Autonomous Equations

- The equations of the form $y' = f(y)$ are called **autonomous equations**.

For example,

- $y' = ay + by^2$, $y' = e^y - 1$

- The logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

Autonomous Equations

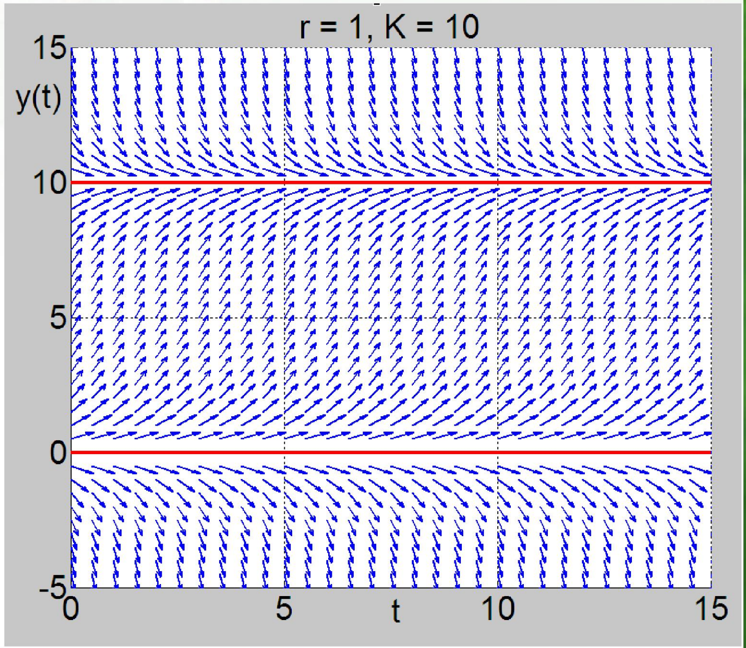
Equilibrium solutions

- **Equilibrium solutions** of a general first order autonomous equation $y' = f(y)$ can be found by locating roots of $f(y) = 0$.
- These roots of $f(y)$ are called **critical points**.
- For example, the critical points of the logistic equation

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

are $y = 0$ and $y = K$.

Thus, critical points are constant functions (equilibrium solutions) in this setting.



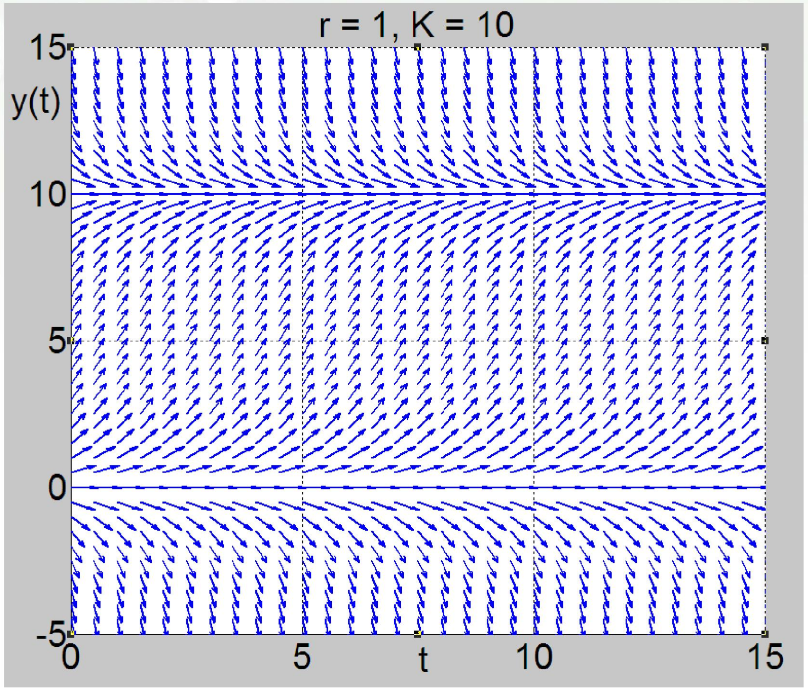
Logistic Equation

- The <u>logistic equation</u> is

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

where $K = r/a,\ r$ is the **intrinsic growth rate**, and K is the **carrying capacity**.

- A direction field for the logistic equation with $r = 1$ and $K = 10$ is given here.



Logistic Equation with Equilibrium Solutions

- Our logistic equation is

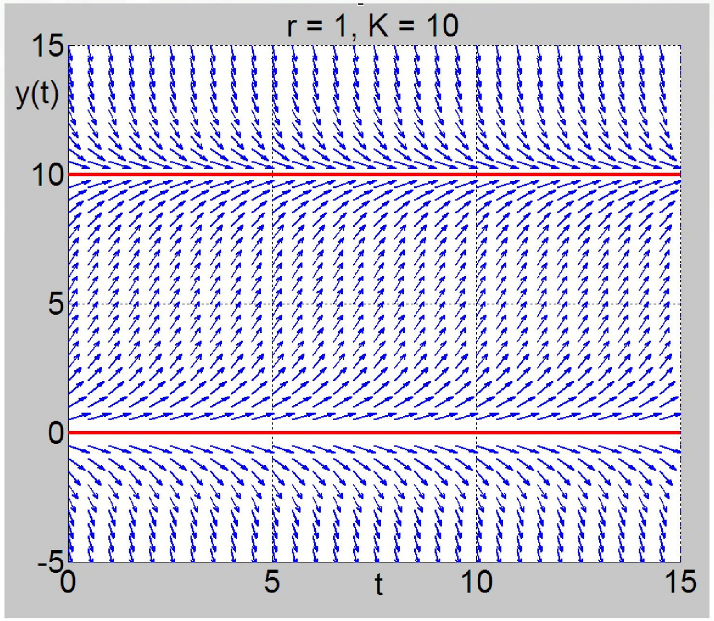
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y, \quad r,\ K > 0$$

- Two equilibrium solutions are $y = \phi_1(t) = 0, \quad y = \phi_2(t) = K.$

  In direction field below, with $r = 1,\ K = 10$, note behavior of solutions near equilibrium solutions:

  $y = 0$ is **unstable**,

  $y = 10$ is **asymptotically stable**.



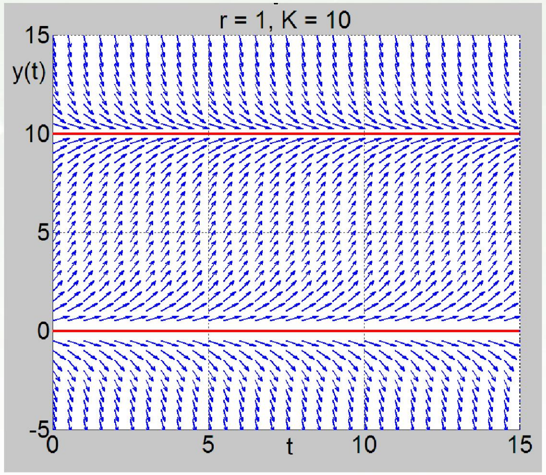
MATLAB Code

```
clear all; clc; clf;
hold on; grid on
r = 1;  K = 10;
t  = 0:0.5:15;                % define grid of values in t-direction
y = -5:0.5:15;                % define grid of values in y-direction

[T,Y] = meshgrid(t, y);       % creates 2D matrices
dT = ones(size(T));           % dt = 1 for all points
dY = r*(1-Y/K).*Y;            % dy = r(1-y/K)y*dt;  this is the ODE

N = sqrt(dT.^2 + dY.^2);      % magnitude of arrows
dT = dT./N; dY = dY./N;       % normalize arrows to get all same length
quiver(T,Y,dT,dY);            % draw arrows (t,y) --> (t+dt,y+dy)

y1 = 0;  y2 = K;              % equilibrium solutions, y1 and y2
plot(t,y1+0*t,'r-','LineWidth',2);
plot(t,y2+0*t,'r-','LineWidth',2);
```



The Solutions of Logistic Equation

- Provided $y \neq 0$ and $y \neq K$, we can rewrite the logistic ODE:

$$\frac{dy}{(1 - y/K)y} = r\,dt$$

- Expanding the left side using partial fractions,

$$\frac{1}{(1 - y/K)y} = \frac{A}{1 - y/K} + \frac{B}{y} \Rightarrow 1 = Ay + B(1 - y/K) \Rightarrow B = 1,\ A = y/K$$

- Thus the logistic equation can be rewritten as

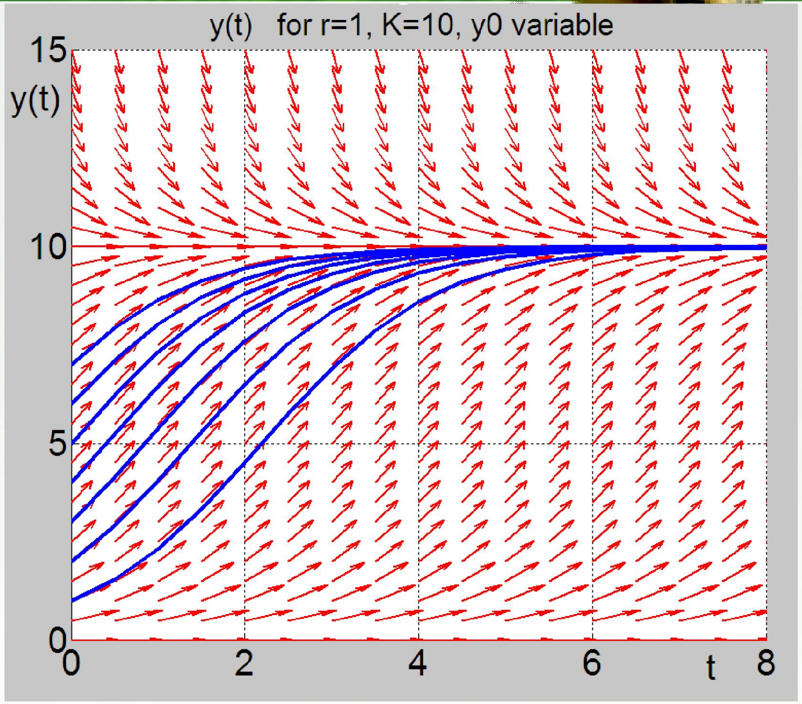
$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right)dy = r\,dt$$

- Integrating the above result, we obtain $\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$

- We have

$$\ln|y| - \ln\left|1 - \frac{y}{K}\right| = rt + C$$

- If $0 < y_0 < K$, then $0 < y < K$ and hence $\ln y - \ln\left(1 - \frac{y}{K}\right) = rt + C$



- Using properties of logs, we rewrite

$$\ln\left[\frac{y}{1 - y/K}\right] = rt + C \iff \frac{y}{1 - y/K} = e^{rt+C} \iff \frac{y}{1 - y/K} = ce^{rt}$$

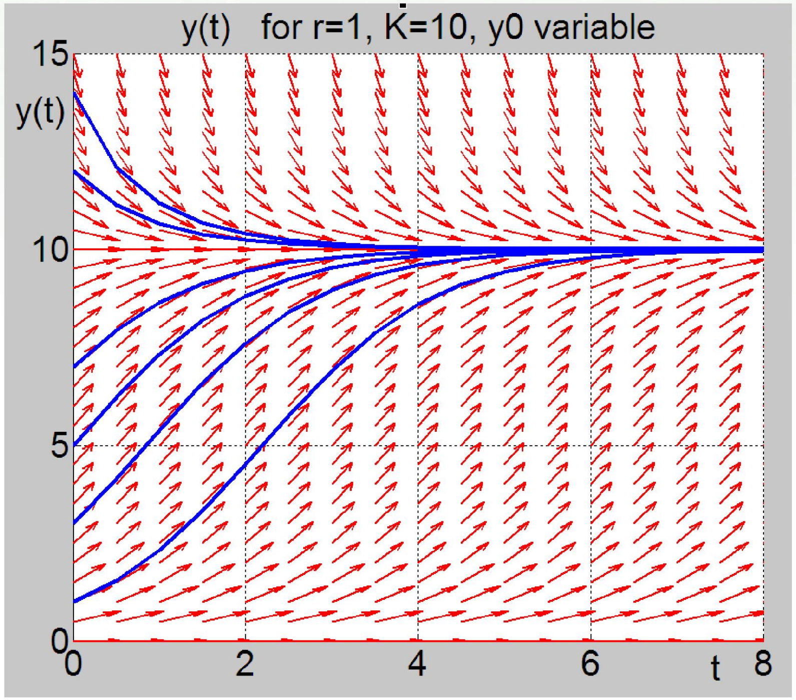
$$\text{or} \quad y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}, \quad \text{where } y_0 = y(0)$$

- We have

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \quad \text{for } 0 < y_0 < K$$

- It can be shown that solution is also valid for $y_0 > K$. Also, this solution contains equilibrium solutions $y = 0$ and $y = K$.
- Hence solution to logistic equation is

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

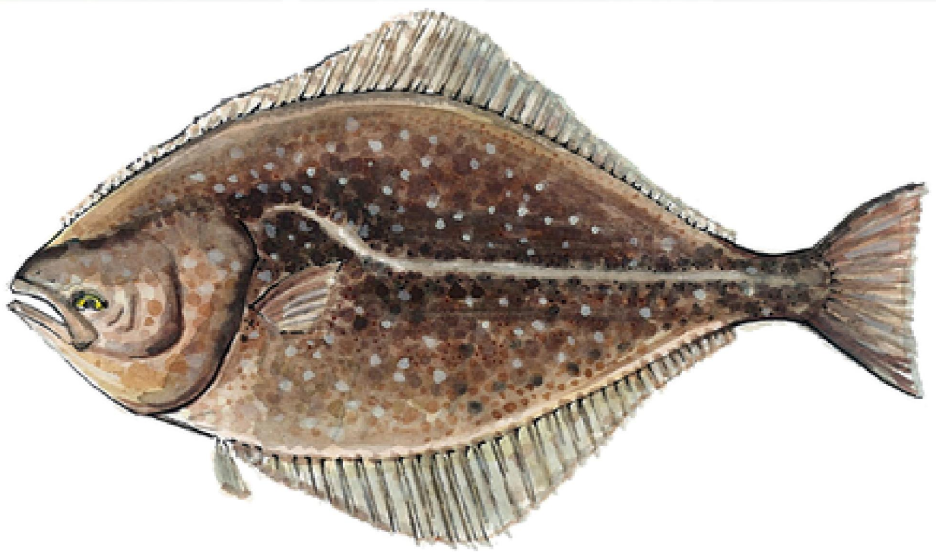


Example

Let y be biomass (in kg) of halibut population at time t, with $r = 0.71/year$ and $K = 80.5 \times 10^6\ kg$. If $y_0 = 0.20K$, find

(a) biomass 4 years later

(b) the time τ such that $y(\tau) = 0.60K$.



Pacific Halibut

Figure source : www.suite101.com/view_image_articles.cfm/1403384

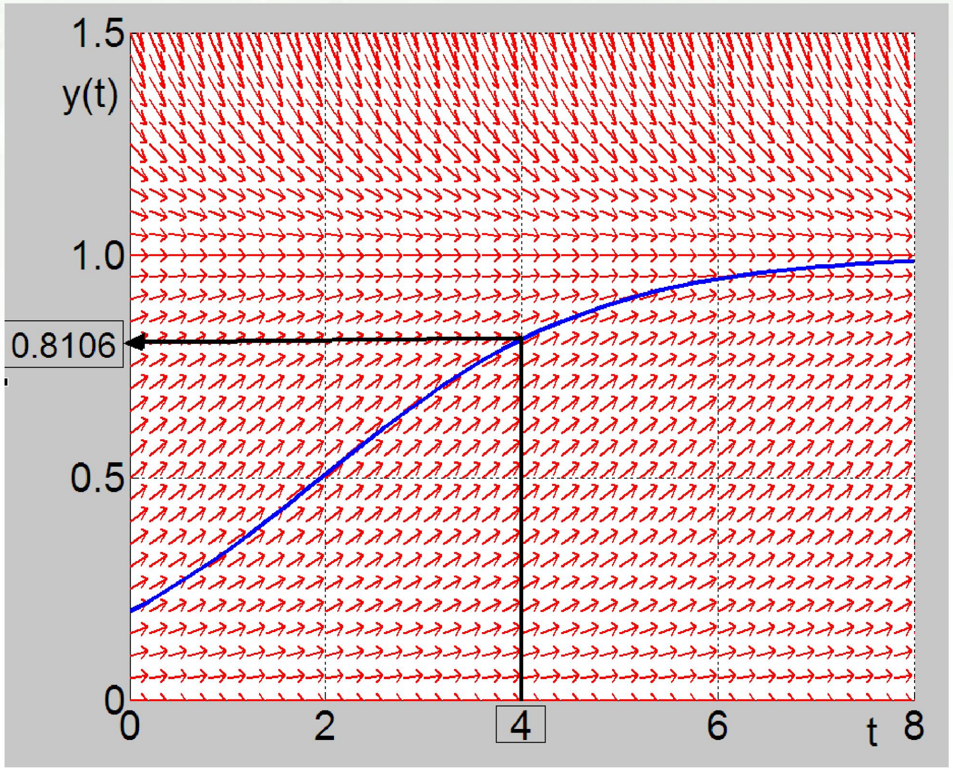
Solution)

(a) For convenience, scale equation $$\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}$$

Then $$\frac{y(4)}{K} = \frac{0.2}{0.2 + 0.8e^{-(0.71)(4)}} \approx 0.8106$$

and hence

$$y(4) \approx 0.8106K \approx 65.25 \times 10^6 \text{ kg}$$



(b) Find time τ for which $y(\tau) = 0.60K$.

$$\frac{y}{K} = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-rt}}$$

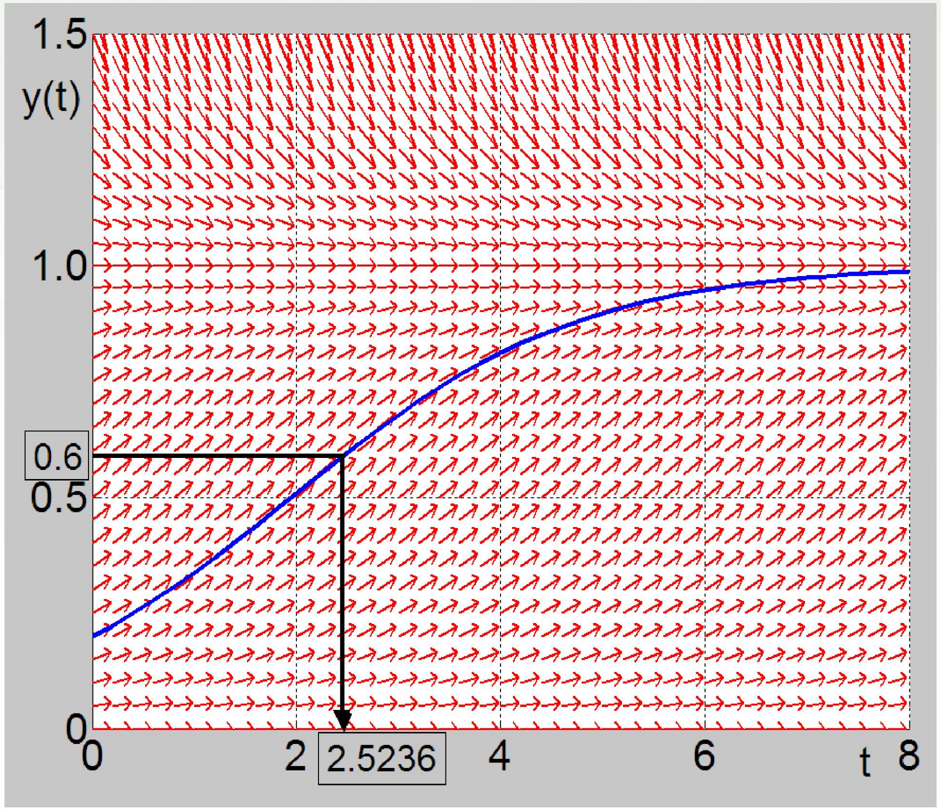
$$0.60 = \frac{y_0/K}{y_0/K + (1 - y_0/K)e^{-r\tau}}$$

$$0.60\left[\frac{y_0}{K} + \left(1 - \frac{y_0}{K}\right)e^{-r\tau}\right] = \frac{y_0}{K}$$

$$0.60\, y_0/K + 0.60(1 - y_0/K)e^{-r\tau} = y_0/K$$

$$e^{-r\tau} = \frac{0.40 y_0/K}{0.60(1 - y_0/K)} = \frac{2\, y_0/K}{3(1 - y_0/K)}$$

$$\tau = \frac{-1}{0.71}\ln\left(\frac{0.40}{3(0.80)}\right) \approx 2.5236 \text{ years}$$



MATLAB Code

```
clear all; clc; clf;
hold on
r = 0.71; K = 80.5*10^6;
K = K/K;  r = 0.71/K;          % redefine scaling factors
t = 0:0.2:8;                   % define grid of values in t-direction
y = 0:0.05:3;                  % define grid of values in y-direction

[T,Y] = meshgrid(t,y);         % creates 2D matrices
dT = ones(size(T));            % dt = 1 for all points
dY = r*(1-Y/K).*Y;             % dy = r(1-y/K)y*dt; this is the ODE
N = sqrt(dT.^2 + dY.^2);       % magnitude of arrows
dT = dT./N; dY = dY./N;        % normalize arrows to get all same length
quiver(T,Y,dT,dY,'r');         % draw arrows (t,y) --> (t+dt,y+dy)

y0 = 0.2*K;
yy = y0*K./(y0+t*0+(K-y0)*exp(-r*t));
plot(t,yy+0*t,'b-','LineWidth',2);
axis([0 8 0 1.5])
```