Bessel's Equation

- Bessel Equation of order n :

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad (1)$$

The general solution of this equation (1) is

$$y = AJ_n(x) + BY_n(x). \quad (2)$$

Modified Bessel Equation 1

- Modified Bessel Equation 1

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0 \tag{3}$$

Let $t = \lambda x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \lambda\frac{dy}{dt}, \tag{4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\lambda\frac{dy}{dt}\right) = \lambda^2\frac{d^2y}{dt^2}. \tag{5}$$

Modified Bessel Equation 1

Substituting the above Eqs. (4) and (5) into Eq. (3), we obtain

$$\frac{t^2}{\lambda^2}\lambda^2\frac{d^2y}{dx^2}+\frac{t}{\lambda}\lambda\frac{dy}{dx}+(t^2-n^2)y=0.$$

$$t^2\frac{d^2y}{dx^2}+t\frac{dy}{dx}+(t^2-n^2)y=0. \qquad (6)$$

By the general solution (2),

$$y=AJ_n(x)+BY_n(x)$$

we obtain the solution of Eq. (6)

$$y=AJ_n(t)+BY_n(t).$$

Modified Bessel Equation 1

After substituting $t = \lambda x$, the above equation becomes

$$y = AJ_n(\lambda x) + BY_n(\lambda x). \tag{7}$$

Modified Bessel Equation 2

- Modified Bessel Equation 2

$$x^2\frac{d^2y}{dx^2} + (1-2\alpha)x\frac{dy}{dx} + \left\{\beta^2\gamma^2x^{2\gamma} + (\alpha^2 - n^2\gamma^2)\right\}y = 0 \qquad (8)$$

Let $y = x^\alpha z$. Then

$$\frac{dy}{dx} = \alpha x^{\alpha-1}z + x^\alpha\frac{dz}{dx}, \qquad (9)$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \alpha(\alpha-1)x^{\alpha-2}z + \alpha x^{\alpha-1}\frac{dz}{dx} + \alpha x^{\alpha-1}\frac{dz}{dx} + x^\alpha\frac{d^2z}{dx^2} \\ &= x^\alpha\frac{d^2z}{dx^2} + 2\alpha x^{\alpha-1}\frac{dz}{dx} + \alpha(\alpha-1)x^{\alpha-2}z. \qquad (10)\end{aligned}$$

Modified Bessel Equation 1

Substituting the above Eqs. (4) and (5) into Eq. (3), we obtain

$$\frac{t^2}{\lambda^2}\lambda^2\frac{d^2y}{dx^2} + \frac{t}{\lambda}\lambda\frac{dy}{dx} + (t^2 - n^2)y = 0.$$

$$t^2\frac{d^2y}{dx^2} + t\frac{dy}{dx} + (t^2 - n^2)y = 0. \qquad (6)$$

By the general solution (2),

$$y = AJ_n(x) + BY_n(x)$$

we obtain the solution of Eq. (6)

$$y = AJ_n(t) + BY_n(t).$$

Modified Bessel Equation 2

Substituting the above Eqs. (9) and (10) into Eq. (8), we obtain

$$x^{\alpha+2}\frac{d^2z}{dx^2} + 2\alpha x^{\alpha+1}\frac{dz}{dx} + \alpha(\alpha-1)x^{\alpha}z + \alpha(1-2\alpha)x^{\alpha}z$$
$$+(1-2\alpha)x^{\alpha+1}\frac{dz}{dx} + \left\{\beta^2\gamma^2x^{2\gamma} + (\alpha^2 - n^2\gamma^2)\right\}x^{\alpha}z = 0.$$

$$x^2\frac{d^2z}{dx^2} + \left[2\alpha x + (1-2\alpha)x\right]\frac{dz}{dx}$$
$$+\left[\alpha(\alpha-1) + \alpha(1-2\alpha) + \beta^2\gamma^2x^{2\gamma} + \alpha^2 - n^2\gamma^2\right]z = 0.$$

The above equation is rewritten

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2)z = 0.$$

Modified Bessel Equation 2

Now, let $t = x^\gamma$. Then $dt = \gamma x^{\gamma-1} dx$. And

$$\frac{dz}{dx} = \frac{dz}{dt}\frac{dt}{dx} = \gamma x^{\gamma-1}\frac{dz}{dt}.$$

$$\frac{d^2z}{dt^2} = \gamma(\gamma-1)x^{\gamma-2}\frac{dz}{dt} + \gamma x^{\gamma-1}\frac{d^2z}{dt^2}\gamma x^{\gamma-1}$$

$$= \gamma^2 x^{2\gamma-2}\frac{d^2z}{dt^2} + \gamma(\gamma-1)x^{\gamma-2}\frac{dz}{dt}.$$

After substituting the above derivatives, and simplifying, Eq. (11) becomes

$$\gamma^2 x^{2\gamma}\frac{d^2z}{dt^2} + \gamma(\gamma-1)x^{\gamma}\frac{dz}{dt} + \gamma x^{\gamma}\frac{dz}{dt} + (\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2)z = 0.$$

Modified Bessel Equation 2

Dividing by γ^2 gives

$$x^{2\gamma}\frac{d^2z}{dt^2}+x^{\gamma}\frac{dz}{dt}+(\beta^2x^{2\gamma}-n^2)z=0.$$

By substituting $t=x^{2\gamma}$ in the above equation we obtain

$$t^2\frac{d^2z}{dt^2}+t\frac{dz}{dt}+(\beta^2t^2-n^2)z=0. \quad (12)$$

Because of the same form of **[Modified Bessel Equation 1]**, Eq. (12) has the following solution

$$z=AJ_n(\beta t)+BY_n(\beta t)$$

Modified Bessel Equation 2

Therefore, the solution of Eq. (8)

$$x^2\frac{d^2y}{dx^2} + (1-2\alpha)x\frac{dy}{dx} + \left\{\beta^2\gamma^2x^{2\gamma} + (\alpha^2 - n^2\gamma^2)\right\} y = 0$$

is

$$y = Ax^\alpha J_n(\beta t) + Bx^\alpha Y_n(\beta t).$$