

# Linear First Order ODE



- We consider linear first order ODEs with variable coefficients:

$$\frac{dy}{dt} + p(t)y = g(t)$$

- **The method of integrating factors** involves multiplying this equation by a function  $\mu(t)$ , chosen so that the resulting equation is easily integrated.

This lecture note is based on the book by William E. boyce and Richard C. DiPrima.  
Those who wants to know more detailed contents, refer to the book,  
“Elementary Differential Equations and Boundary Value Problems”.

# Method of Integrating Factors



## (1) Variable Right Side

In general, for variable right side  $g(t)$ ,

$$y' + ay = g(t)$$

the solution can be found as follows:

- Multiplying both sides by  $\mu(t)$ , we obtain

$$\mu(t) \frac{dy}{dt} + a\mu(t)y = \mu(t)g(t)$$

- We will choose  $\mu(t)$  so that left side is derivative of known quantity. Consider the following, and recall product rule:

$$\frac{d}{dt}[\mu(t)y] = \mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt} y$$

- Choose  $\mu(t)$  so that

$$\mu'(t) = a\mu(t) \Rightarrow \mu(t) = e^{at}$$

- With  $\mu(t) = e^{at}$ , we solve the original equation as follows:

$$e^{at} \frac{dy}{dt} + ae^{at} y = e^{at} g(t)$$

$$\frac{d}{dt} [e^{at} y] = e^{at} g(t)$$

$$e^{at} y = \int e^{at} g(t) dt$$

- Thus,

$$y = e^{-at} \int e^{at} g(t) dt + Ce^{-at}$$

# Example 1



Consider the following equation

$$y' - y = e^{2t}$$

(1) Find the integrating factor

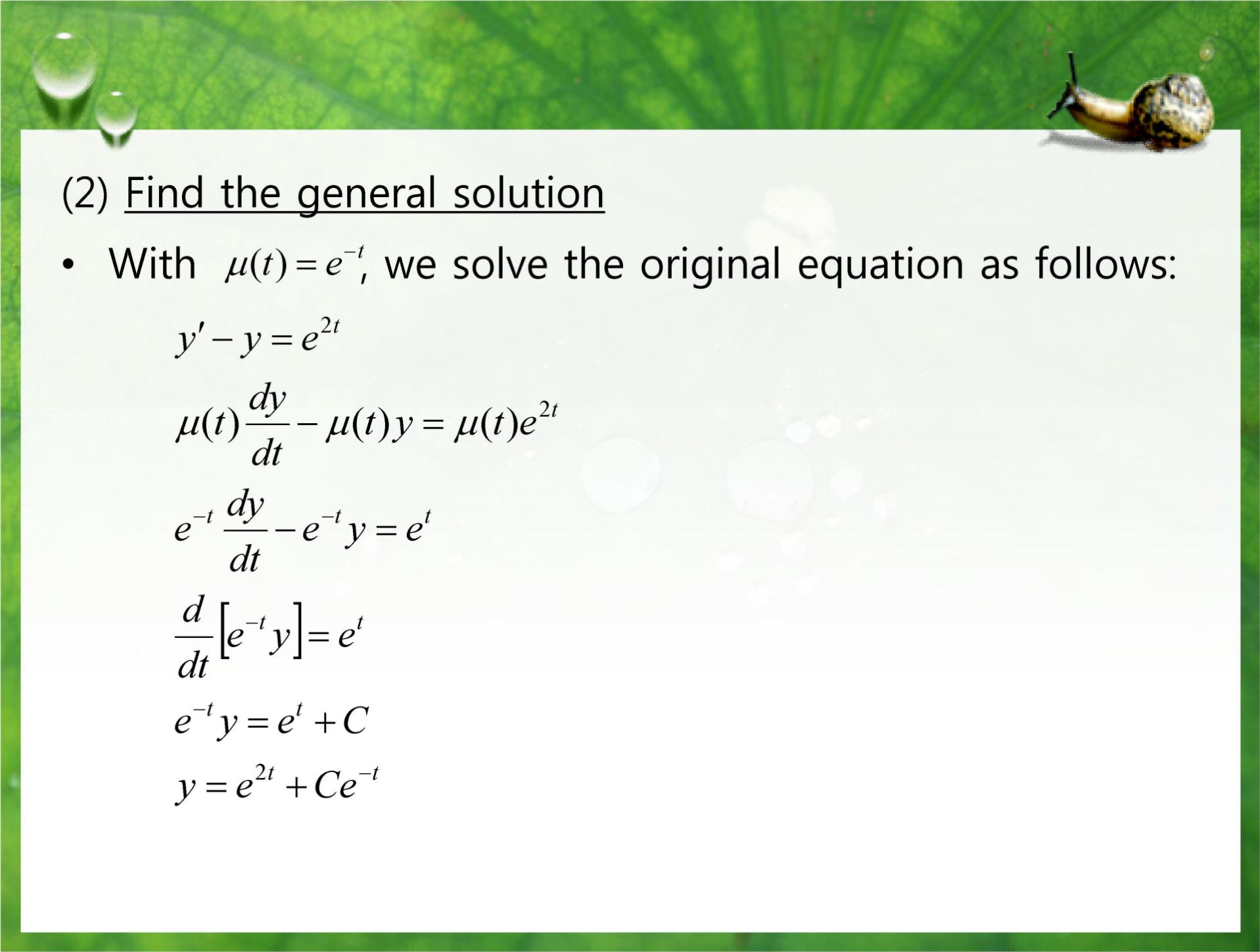
- Multiplying both sides by  $\mu(t)$ , we obtain

$$\mu(t) \frac{dy}{dt} - \mu(t)y = e^{2t} \mu(t)$$

- We will choose  $\mu(t)$  so that left side is derivative of known quantity. Consider the following, and recall product rule:

$$\frac{d}{dt}[\mu(t)y] = \mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt} y$$

- Choose  $\mu(t)$  so that  $\mu'(t) = -\mu(t) \Rightarrow \mu(t) = e^{-t}$



(2) Find the general solution

- With  $\mu(t) = e^{-t}$ , we solve the original equation as follows:

$$y' - y = e^{2t}$$

$$\mu(t) \frac{dy}{dt} - \mu(t)y = \mu(t)e^{2t}$$

$$e^{-t} \frac{dy}{dt} - e^{-t}y = e^t$$

$$\frac{d}{dt}[e^{-t}y] = e^t$$

$$e^{-t}y = e^t + C$$

$$y = e^{2t} + Ce^{-t}$$

## Example 2



Consider the following equation

$$y' + 2y = t + 3$$

(1) Find the integrating factor

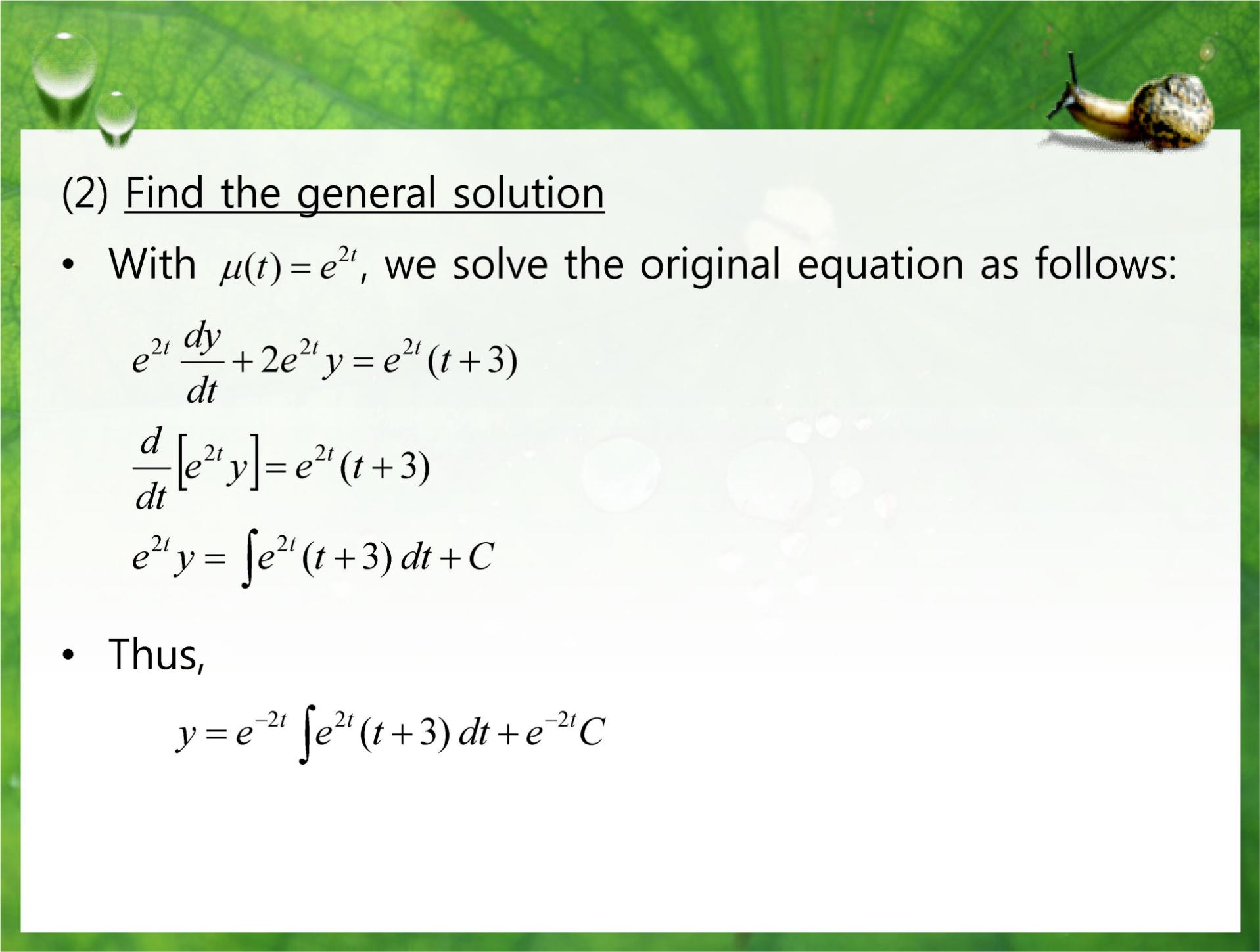
- Multiplying both sides by  $\mu(t)$ , we obtain

$$\mu(t) \frac{dy}{dt} + 2\mu(t)y = (t+3)\mu(t)$$

- We will choose  $\mu(t)$  so that left side is derivative of known quantity. Consider the following, and recall product rule:

$$\frac{d}{dt}[\mu(t)y] = \mu(t) \frac{dy}{dt} + \frac{d\mu(t)}{dt} y$$

- Choose  $\mu(t)$  so that  $\mu'(t) = 2\mu(t) \Rightarrow \mu(t) = e^{2t}$



(2) Find the general solution

- With  $\mu(t) = e^{2t}$ , we solve the original equation as follows:

$$e^{2t} \frac{dy}{dt} + 2e^{2t} y = e^{2t} (t + 3)$$

$$\frac{d}{dt} [e^{2t} y] = e^{2t} (t + 3)$$

$$e^{2t} y = \int e^{2t} (t + 3) dt + C$$

- Thus,

$$y = e^{-2t} \int e^{2t} (t + 3) dt + e^{-2t} C$$

- Integrating by parts,

$$\begin{aligned}\int e^{2t}(t+3)dt &= 3 \int e^{2t} dt + \int te^{2t} dt \\ &= \frac{3}{2}e^{2t} + \left[ \frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t} dt \right] \\ &= \frac{5}{4}e^{2t} + \frac{1}{2}te^{2t}\end{aligned}$$

Thus,

$$\begin{aligned}y &= e^{-2t} \left( \frac{5}{4}e^{2t} + \frac{1}{2}te^{2t} \right) + Ce^{-2t} \\ &= \frac{5}{4} + \frac{1}{2}t + Ce^{-2t}\end{aligned}$$

# Method of Integrating Factors



## (2) General first order linear equation

We consider the general first order linear equation

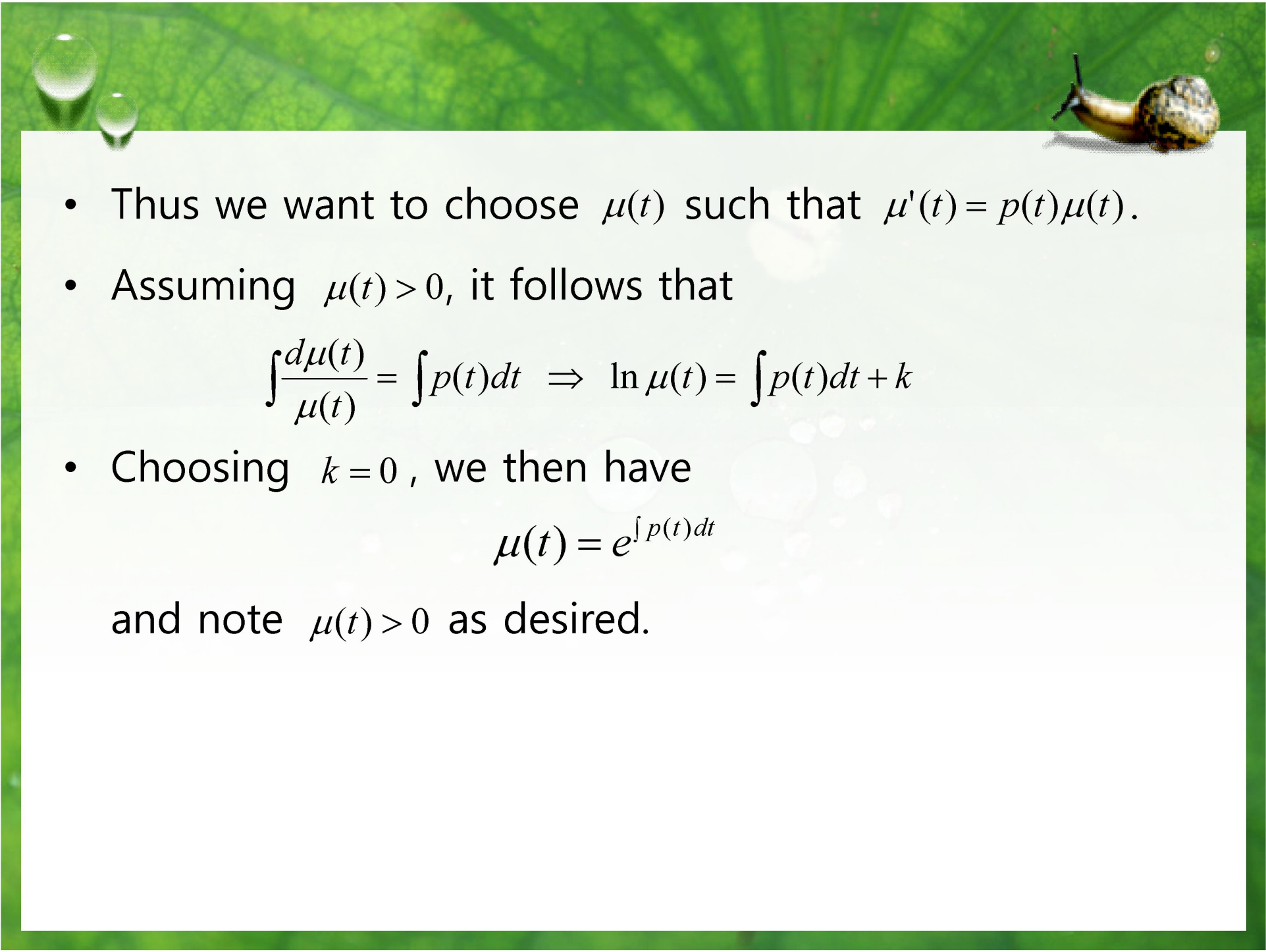
$$y' + p(t)y = g(t)$$

- Multiplying both sides by  $\mu(t)$ , we obtain

$$\mu(t) \frac{dy}{dt} + p(t)\mu(t)y = g(t)\mu(t)$$

- Next, we want  $\mu(t)$  such that  $\mu'(t) = p(t)\mu(t)$ , from which it will follow that

$$\frac{d}{dt}[\mu(t)y] = \mu(t) \frac{dy}{dt} + p(t)\mu(t)y$$

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- Thus we want to choose  $\mu(t)$  such that  $\mu'(t) = p(t)\mu(t)$ .
  - Assuming  $\mu(t) > 0$ , it follows that

$$\int \frac{d\mu(t)}{\mu(t)} = \int p(t)dt \Rightarrow \ln \mu(t) = \int p(t)dt + k$$

- Choosing  $k = 0$ , we then have

$$\mu(t) = e^{\int p(t)dt}$$

and note  $\mu(t) > 0$  as desired.

- Thus we have the following:

$$y' + p(t)y = g(t)$$

$$\mu(t) \frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t), \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

- Then

$$\frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

$$\mu(t)y = \int \mu(t)g(t)dt + c$$

$$y = \frac{\int \mu(t)g(t)dt + c}{\mu(t)}, \quad \text{where } \mu(t) = e^{\int p(t)dt}$$

## Example 3



Consider the initial value problem

$$ty' + 4y = 4t^2, \quad y(1) = 3.$$

- First, we put into the standard form

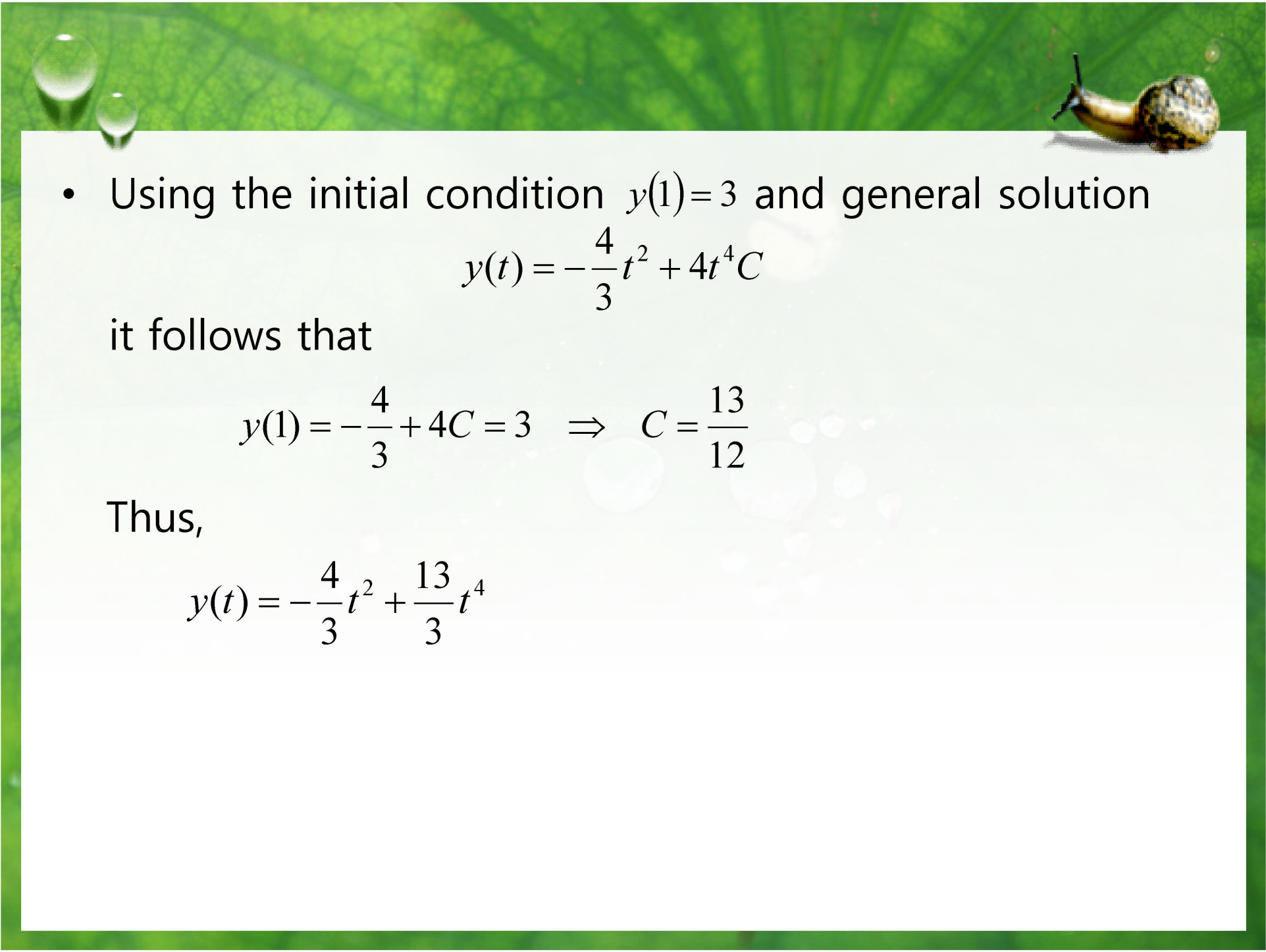
$$y' + \frac{4}{t}y = 4t \quad \text{for } t \neq 0$$

- Then,

$$\mu(t) = e^{\int p(t)dt} = e^{\int \frac{4}{t}dt} = e^{4\ln|t|} = e^{\ln\left(\frac{1}{t^4}\right)} = \frac{1}{t^4}$$

and hence

$$y = \frac{\int \mu(t)g(t)dt + C}{\mu(t)} = \frac{\int \frac{1}{t^4} 4t dt + C}{\frac{1}{t^4}} = 4t^2 \left[ \int \frac{1}{t^3} dt + C \right] = -\frac{4}{3}t^2 + 4Ct^4$$

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- Using the initial condition  $y(1)=3$  and general solution

$$y(t) = -\frac{4}{3}t^2 + 4t^4C$$

it follows that

$$y(1) = -\frac{4}{3} + 4C = 3 \Rightarrow C = \frac{13}{12}$$

Thus,

$$y(t) = -\frac{4}{3}t^2 + \frac{13}{3}t^4$$