.27$, $\rho_{23} = 0.45$, $\rho_{24} = 0.3$, $\rho_{34} = 0.5$, coupon rates $c_1 = 0.05$, $c_2 = 0.1$, $c_3 = 0.15$, $c_4 = 0.2$, $c_5 = 0.25$, $c_6 = 0.3$, the risk-free interest free $r = 0.01$, and the dummy rate $d = c_6$. Figure 5 shows the schematic illustration of the four underlying asset ELS option payoff at times (a) $t = T_1, T_2, T_3, T_4, T_5$ and (b) $t = T_6$.

4.2. **Convergence test.** Figure 6 shows the convergence of the option values of the 4-asset ELS with the increasing number of samples. Open circles and plus marks are the results of the ELS prices from the Brownian bridge and standard MCS, respectively. For each sample number, 100 simulation results are shown. The computational results indicate both the two methods converge to the same value.

Table 1 shows the mean and variance of the 4-asset ELS price from the two difference methods with 10^5 samples. Here, the mean and variance are computed with 100 simulations. The numerical results indicates the equivalency of Brownian bridge and standard MCS.

TABLE 1. Mean and variance of the 4-asset ELS prices with two different approaches.

Case	Mean	Variance
Standard MCS	98.3956	0.0066
Brownian bridge MCS	98.4000	0.0064

Algorithm 2 Brownian bridge MCS algorithm for four asset ELS

for iteration $= 1$ to N_s **do**
 ▷ Generate scaled stock paths on only T_i
 for $i = 0$ to 5 **do**
  for $k = 1$ to 4 **do**
   $X_k(T_{i+1}) = X_k(T_i)\exp((r - 0.5\sigma_k)(T_{i+1} - T_i) + \sigma_k\sqrt{T_{i+1} - T_i}Z^*_{ki})$,
   $Z^*_{ki} \sim N(0,1)$
  end for
  $WP(T_{i+1}) = \min(X_1(T_{i+1}), X_2(T_{i+1}), X_3(T_{i+1}), X_4(T_{i+1}))$
 end for
 ▷ Check the value of the stock path at checking days
 if $WP(T_1) \geq K_1$ **then** $M_1 = M_1 + (1 + c_1)F$
 else if $WP(T_2) \geq K_2$ **then** $M_2 = M_2 + (1 + c_2)F$
 else if $WP(T_3) \geq K_3$ **then** $M_3 = M_3 + (1 + c_3)F$
 else if $WP(T_4) \geq K_4$ **then** $M_4 = M_4 + (1 + c_4)F$
 else if $WP(T_5) \geq K_5$ **then** $M_5 = M_5 + (1 + c_5)F$
 else if $WP(T_6) \geq K_6$ **then** $M_6 = M_6 + (1 + c_6)F$
 else if $\min_{1\leq i\leq 6}\{WP(T_i)\} \leq D$ **then** $M_6 = M_6 + (WP(T_6)/100)F$
 else
  ▷ Generate daily stock paths passing through $X_k(T_i)$ using the Brownian bridge technique as
  for $i = 0$ to 5 **do**
   Set $Y_1(T_i) = X_1(T_i)$, $Y_2(T_i) = X_2(T_i)$, $Y_3(T_i) = X_3(T_i)$, and $Y_4(T_i) = X_4(T_i)$
   for $k = 1$ to 4 **do**
    for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
     $Y_k(t_{j+1}) = Y_k(t_j)\exp(w^j_k)$,
     $w^j_k = (r - 0.5\sigma^2_k)\Delta t + \sigma_k\sqrt{\Delta t}Z^*_{kj}$, $Z^*_{kj} \sim N(0,1)$
    end for
   end for
  end for
  ▷ Apply the Brownian bridge technique
  for $k = 1$ to 4 **do**
   for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
    $Y_k(t_{j+1}) = Y_k(T_i)\exp(W^j_k),\ W^j_k = \sum_{p=T_i/\Delta t}^{j} w^p_k$
   end for
  end for
  for $j = T_i/\Delta t$ to $T_{i+1}/\Delta t - 1$ **do**
   for $k = 1$ to 4 **do**
    $X_k(t_j) = X_k(T_j)\exp(B^j_k),\ B^j_k = W^j_k + \frac{t_j - T_i}{T_{i+1} - T_i}\log\frac{X_k(T_{i+1})}{Y_k(T_{i+1})}$
   end for
   $WP(t_j) = \min(X_1(t_j), X_2(t_j), X_3(t_j), X_4(t_j))$
  end for
  if $\min_{1\leq j\leq T_6/\Delta t}\{WP(t_j)\} \leq D$ **then** $M_6 = M_6 + (WP(T_6)/100)F$
  else $M_6 = M_6 + (1 + d)F$
  end if
 end if
end for
▷ Take average and discount to present value.
$V = \sum_{i=1}^{6} e^{-rT_i}M_i/N_s$

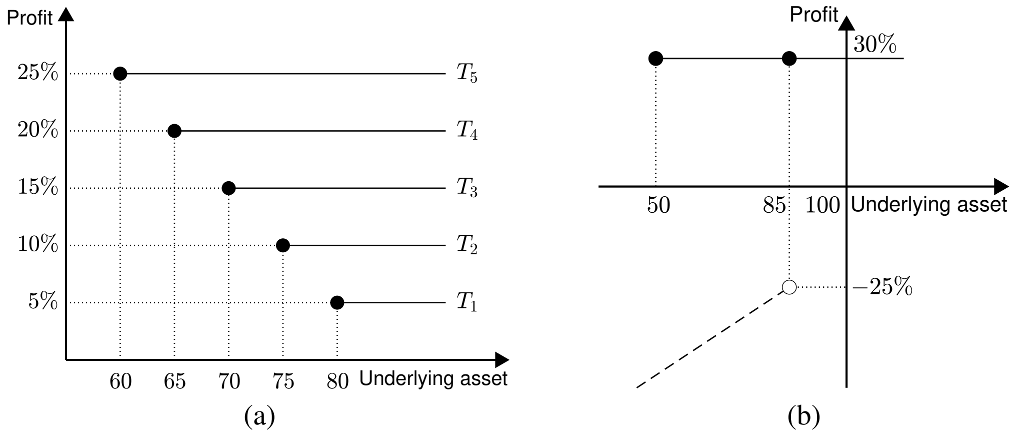

FIGURE 5. Payoff structures of the four-asset step-down ELS at times (a) $t = T_1, T_2, T_3, T_4, T_5$ and (b) $t = T_6$.

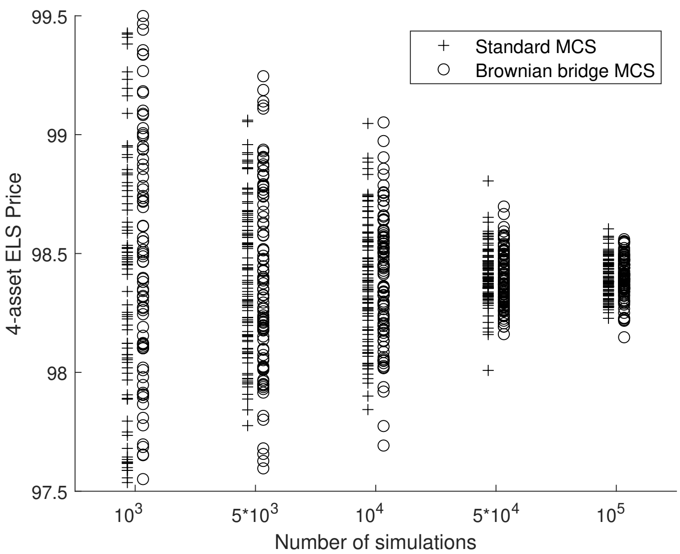

FIGURE 6. 4-asset ELS price versus the number of samples. Here, we plot 100 simulation results for each case.

4.3. **CPU time.** We compute the amount of CPU time needed to evaluate the 4-asset ELS price using the Brownian bridge and standard MCS methods with the number of samples: 10^3, 5×10^3, 10^4, 5×10^4, and 10^5. Table 2 lists the CPU times for the Brownian bridge MCS and standard MCS for 4-asset ELS with different number of samples. Furthermore, it shows the ratio of the CPU times for both the approaches. Figure 7 shows the numerical results in Table 2 and demonstrates the Brownian bridge MCS is about 25 times faster than the standard MCS.

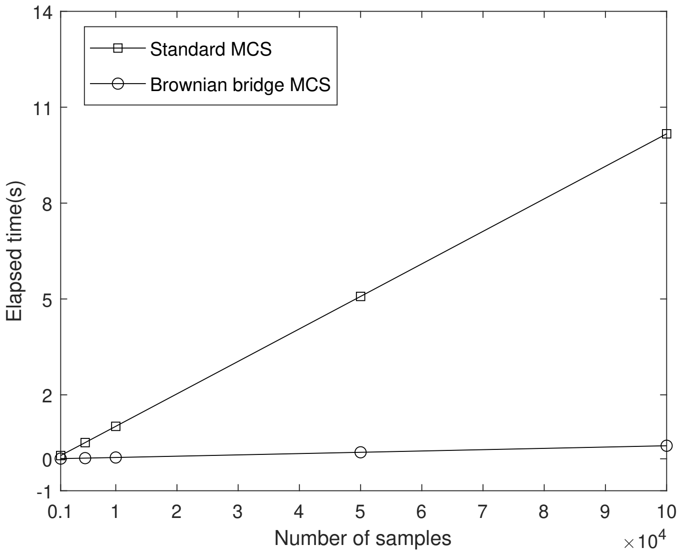

FIGURE 7. Comparison of the elapsed time (in seconds).

TABLE 2. Comparison of the elapsed CPU time in seconds for four asset ELS.

M	10^3	5×10^3	10^4	5×10^4	10^5
Brownian bridge MCS	0.0058	0.0203	0.0404	0.2029	0.4077
Standard MCS	0.1016	0.5064	1.0150	5.0801	10.1682
Ratio	17.5089	24.9103	25.1441	25.0418	24.9412

5. CONCLUSION

In this paper, we developed a fast option pricing method for four asset ELS using Brownian bridge and Monte Carlo simulation. Even though three asset ELS is the most popular ELS among multi-asset ELSs, however, four asset ELS emerged as an alternative to three asset ELS under low interest rate environment to give higher coupon rate to investors. We described in detail the computational solution algorithm for the four underlying asset step-down ELS. The numerical tests confirmed the accuracy and speed of the method compared to the standard MCS.

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