Complex Roots of Characteristic Equation

- Consider 2nd order differential equation

$$ay'' + by' + cy = 0$$

where *a*, *b* and *c* are constants.

(1) Assume exponential solution and characteristic equation :

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

(2) Then we have two solutions, r_1 & r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac < 0$, then complex roots:

  $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$.

- Thus $y_1(t) = e^{(\lambda + i\mu)t}, \; y_2(t) = e^{(\lambda - i\mu)t}$

Euler Formula; Complex Valued Solutions

- **Euler's formula**:

$$e^{it} = \cos t + i \sin t$$

Then

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left[\cos \mu t + i \sin \mu t\right] = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

- Therefore

$$y_1(t) = e^{(\lambda+i\mu)t} = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$
$$y_2(t) = e^{(\lambda-i\mu)t} = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

Real Valued Solutions

$$y_1(t) = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

$$y_2(t) = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

- We want real-valued solutions, since our differential equation has real coefficients.
- Therefore, consider linear combinations of solutions as follows:

$$y_1(t) + y_2(t) = 2e^{\lambda t} \cos \mu t$$

$$y_1(t) - y_2(t) = 2ie^{\lambda t} \sin \mu t$$

$$\Rightarrow y_3(t) = e^{\lambda t} \cos \mu t, \;\; y_4(t) = e^{\lambda t} \sin \mu t$$

Real Valued Solutions: The Wronskian

- We have real-valued functions:

$$y_3(t) = e^{\lambda t}\cos\mu t,\ \ y_4(t) = e^{\lambda t}\sin\mu t$$

- Wronskian is as follows:

$$W = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ e^{\lambda t}\left(\lambda\cos\mu t - \mu\sin\mu t\right) & e^{\lambda t}\left(\lambda\sin\mu t + \mu\cos\mu t\right) \end{vmatrix}$$
$$= \mu e^{2\lambda t} \neq 0$$

- Therefore, y_3 and y_4 form a fundamental solution as follows:

$$y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t$$

Example 1

Consider the equation

$$y'' + 2y' + 4y = 0$$

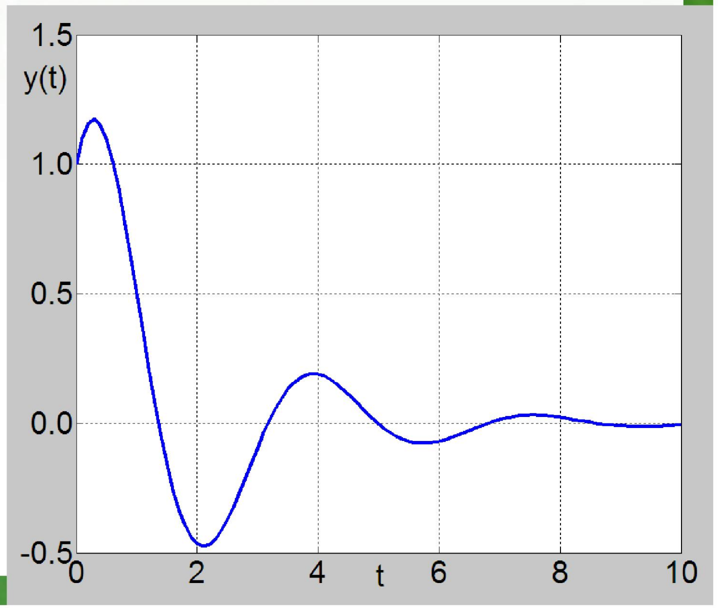
Assume exponential solution and characteristic equation :

$$y(t) = e^{rt} \;\Rightarrow\; r^2 + 2r + 4 = 0 \;\Leftrightarrow\; r = \frac{-2 \pm \sqrt{2^2 - 16}}{2} = \frac{-2 \pm \sqrt{12}\, i}{2} = -1 \pm \sqrt{3}\, i$$

Therefore,

$$y(t) = e^{-t/2}\cos\left(\sqrt{3}t\right) + e^{-t/2}\sin\left(\sqrt{3}t\right)$$

$$\lambda = -1,\; \mu = \sqrt{3}$$



The general solution is :

$$y(t) = c_1 e^{-t/2}\cos\left(\sqrt{3}t\right) + c_2 e^{-t/2}\sin\left(\sqrt{3}t\right)$$

Example 2

Consider the equation

$$y'' + 9y = 0$$

Assume exponential solution and characteristic equation :

$$y(t) = e^{rt} \quad \Rightarrow \quad r^2 + 4 = 0 \;\Leftrightarrow\; r = \pm 2i$$

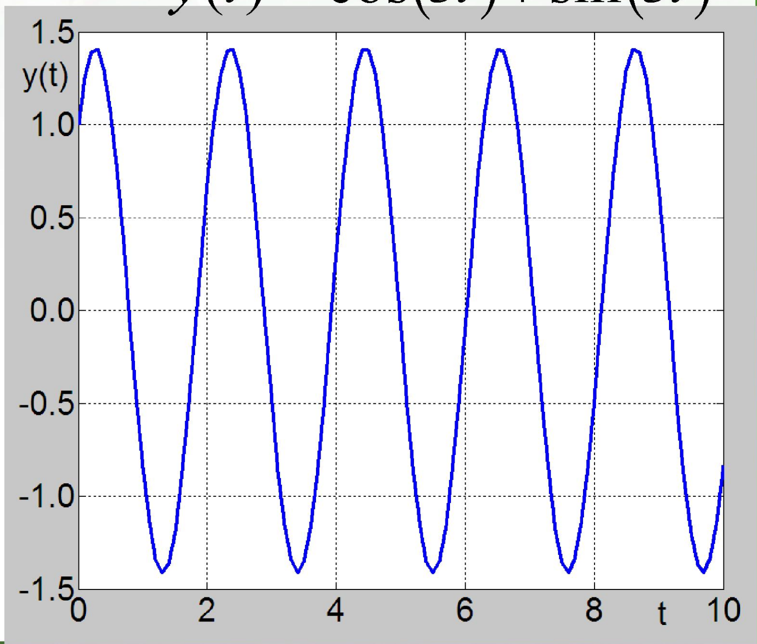
Therefore

$$\lambda = 0, \; \mu = 3$$

The general solution is :

$$y(t) = c_1 \cos(3t) + c_2 \sin(3t).$$

$$y(t) = \cos(3t) + \sin(3t)$$



Example 3

Consider the equation

$$4y'' - 3y' + y = 0.$$

Assume exponential solution and characteristic equation : :

$$y(t) = e^{rt} \Rightarrow 4r^2 - 3r + 1 = 0 \Leftrightarrow r = \frac{3 \pm \sqrt{9-16}}{8} = \frac{3}{8} \pm \frac{\sqrt{7}}{8} i$$

Therefore the general solution is

$$y(t) = c_1 e^{3t/8} \cos\left(\sqrt{7}t/8\right) + c_2 e^{3t/8} \sin\left(\sqrt{7}t/8\right)$$

$$y(t) = e^{3t/8}\cos\left(\sqrt{7}t/8\right) + e^{3t/8}\sin\left(\sqrt{7}t/8\right)$$