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On stochastic partial differential equations with variable coefficients in C^1 domains

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Abstract

Stochastic partial differential equations with variable coefficients are considered in C^1 domains. Existence and uniqueness results are given in Sobolev spaces with weights allowing the derivatives of the solutions to blow up near the boundary. The number of derivatives of the solution can be negative and fractional, and the coefficients of the equations are allowed to substantially oscillate or blow up near the boundary.

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1. Introduction

The main goal of this article is to extend the results of Kim and Krylov (2004a) to multi-dimensional cases. We are dealing with an L_p-theory of the following Itô stochastic partial differential equations (SPDEs) in a domain $G \subset \mathbb{R}^d$:

$$du = (a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f)\,dt + (\sigma^{ik}u_{x^i} + v^ku + g^k)\,dw_t^k, \quad t > 0. \tag{1.1}$$

Here w_t^k are independent one-dimensional Wiener processes, i and j go from 1 to d, and k runs through $1, 2, \ldots$. The coefficients a^{ij}, b^i, c, σ^{ik}, v^k and the free terms f, g^k are random functions depending on (t,x). As mentioned in Krylov and Lototsky (1999), such equations with a finite number of the processes w_t^k appear, for instance,

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in nonlinear filtering problems for partially observable diffusions, and considering infinitely many w_t^k is instrumental in treating equations for measure-valued processes, for instance, driven by space–time white noise (see Krylov, 1999a).

Our approach is based on Sobolev spaces with weights allowing the derivatives of the solution to blow up near the boundary of G, and our goal is to prove existence and uniqueness theorems in Sobolev classes with fractional positive or negative number of derivatives summable to the power $p \geqslant 2$. The motivation for such setting is discussed at length in Krylov (1999a) and Krylov and Lototsky (1999). We only mention that, unless certain compatibility conditions (see, for instance, Flandoli, 1990) are fulfilled, the derivatives of the solution may blow up near the boundary, and this blow-up can be controlled with the help of appropriate weights.

In Krylov (1999a) and Krylov and Lototsky (1999), equations of type (1.1) are considered either in the whole space with variable coefficients or in half spaces with constant coefficients. Quite often (although not always) in the theory of partial differential equations, once we know how to solve equations with constant coefficients in the whole space and in half spaces, then constructing a solvability theory even for nonlinear equations with variable coefficients becomes a standard and rather unrewarding task, especially if one is satisfied with quasi-linear equations and somewhat sloppy assumptions on smoothness of the coefficients. The case of L_p-theory of SPDEs turns out to be one more exception to the usual situation if we want only to impose almost necessary conditions. Actually, in Krylov (1994) we already saw this even for $p=2$ and nonnegative integral number of derivatives. It is also worth noting that, if there are no stochastic terms in (1.1), the corresponding L_p-theory is developed in Kim and Krylov (2004b).

A version of L_p-solvability theory ($p \geqslant 2$) for Eq. (1.1) with "uniformly" continuous leading coefficients in sufficiently smooth domains in $\mathbb{R}^d$ is presented in Lototsky (1999). However, while reading somewhat sketchy proofs in Lototsky (1999) we could not reconstruct the argument based on renormalization of spaces and came to the conclusion that the argument may be wrong. The trouble we had is that the renormalization may dramatically change constants in estimates for equations with constant coefficients. In this article we give independent proofs in much more general situations.

Our main results are stated in Section 2 and consist of Theorems 2.9 and 2.10, on solvability of SPDEs in domains and half space, respectively. Notice that in Theorem 2.9 we only consider bounded domains, however, actually the result is also true for the domains G which are uniformly C^1 smooth in a natural sense. It is assumed usually in L_p-theory of parabolic partial differential equations that the leading coefficients are continuous in the closure of the domain. But in our results the coefficients are only assumed to be measurable in (ω, t) and may substantially oscillate near the boundary. For instance, if $G = \mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$, then we allow a to behave near $x=0$ like $2+\sin(|\ln x|^\alpha), \alpha \in (0,1)$ (see Remark 2.4).

In Section 3, we prove some auxiliary results, and in Section 4 we investigate the solvability in half spaces and prove Theorem 2.10. Finally, in Section 5 we prove Theorem 2.9.

We finish the Introduction with some notations. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x=(x^1,\dots,x^d)$, $B_r(x) = \{y \in \mathbb{R}^d : |x-y| < r\}$. For $i=1,\dots,d$, multi-indices $\alpha=(\alpha_1,\dots,\alpha_d)$, $\alpha_i\in\{0,1,2,\dots\}$, and functions $u(x)$ we set

$$u_{x^i}=\partial u/\partial x^i=D_i u, \quad D^\alpha u=D_1^{\alpha_1}\cdot\dots\cdot D_d^{\alpha_d}u, \quad |\alpha|=\alpha_1+\dots+\alpha_d.$$

2. Main results

Let $(\Omega,\mathscr{F},P)$ be a complete probability space, and $\{\mathscr{F}_t,t\geqslant 0\}$ be an increasing filtration of σ-fields $\mathscr{F}_t\subset\mathscr{F}$, each of which contains all $(\mathscr{F},P)$-null sets. By $\mathscr{P}$ we denote the predictable σ-field generated by $\{\mathscr{F}_t,t\geqslant 0\}$ and we assume that on Ω we are given independent one-dimensional Wiener processes $w_t^1,w_t^2,\dots$, each of which is a Wiener process relative to $\{\mathscr{F}_t,t\geqslant 0\}$.

Let G be an open set in $\mathbb{R}^d, G\neq\mathbb{R}^d$. We are going to consider the equation

$$\begin{aligned}\mathrm{d}u(t,x)=&(a^{ij}(t,x)u_{x^ix^j}(t,x)+b^i(t,x)u_{x^i}+c(t,x)u(t,x)+f(t,x))\,\mathrm{d}t\\&+\sum_{k=1}^{\infty}(\sigma^{ik}(t,x)u_{x^i}(t,x)+v^k(t,x)u(t,x)+g^k(t,x))\,\mathrm{d}w_t^k,\end{aligned}\tag{2.1}$$

where a^{ij},b^i,c,f are real-valued, and σ^i,v,g are ℓ_2-valued functions defined for $\omega\in\Omega$, $t\geqslant 0,x\in G$.

Fix an increasing function κ_0 defined on $[0,\infty)$ such that $\kappa_0(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$.

Assumption 2.1. The domain $G\subset\mathbb{R}^d$ is of class C_u^1. In other words, there exist constants $r_0,K_0>0$ such that for any $x_0\in\partial G$ there exists a one-to-one continuously differentiable mapping Ψ from $B_{r_0}(x_0)$ onto a domain $J\subset\mathbb{R}^d$ such that

(i) $J_+:=\Psi(B_{r_0}(x_0)\cap G)\subset\mathbb{R}_+^d:=\{x\in R^d:x^1>0\}$ and $\Psi(x_0)=0$;
(ii) $\Psi(B_{r_0}(x_0)\cap\partial G)=J\cap\{y\in\mathbb{R}^d:y^1=0\}$;
(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))}\leqslant K_0$ and $|\Psi^{-1}(y_1)-\Psi^{-1}(y_2)|\leqslant K_0|y_1-y_2|$ for any $y_i\in J$;
(iv) $|\Psi_x(x_1)-\Psi_x(x_2)|\leqslant\kappa_0(|x_1-x_2|)$ for any $x_i\in B_{r_0}(x_0)$.

To state our assumptions on the coefficients, we take some notations from Kim and Krylov (2004b). Denote $\rho(x)=\rho_G(x)=\mathrm{dist}(x,\partial G)$, and $\rho(x,y)=\rho_G(x,y)=\rho(x)\wedge\rho(y)$. For $\sigma\in\mathbb{R}$, $\alpha\in(0,1)$, and $k=0,1,2,\dots$, as in Douglis and Nirenberg (1955) and Gilbarg and Trudinger (1983), define

$$[f]_k^{(\sigma)}=[f]_{k,G}^{(\sigma)}=\sup_{\substack{x\in G\\|\beta|=k}}\rho^{k+\sigma}(x)|D^\beta f(x)|,$$

$$[f]_{k+\alpha}^{(\sigma)}=[f]_{k+\alpha,G}^{(\sigma)}=\sup_{\substack{x,y\in G\\|\beta|=k}}\rho^{k+\alpha+\sigma}(x,y)\frac{|D^\beta f(x)-D^\beta f(y)|}{|x-y|^\alpha},$$

$$|f|_k^{(\sigma)}=|f|_{k,G}^{(\sigma)}=\sum_{j=0}^{k}[f]_{j,G}^{(\sigma)}, \quad |f|_{k+\alpha}^{(\sigma)}=|f|_{k+\alpha,G}^{(\sigma)}=|f|_{k,G}^{(\sigma)}+[f]_{k+\alpha,G}^{(\sigma)}.$$

By $D^\beta f$ we mean either classical derivatives or Sobolev ones and in the latter case sup's in the above are understood as ess sup's. We also use the same notations for ℓ_2-valued functions.

Fix a function $\delta_0(\tau) \geqslant 0$ defined on $[0,\infty)$ such that $\delta_0(\tau) > 0$ unless $\tau \in \{0,1,2,\dots\}$. For $\tau \geqslant 0$ define

$$\tau + = \tau + \delta_0(\tau)$$

and fix some constants

$$\delta, K \in (0,\infty), \quad \gamma \in \mathbb{R}, \quad p \in [2,\infty).$$

Assumption 2.2(γ). (i) For each $x \in G$, the functions $a^{ij}(t,x)$, $b^i(t,x)$, $c(t,x)$, $\sigma^{ik}(t,x)$ and $v^k(t,x)$ are predictable functions of (ω,t).

(ii) For any x, t, ω and $\lambda \in \mathbb{R}^d$,

$$(a^{ij}(t,x) - \alpha^{ij}(t,x))\lambda^i\lambda^j \geqslant \delta|\lambda|^2, \tag{2.2}$$

where $\alpha^{ij} = \frac{1}{2}\sigma^{ik}\sigma^{jk}$.

(iii) For any $t > 0$ and $\omega \in \Omega$,

$$|a^{ij}(t,\cdot)|^{(0)}_{|\gamma|+} + |b^i(t,\cdot)|^{(1)}_{|\gamma|+} + |c(t,\cdot)|^{(2)}_{|\gamma|+} + |\sigma^i(t,\cdot)|^{(0)}_{|\gamma+1|+} + |v(t,\cdot)|^{(1)}_{|\gamma+1|+} \leqslant K.$$

Assumption 2.3. (i) The functions $a^{ij}(t,\cdot)$, $\sigma^i(t,\cdot)$ are continuous at any point $x \in G$ uniformly with respect to (t,ω).

(ii) There is control on the behavior of a^{ij}, b^i, c, σ^i and v near ∂G, namely,

$$\lim_{\substack{\rho(x)\to 0\\ x\in G}} \sup_{\substack{y\in G\\ |x-y|\leqslant\rho(x,y)}} \sup_{t,\omega} [|a^{ij}(t,x) - a^{ij}(t,y)| + |\sigma^i(t,x) - \sigma^i(t,x)|] = 0. \tag{2.3}$$

$$\lim_{\substack{\rho(x)\to 0\\ x\in G}} \sup_{t,\omega} [\rho(x)|b^i(t,x)| + \rho^2(x)|c(t,x)| + \rho(x)|v(t,x)|] = 0. \tag{2.4}$$

Remark 2.4. In Lototsky (1999), the uniform continuity of $a(t,x)$ and $\sigma(t,x)$ in G is assumed instead of (2.3). It is easy to see that this is much stronger than (2.3). For instance, if $\delta \in (0,1)$, $d=1$, and $G = \mathbb{R}_+$, then the function $a(x)$ equal to $2+\sin(|\ln x|^\delta)$ for $0 < x \leqslant \frac{1}{2}$ satisfies (2.3).

Indeed, if $x, y > 0$ and $|x-y| \leqslant x \wedge y$, then

$$|a(x) - a(y)| = |x-y||a'(\xi)|,$$

where ξ lies between x and y. In addition, $|x-y| \leqslant x \wedge y \leqslant \xi \leqslant 2(x \wedge y)$, and $\xi|a'(\xi)| \leqslant |\ln[2(x\wedge y)]|^{\delta-1} \to 0$ as $x \wedge y \to 0$.

The function $a(x)$ also satisfies Assumption 2.2 for any γ if we change it appropriately for $x > \frac{1}{2}$. Also observe that (2.4) allows the coefficients b^i, c and v to blow up near the boundary at a certain rate.

To proceed further we introduce some well-known results from Gilbarg and Hörmander (1980) and Kim and Krylov (2004b) (also, see Lapic, 1994 for details).

Lemma 2.5. *Let the domain G be of class C^1_u. Then*

(i) *there is a bounded real-valued function ψ defined in $\bar{G}$ such that for any multi-index α,*

$$\sup_G \rho^{|\alpha|}(x)|D^\alpha \psi_x(x)| < \infty \tag{2.5}$$

and the functions ψ and ρ are comparable in the part of a neighborhood of ∂G lying in G. In other words, if $\rho(x)$ is sufficiently small then $N^{-1}\rho(x) \leqslant \psi(x) \leqslant N\rho(x)$ with some constant N independent of x,

(ii) *the function Ψ in Assumption 2.1 can be chosen in such a way that for any nonnegative integer n*

$$|\Psi_x|^{(0)}_{n,B_{r_0}(x_0)\cap G} + |\Psi_x^{-1}|^{(0)}_{n,J_+} < N(n) < \infty \tag{2.6}$$

and

$$\rho(x)\Psi_{xx}(x) \to 0 \text{ as } x \in B_{r_0}(x_0)\cap G \quad \text{and } \rho(x) \to 0, \tag{2.7}$$

where the constants $N(n)$ and the convergence in (2.7) are independent of x_0.

To describe the assumptions of f and g we use the Banach spaces introduced in Kim and Krylov (2004b) and Lototsky (2000). Let $\xi \in C_0^\infty(\mathbb{R}_+)$ be a function satisfying

$$\sum_{n=-\infty}^{\infty} \xi(\mathrm{e}^{n+t}) > 0 \quad \forall t \in \mathbb{R}.$$

For $x \in G$ and $n \in \mathbb{Z} = \{0, \pm 1, \dots\}$ define

$$\zeta_n(x) = \xi(\mathrm{e}^n \psi(x)).$$

Then we have $\sum_n \zeta_n \geqslant \text{const} > 0$ in G and

$$\zeta_n \in C_0^\infty(G), \quad |D^m \zeta_n(x)| \leqslant N(m)\mathrm{e}^{mn}.$$

For $\theta, \gamma \in \mathbb{R}$, let $H^\gamma_{p,\theta}(G)$ be the set of all distributions u on G such that

$$\|u\|^p_{H^\gamma_{p,\theta}(G)} := \sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta} \|\zeta_{-n}(\mathrm{e}^n \cdot)u(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p} < \infty, \tag{2.8}$$

where $H^\gamma_p = H^\gamma_p(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2}L_p$ is the space of Bessel potential. This definition is also used for ℓ_2-valued function $g = (g^1, g^2, \dots)$, in which case,

$$\|g\|^p_{H^\gamma_{p,\theta}(G)} = \sum_{n\in\mathbb{Z}} \mathrm{e}^{n\theta} \|\zeta_{-n}(\mathrm{e}^n \cdot)g(\mathrm{e}^n \cdot)\|^p_{H^\gamma_p(\ell_2)},$$

where $\|h\|_{H^\gamma_{p,\theta}(\ell_2)} := \| |(1-\Delta)^{\gamma/2}h|_{\ell_2} \|_{L_p}$ for any $h \in H^\gamma_{p,\theta}(\ell_2)$.

It is known (see, for instance, Lototsky (2000)) that up to equivalent norms the space $H^\gamma_{p,\theta}(G)$ is independent of the choice of ξ and ψ if G is bounded. Moreover if $\gamma = n$ is a nonnegative integer then

$$\|u\|^p_{H^\gamma_{p,\theta}(G)} \sim \sum_{i=0}^{n} \sum_{|\alpha|=i} \int_G |D^\alpha u(x)|^p \rho^{\theta-d+ip}(x)\,\mathrm{d}x.$$

For any $\mathscr{F}_t$-stopping time τ, denote

$$\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)=L_p((|0,\tau]\!], \mathscr{P}, H^{\gamma}_{p,\theta}(G)),$$

$$U^{\gamma}_{p,\theta}(G)=\psi^{1-2/p}L_p(\Omega,\mathscr{F}_0,H^{\gamma-2/p}_{p,\theta}(G)),\qquad \mathbb{L}_{p,\theta}(G,\tau)=\mathbb{H}^0_{p,\theta}(G,\tau)$$

and by $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ we denote the space of all functions $u\in\psi\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)$ such that $u(0,\cdot)\in U^{\gamma}_{p,\theta}(G)$ and for some $f\in\psi^{-1}\mathbb{H}^{\gamma-2}_{p,\theta}(G,\tau)$, $g\in\mathbb{H}^{\gamma-1}_{p,\theta}(G,\tau)$,

$$\mathrm{d}u=f\,\mathrm{d}t+g^k\,\mathrm{d}w^k_t,$$

in the sense of distributions. In other words, for any $\phi\in C^{\infty}_0(G)$, the equality

$$(u(t,\cdot),\phi)=(u(0,\cdot),\phi)+\int_0^t (f(s,\cdot),\phi)\,\mathrm{d}s+\sum_{k=1}^{\infty}\int_0^t (g^k(s,\cdot),\phi)\,\mathrm{d}w^k_s$$

holds for all $t\leqslant\tau$ with probability 1. Let

$$\mathfrak{H}^{\gamma}_{p,\theta,0}(G,\tau)=\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)\cap\{u:u(0,\cdot)=0\}.$$

The norm in $\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta}(G,\tau)}=\|\psi^{-1}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)}+\|\psi f\|_{\mathbb{H}^{\gamma-2}_{p,\theta}(G,\tau)}$$
$$+\|g\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(G,\tau)}+\|u(0,\cdot)\|_{U^{\gamma}_{p,\theta}(G)}.$$

In the case $G=\mathbb{R}^d_+$, we also use the Banach spaces $H^{\gamma}_{p,\theta}$, $\mathbb{H}^{\gamma}_{p,\theta}(\tau)$, $\mathfrak{H}^{\gamma}_{p,\theta}(\tau)$ and $\mathfrak{H}^{\gamma}_{p,\theta,0}(\tau)$ introduced in Krylov (1999b). They are defined on the basis of (2.8) by formally taking $\psi(x)=x^1$, so that $\zeta_{-n}(\mathrm{e}^n x)=\xi(x^1)=:\zeta(x)$ and

$$\|u\|^p_{H^{\gamma}_{p,\theta}}:=\sum_{n\in\mathbb{Z}}\mathrm{e}^{n\theta}\|u(\mathrm{e}^n\cdot)\zeta\|^p_{H^{\gamma}_p}<\infty.$$

Observe that the spaces $H^{\gamma}_{p,\theta}(\mathbb{R}^d_+)$ and $H^{\gamma}_{p,\theta}$ are different since ψ is bounded. We drop τ in the notations of appropriate Banach spaces if $\tau\equiv\infty$.

From this point on, we assume that

$$d-1<\theta<d-1+p. \tag{2.9}$$

As in Krylov (1999b), by M^{α} we denote the operator of multiplying by $(x^1)^{\alpha}$ and $M=M^1$. We denote $\mathscr{M}_{d\times d},\mathscr{M}_{d\times\infty}$ the set of all real-valued $d\times d$, $d\times\infty$-matrices, respectively. For $a\in\mathscr{M}_{d\times d},\sigma\in\mathscr{M}_{d\times\infty}$, define $|a|,|\sigma|$ from

$$|a|^2=\sum_{i,j=1}^{d}(a^{ij})^2,\qquad |\sigma|^2=\sum_{i=1}^{d}\sum_{k=1}^{\infty}(\sigma^{ik})^2.$$

Finally we denote I the $d\times d$ identity matrix.

Definition 2.6. Let $\mathscr{A}$ be a set of (a,σ), where $a=(a^{ij})\in\mathscr{M}_{d\times d}$ and $\sigma=(\sigma^{ik})\in\mathscr{M}_{d\times\infty}$. We call $\mathscr{A}$ to be of $\mathscr{A}_{p,\theta}$-type if

(i) $(I,0)\in\mathscr{A}$ and

$$(a,\sigma)\in\mathscr{A},\lambda\in(0,1)\Rightarrow\lambda(a,\sigma)+(1-\lambda)(I,0)\in\mathscr{A};$$

(ii) the set $\mathscr{A}$ is invariant under rotation, that is, for any orthogonal matrix $O \in \mathscr{M}_{d\times d}$,

$$(a,\sigma)\in\mathscr{A} \Rightarrow (OaO^*, O\sigma)\in\mathscr{A};$$

(iii) for any $v\in\mathbb{R}$ and any bounded predictable $\mathscr{A}$-valued function $(a,\sigma)=(a(t),\sigma(t))=(a(\omega,t),\sigma(\omega,t))$, $f\in M^{-1}\mathbb{H}^{v}_{p,\theta}$, and $g=(g^1,g^2,\ldots)\in\mathbb{H}^{v+1}_{p,\theta}$, Eq. (2.1) with $a(t,x)=a(t)$, $\sigma(t,x)=\sigma(t)$ and $b^i=c=v^k=0$ admits a unique solution u in the class $\mathfrak{H}^{v+2}_{p,\theta,0}$;

(iv) there exists a finite function $N_0(v)$, $v\in\mathbb{R}$, such that for the solution from (iii) we have

$$\|M^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}} \leqslant N_0(v)(\|Mf\|^p_{\mathbb{H}^{v}_{p,\theta}} + \|g\|^p_{\mathbb{H}^{v+1}_{p,\theta}}). \tag{2.10}$$

Remark 2.7. The widest $\mathscr{A}_{p,\theta}$-type sets known so far are given in Krylov and Lototsky (1999). For $\delta_1,\delta_2\in(0,1]$, let us denote B_{δ_1,δ_2} the set of all (a,σ) such that for any $\lambda\in\mathbb{R}^d$,

$$\delta_1|\lambda|^2 \leqslant \delta_2 a^{ij}\lambda^i\lambda^j \leqslant \bar{a}^{ij}\lambda^i\lambda^j \leqslant a^{ij}\lambda^i\lambda^j \leqslant \delta_1^{-1}|\lambda|^2, \tag{2.11}$$

where $\bar{a}^{ij}=a^{ij}-\frac{1}{2}\sigma^{ik}\sigma^{jk}$. It turns out that if p and θ satisfy

$$d-1+p\left[1-\frac{1}{p(1-\delta_2)+\delta_2}\right] < \theta < d-1+p \tag{2.12}$$

then the set B_{δ_1,δ_2} is of $\mathscr{A}_{p,\theta}$-type. If $\sigma\equiv 0$ then one can take $\delta_2=1$ and then (2.12) becomes $d-1<\theta<d-1+p$. But in general B_{δ_1,δ_2} is not of $\mathscr{A}_{p,\theta}$-type, so in addition to (2.9) one needs to impose stronger restrictions on θ like (2.12). Note that (2.12) is satisfied for any $\delta_2\in(0,1]$ if $d-2+p\leqslant\theta<d-1+p$.

Assumption 2.8. There exists a set $\mathscr{A}$ of $\mathscr{A}_{p,\theta}$-type such that

$$(a(t,x),\sigma(t,x))\in\mathscr{A} \quad \forall\omega,t,x.$$

Here is our main result.

Theorem 2.9. *Let G be bounded and $\tau\leqslant T$, for some constant $T\in[0,\infty)$. Then under the above assumptions,*

(i) *for any $f\in\psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)$, $g\in\mathbb{H}^{\gamma+1}_{p,\theta}(G,\tau)$ and $u_0\in U^{\gamma+2}_{p,\theta}(G)$ Eq. (2.1) with initial data u_0 admits a unique solution u in the class $\mathfrak{H}^{\gamma+2}_{p,\theta}(G,\tau)$,*

(ii) *for this solution*

$$\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,\tau)} \leqslant N(\|u_0\|_{U^{\gamma+2}_{p,\theta}(G)} + \|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,\tau)} + \|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,\tau)}), \tag{2.13}$$

where the constant N is independent of f,g and u_0.

We will see that the proof of Theorem 2.9 is based on the following result for $\mathbb{R}^d_+$, in which Assumption 2.8 is relaxed to an assumption that (a,σ) is sufficiently close to the set $\mathscr{A}$ instead of belonging to it.

Theorem 2.10. *Let* $G = \mathbb{R}^d_+$, $\beta, \tilde{\beta} \in (0,\infty)$ *and* $\mathscr{A}$ *be a set of* $\mathscr{A}_{p,\theta}$*-type. Let Assumption* 2.2(γ) *be satisfied and let, for each* x, $(\tilde{a}^{ij}(t,x), \tilde{\sigma}^{ik}(t,x))$ *be a bounded* $\mathscr{A}$*-valued predictable function of* (ω, t) *such that*

$$\sup_{\omega,t,x} (|\tilde{a}(t,x) - a(t,x)| + |\tilde{\sigma}(t,x) - \sigma(t,x)|) \leqslant \tilde{\beta}.$$

Also (*instead of Assumption* 2.3) *assume*

$$|a^{ij}(t,x) - a^{ij}(t,y)| + x^1|b^i(t,x)| + (x^1)^2|c(t,x)|$$
$$+|\sigma^i(t,x) - \sigma^i(t,y)| + x^1|\nu(t,x)| \leqslant \beta \tag{2.14}$$

whenever $t > 0$, $x, y \in \mathbb{R}^d_+$, *and* $|x - y| \leqslant x^1 \wedge y^1$.

Then there exist $\tilde{\beta}_0 = \tilde{\beta}_0(\gamma, N_0(\gamma)) > 0$ *and* $\beta_0 \in (0,1)$ *depending only on* N_0, δ_0, p, δ, θ, γ *and* K, *such that, if*

$$\beta \leqslant \beta_0, \qquad \tilde{\beta} \leqslant \tilde{\beta}_0, \tag{2.15}$$

then

(i) *for any* $f \in M^{-1}\mathbb{H}^\gamma_{p,\theta}(\tau), g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\tau)$ *and* $u_0 \in U^{\gamma+2}_{p,\theta}$ *Eq.* (2.1) *with initial data* u_0 *admits a unique solution* u *in the class* $\mathfrak{H}^{\gamma+2}_{p,\theta}(\tau)$,

(ii) *for this solution*

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\tau)} \leqslant N(\|u_0\|_{U^{\gamma+2}_{p,\theta}} + \|Mf\|_{\mathbb{H}^\gamma_{p,\theta}(\tau)} + \|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau)}), \tag{2.16}$$

where the constant N *depends only on* $N_0, p, \delta, \theta, \gamma, \delta_0$ *and* K.

3. Auxiliary results

Here we extend properties (iii) and (iv) in Definition 2.6 and introduce results about partitions of unity and pointwise multipliers.

For any unit vector $\xi \in \mathbb{R}^d$, denote $\mathbb{R}^d_+(\xi) = \{x \in \mathbb{R}^d : x \cdot \xi > 0\}$, $\mathbb{R}^d_+ = \mathbb{R}^d_+(e_1)$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. We define Banach space $H^\gamma_{p,\theta}(\xi)$ corresponding to $H^\gamma_{p,\theta}$ as follows. Let O be an orthogonal matrix which induces a bijective linear mapping from $\mathbb{R}^d_+$ to $\mathbb{R}^d_+(\xi)$. For $\alpha \in \mathbb{R}$ consider operators

$$O : f(x) \to f(Ox), \quad M^\alpha_\xi : f(x) \to (x \cdot \xi)^\alpha f(x).$$

We write $u \in \mathbb{H}^\gamma_{p,\theta}(\xi)$ if and only if $Ou \in H^\gamma_{p,\theta}$ and define

$$\|u\|_{H^\gamma_{p,\theta}(\xi)} = \|Ou\|_{H^\gamma_{p,\theta}}. \tag{3.1}$$

By Lemma 1.9 in Krylov (1999b) it follows that $H^\gamma_{p,\theta}(\xi)$ is independent of O, and the norms in (3.1) constructed from different O are equivalent.

Similarly, we introduce the spaces

$$\mathbb{H}^\gamma_{p,\theta}(\xi,\tau), \quad \mathfrak{H}^\gamma_{p,\theta}(\xi,\tau), \quad \mathfrak{H}^\gamma_{p,\theta,0}(\xi,\tau)$$

which correspond to $\mathbb{H}^\gamma_{p,\theta}(\tau), \mathfrak{H}^\gamma_{p,\theta}(\tau)$ and $\mathfrak{H}^\gamma_{p,\theta,0}(\tau)$, respectively.

Lemma 3.1. *Let $\mathscr{A}$ be a set of $\mathscr{A}_{p,\theta}$-type, and let $(a_0(\omega,t),\sigma_0(\omega,t))$, $(a_1(\omega,t),\sigma_1(\omega,t))$ be bounded $\mathscr{M}_{d\times d}\times\mathscr{M}_{d\times\infty}$-valued predictable functions such that*

$$(a_0(\omega,t),\sigma_0(\omega,t))\in\mathscr{A} \quad \forall\omega,t,$$
$$\sup_{\omega,t}(|a_1(\omega,t)-a_0(\omega,t)|+|\sigma_1(\omega,t)-\sigma_0(\omega,t)|)\leqslant\alpha. \tag{3.2}$$

Then for any $v\in\mathbb{R}$,

(i) *there is $\alpha_0=\alpha_0(v)>0$ depending only on d,p,θ,v,N_0 such that if $\alpha\leqslant\alpha_0$, then for any invertible matrix $P\in\mathscr{M}_{d\times d}$, $f\in M_\xi^{-1}\mathbb{H}^v_{p,\theta}(\xi), g\in\mathbb{H}^{v+1}_{p,\theta}(\xi)$ Eq. (2.1) with coefficients $a(t):=Pa_1(t)P^*,\sigma(t):=P\sigma_1(t)$, $b^i=c=v^k=0$ admits a unique solution u in the class $\mathfrak{H}^{v+2}_{p,\theta,0}(\xi)$,*

(ii) *for this solution,*

$$\|M_\xi^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}(\xi)}\leqslant N(\|M_\xi f\|^p_{\mathbb{H}^v_{p,\theta}(\xi)}+\|g\|^p_{\mathbb{H}^{v+1}_{p,\theta}(\xi)}),$$

where N depends only on $\theta,p,d,v,|P|,|P^{-1}|$ and $N_0(v)$.

Proof. *Step* 1: First assume $P=I,\xi=e_1$. Since the coefficients are independent of x, the result in this case just follows from standard perturbation arguments. Indeed, assume $u\in\mathfrak{H}^{v+2}_{p,\theta,0}$ is a solution of the equation

$$\begin{aligned}\mathrm{d}u&=(a_1^{ij}u_{x^ix^j}+f)\,\mathrm{d}t+(\sigma_1^{ik}u_{x^i}+g^k)\,\mathrm{d}w_t^k\\&=(a_0^{ij}u_{x^ix^j}+(a_1^{ij}-a_0^{ij})u_{x^ix^j}+f)\,\mathrm{d}t+(\sigma_0^{ik}u_{x^i}+(\sigma_1^{ik}-\sigma_0^{ik})u_{x^i}+g^k)\,\mathrm{d}w_t^k.\end{aligned}$$

Then by Definition 2.6(iii),

$$\begin{aligned}\|M^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}}&\leqslant N_0(v)(\|(a_1^{ij}-a_0^{ij})Mu_{x^ix^j}+Mf\|^p_{\mathbb{H}^v_{p,\theta}}+\|(\sigma_1^{ik}-\sigma_0^{ik})u_{x^i}+g\|^p_{\mathbb{H}^{v+1}_{p,\theta}})\\&\leqslant\alpha^p2^{p-1}N_0(v)(\|Mu_{x^ix^j}\|^p_{\mathbb{H}^v_{p,\theta}}+\|u_{x^i}\|^p_{\mathbb{H}^{v+1}_{p,\theta}})\\&\quad+2^{p-1}N_0(v)(\|Mf\|^p_{\mathbb{H}^v_{p,\theta}}+\|g\|^p_{\mathbb{H}^{v+1}_{p,\theta}}).\end{aligned}$$

Moreover we know (see Krylov, 1999b)

$$\|Mu_{xx}\|_{H^v_{p,\theta}}+\|u_x\|_{H^{v+1}_{p,\theta}}\leqslant N\|M^{-1}u\|_{H^{v+2}_{p,\theta}}.$$

Thus,

$$\|M^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}}\leqslant\alpha^pNN_0(v)\|M^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}}+2^{p-1}N_0(v)(\|Mf\|^p_{\mathbb{H}^v_{p,\theta}}+\|g\|^p_{\mathbb{H}^{v+1}_{p,\theta}}). \tag{3.3}$$

Now we choose α_0 such that $\alpha N N_0(v) \leqslant \frac{1}{2}$ for $\alpha \leqslant \alpha_0$. Then from (3.3) we get

$$\|M^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}} \leqslant 2^p N_0(v)(\|Mf\|^p_{\mathbb{H}^v_{p,\theta}} + \|g\|^p_{\mathbb{H}^{v+1}_{p,\theta}}).$$

Note that for any $\lambda \in [0,1]$, $a_\lambda := \lambda a_1 + (1-\lambda)a_0$, $\sigma_\lambda := \lambda\sigma_1 + (1-\lambda)\sigma_0$ satisfies (3.2) since

$$|a_\lambda - a_0| + |\sigma_\lambda - \sigma_0| \leqslant |a_1 - a_0| + |\sigma_1 - \sigma_0| \leqslant \alpha.$$

Therefore the method of continuity yields the result in the case under consideration.

For the rest of the proof we assume that (3.2) holds with α_0 chosen above.

Step 2: Assume P is an orthogonal matrix. Let O be the orthogonal matrix which is used to construct the space $H^\gamma_{p,\theta}(\xi)$. Take $f \in M_\xi^{-1}\mathbb{H}^v_{p,\theta}(\xi)$, $g \in \mathbb{H}^{v+1}_{p,\theta}(\xi)$. Then by (3.1)

$$Of \in M^{-1}\mathbb{H}^v_{p,\theta}, \quad Og \in \mathbb{H}^{v+1}_{p,\theta}.$$

Observe that $u \in \mathfrak{H}^{v+2}_{p,\theta,0}(\xi)$ satisfies

$$\mathrm{d}u = ((Pa_1P^*)^{ij}u_{x^ix^j} + f)\,\mathrm{d}t + ((P\sigma_1)^{ik}u_{x^i} + g^k)\,\mathrm{d}w^k_t \tag{3.4}$$

if and only if $v(t,x) := u(t,Ox) \in \mathfrak{H}^{v+2}_{p,\theta,0}$ and it satisfies

$$\mathrm{d}v = (\tilde{a}^{ij}_1 v_{x^ix^j} + Of)\,\mathrm{d}t + (\tilde{\sigma}_1 v_{x^i} + Og^k)\,\mathrm{d}w^k_t, \tag{3.5}$$

where $(\tilde{a}_1(t), \tilde{\sigma}_1(t)) = (O^*Pa_1(t)P^*O, O^*P\sigma(t))$.

By assumption

$$(O^*Pa_0(t)P^*O, O^*P\sigma_0(t)) \in \mathscr{A} \quad \forall\omega, t$$

and obviously

$$|\tilde{a}_1 - O^*Pa_0(t)P^*O| + |\tilde{\sigma}_1 - O^*P\sigma_0| \leqslant \alpha_0. \tag{3.6}$$

Therefore, by the results of step 1, (3.5) is uniquely solvable in the class $\mathfrak{H}^{v+2}_{p,\theta,0}$, and consequently, the solvability of Eq. (3.4) in the class $\mathfrak{H}^{v+2}_{p,\theta,0}(\xi)$ follows. Also for the solution,

$$\begin{aligned}\|M_\xi^{-1}u\|^p_{\mathbb{H}^{v+2}_{p,\theta}(\xi)} &= \|M^{-1}v\|^p_{\mathbb{H}^{v+2}_{p,\theta}} \leqslant 2^pN_0(v)(\|MOf\|^p_{\mathbb{H}^v_{p,\theta}} + \|Og\|^p_{\mathbb{H}^{v+1}_{p,\theta}})\\ &= 2^pN_0(v)(\|M_\xi f\|^p_{\mathbb{H}^v_{p,\theta}(\xi)} + \|g\|^p_{\mathbb{H}^{v+1}_{p,\theta}(\xi)}).\end{aligned} \tag{3.7}$$

Step 3: Assume that $P = \Lambda$ is a diagonal matrix with positive diagonal entries. Consider the operator

$$\Lambda : h(x) \to h(\Lambda x).$$

Using Corollary 1.6 and Lemma 1.9 in Krylov (1999b) one can easily check that

$$\Lambda f \in M_\xi^{-1}\mathbb{H}^v_{p,\theta}(\xi), \quad \Lambda g \in \mathbb{H}^{v+1}_{p,\theta}(\xi).$$

Thus, by the results of step 2, one can define the solution $v \in \mathfrak{H}^{v+2}_{p,\theta,0}(\xi)$ of

$$\mathrm{d}v = (a^{ij}_1 v_{x^ix^j} + \Lambda f)\,\mathrm{d}t + (\sigma^{ik}_1 v_{x^i} + \Lambda g^k)\,\mathrm{d}w^k_t. \tag{3.8}$$

Also, it is easy to see that $u(t,x)=v(t,\Lambda^{-1}x)$ is a unique solution of (3.4) in the class $\mathfrak{H}^{\nu+2}_{p,\theta,0}(\xi)$, and by Corollary 1.6, Lemma 1.9 in Krylov (1999b) and (3.7),

$$\|M_{\xi}^{-1}u\|^p_{\mathbb{H}^{\nu+2}_{p,\theta}(\xi)} \leqslant N\|M_{\xi}^{-1}v\|^p_{\mathbb{H}^{\nu+2}_{p,\theta}(\xi)}$$

$$\leqslant NN_0(\nu)(\|M_{\xi}\Lambda f\|^p_{\mathbb{H}^{\nu}_{p,\theta}(\xi)} + \|\Lambda g\|^p_{\mathbb{H}^{\nu+1}_{p,\theta}(\xi)})$$

$$\leqslant NN_0(\nu)(\|M_{\xi} f\|^p_{\mathbb{H}^{\nu}_{p,\theta}(\xi)} + \|g\|^p_{\mathbb{H}^{\nu+1}_{p,\theta}(\xi)}),$$

where N depends only on θ,ν,p,d, $|\Lambda|$ and $|\lambda^{-1}|$.

Step 4: Note that in general we can put $P=O_1\Lambda O_2$, where O_i are orthogonal matrices and Λ is a diagonal matrix with positive diagonal entries. Denote $\tilde{\xi}=O_1^*\xi$. As before, $u\in\mathfrak{H}^{\nu+2}_{p,\theta,0}(\xi)$ satisfies (3.4) if and only if $v(t,x):=u(t,O_1x)\in\mathfrak{H}^{\nu+2}_{p,\theta,0}(\tilde{\xi})$ satisfies

$$dv=(\tilde{a}_1^{ij}v_{x^ix^j}+O_1f)\,dt+(\tilde{\sigma}_1^{ik}v_{x^i}+O_1g^k)\,dw_t^k, \tag{3.9}$$

where $\tilde{a}_1(t)=\Lambda O_2a_1(t)O_2^*\Lambda, \tilde{\sigma}_1(t)=\Lambda O_2\sigma_1(t)$.

Observing

$$|O_2a_1O_2^*-O_2a_0O_2^*|+|O_2\sigma_1-O_2\sigma_0|\leqslant\alpha_0,$$

$$(O_2a_0(t)O_2^*,O_2\sigma_0(t))\in\mathscr{A}\quad\forall\omega,t$$

and using the results of step 3, one easily gets the unique solvability of (3.9) in the class $\mathfrak{H}^{\nu+2}_{p,\theta,0}(\tilde{\xi})$ and the unique solvability of (3.4) in the class $\mathfrak{H}^{\nu+2}_{p,\theta,0}(\xi)$. And for the solution

$$\|M_{\xi}^{-1}u\|^p_{\mathbb{H}^{\nu+2}_{p,\theta}(\xi)} \leqslant N\|M_{\tilde{\xi}}^{-1}v\|^p_{\mathbb{H}^{\nu+2}_{p,\theta}(\tilde{\xi})} \leqslant N(\|M_{\tilde{\xi}}O_1f\|^p_{\mathbb{H}^{\nu}_{p,\theta}(\tilde{\xi})}+\|Og\|^p_{\mathbb{H}^{\nu+1}_{p,\theta}(\tilde{\xi})})$$

$$=N(\|M_{\xi}f\|^p_{\mathbb{H}^{\nu}_{p,\theta}(\xi)}+\|g\|^p_{\mathbb{H}^{\nu+1}_{p,\theta}(\xi)}),$$

where N depends only on $\theta,p,d,\nu,N_0(\nu)$, $|P|$ and $|P^{-1}|$. The lemma is proved. □

The following lemmas are taken from Kim and Krylov (2004b).

Lemma 3.2. *Let constants* $C,\delta\in(0,\infty)$, *a function* $u\in H^{\gamma}_{p,\theta}$, *and* q *be the smallest integer such that* $|\gamma|+2\leqslant q$.

(i) *Let* $\eta_n\in C^{\infty}(\mathbb{R}^d_+)$, $n=1,2,\ldots$, *satisfy*

$$\sum_n M^{|\alpha|}|D^{\alpha}\eta_n|\leqslant C\quad \text{in } \mathbb{R}^d_+ \tag{3.10}$$

for any multi-index α *such that* $0\leqslant|\alpha|\leqslant q$. *Then*

$$\sum_n\|\eta_nu\|^p_{H^{\gamma}_{p,\theta}}\leqslant NC^p\|u\|^p_{H^{\gamma}_{p,\theta}},$$

where the constant N *is independent of* u, θ, *and* C.

(ii) *If in addition to the condition in* (i)

$$\sum_n\eta_n^2\geqslant\delta\quad\text{on } \mathbb{R}^d_+, \tag{3.11}$$

then

$$\|u\|^p_{H^\gamma_{p,\theta}} \leqslant N \sum_n \|\eta_n u\|^p_{H^\gamma_{p,\theta}}, \tag{3.12}$$

where the constant N *is independent of* u *and* θ.

The reason the first inequality in (3.13) below is written for η_n^4 (not for η_n^2) as in the above lemma is to have the possibility to apply Lemma 3.2 to η_n^2. Also observe that obviously $\sum a^2 \leqslant (\sum |a|)^2$.

Lemma 3.3. *For each* $\varepsilon > 0$ *and* $q = 1, 2, \ldots$ *there exist nonnegative functions* $\eta_n \in C_0^\infty(\mathbb{R}^d_+)$, $n = 1, 2, \ldots$ *such that*

(i) *on* $\mathbb{R}^d_+$ *for each multi-index* α *with* $1 \leqslant |\alpha| \leqslant q$ *we have*

$$\sum_n \eta_n^4 \geqslant 1, \quad \sum_n \eta_n \leqslant N(d), \quad \sum_n M^{|\alpha|}|D^\alpha \eta_n| \leqslant \varepsilon; \tag{3.13}$$

(ii) *for any* n *and* $x, y \in \operatorname{supp} \eta_n$ *we have* $|x - y| \leqslant N(x^1 \wedge y^1)$, *where* $N = N(d, q, \varepsilon) \in [1, \infty)$.

Lemma 3.4. *Let* $p \in (1, \infty)$, $\gamma, \theta \in \mathbb{R}$. *Then there exists a constant* $N = N(\gamma, |\gamma|+, p, d)$ *such that if* $f \in H^\gamma_{p,\theta}$ *and* a *is a function with finite norm* $|a|^{(0)}_{|\gamma|+,\mathbb{R}^d_+}$, *then*

$$\|af\|_{H^\gamma_{p,\theta}} \leqslant N|a|^{(0)}_{|\gamma|+,\mathbb{R}^d_+}\|f\|_{H^\gamma_{p,\theta}}. \tag{3.14}$$

In addition,

(i) *if* $\gamma = 0, 1, 2, \ldots$, *then*

$$\|af\|_{H^\gamma_{p,\theta}} \leqslant N \sup_{\mathbb{R}^d_+} |a| \, \|f\|_{H^\gamma_{p,\theta}} + N\|f\|_{H^{\gamma-1}_{p,\theta}} \sup_{\mathbb{R}^d_+} \sup_{1 \leqslant |\alpha| \leqslant \gamma} |M^{|\alpha|} D^\alpha a| \tag{3.15}$$

(ii) *if* γ *is not integer, then*

$$\|af\|_{H^\gamma_{p,\theta}} \leqslant N \left(\sup_{\mathbb{R}^d_+} |a| \right)^s (|a|^{(0)}_{|\gamma|+})^{1-s} \|f\|_{H^\gamma_{p,\theta}}, \tag{3.16}$$

where $s := 1 - |\gamma|/|\gamma|+ > 0$.

The same assertions hold true for ℓ_2*-valued* a.

4. Proof of Theorem 2.10

We closely follow the proof of Theorem 2.16 of Kim and Krylov (2004a). As usual we assume $u_0 = 0$ and $\tau = \infty$. Denote, for $\lambda \in [0, 1]$, $a_\lambda = \lambda a + (1 - \lambda)I$, $\sigma_\lambda = \lambda\sigma$ and define $\tilde{a}_\lambda, \tilde{\sigma}_\lambda$ similarly. Note

$$a_\lambda = Ia_\lambda I^*, \quad \sigma_\lambda = I\sigma_\lambda,$$

$$(\tilde{a}_\lambda(t,x), \tilde{\sigma}_\lambda(t,x)) \in \mathscr{A} \quad \forall \omega, t, x, \tag{4.1}$$

$$|a_\lambda - \tilde{a}_\lambda| + |\sigma_\lambda - \tilde{\sigma}_\lambda| \leqslant \tilde{\beta} \quad \forall \omega, t, x$$

and the functions $a_\lambda, \sigma_\lambda$, λc, $\lambda \nu$ satisfy Assumption 2.2. Therefore having the method of continuity in mind, we convince ourselves that to prove the theorem it suffices to show that there exist β_0, $\tilde\beta_0$ such that the a priori estimate (2.16) holds given that the solution already exists and $\beta \leqslant \beta_0$, $\tilde\beta \leqslant \tilde\beta_0$. Below, by N we denote various constants depending only on the same data as in the statement of the theorem. We divide the proof into several cases. The reason for this is that if γ is not an integer we use (3.16) and if γ is a nonnegative integer we use (3.15), but if γ is a negative integer we use the somewhat different approaches used in Kim and Krylov (2004a). We also mention that if extra (unnecessary) smoothness of the coefficients is assumed, then the arguments in case 1 work for all the cases, since then using (3.14) and interpolation inequalities one can get an inequality similar to (3.16).

Case 1: $|\gamma| \notin \{0,1,2,\ldots\}$. Take the least integer $q \geqslant |\gamma|+4$. Also take an $\varepsilon \in (0,1)$ to be specified later and take a sequence of functions η_n, $n=1,2,\ldots$, from Lemma 3.3 corresponding to ε, q. Then by Lemma 3.2, we have

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}} \leqslant N\sum_{n=1}^{\infty}\|M^{-1}u\eta_n^2\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}}. \tag{4.2}$$

For any n let x_n be a point in $\operatorname{supp}\eta_n$ and $a_n(t)=a(t,x_n)$, $\sigma_n(t)=\sigma(t,x_n)$. From (2.1), we easily have

$$\mathrm{d}(u\eta_n^2) = (a_n^{ij}(u\eta_n^2)_{x^ix^j} + M^{-1}f_n)\,\mathrm{d}t + (\sigma_n^{ik}(u\eta_n^2)_{x^i} + g_n^k)\,\mathrm{d}w_t^k,$$

where

$$\begin{aligned} f_n &= (a^{ij}-a_n^{ij})\eta_n^2 M u_{x^ix^j} - 2a_n^{ij}M(\eta_n^2)_{x^i}u_{x^j} - a_n^{ij}M^{-1}uM^2(\eta_n^2)_{x^ix^j}\\ &\quad + \eta_n^2 M b^i u_{x^i} + \eta_n^2 M^2 c M^{-1}u + Mf\eta_n^2,\\ g_n^k &= (\sigma^{ik}-\sigma_n^{ik})\eta_n^2 u_{x^i} - \sigma_n^{ik}M^{-1}uM(\eta_n^2)_{x^i} + M\nu^k M^{-1}u\eta_n^2 + g^k\eta_n^2. \end{aligned}$$

Take $\alpha_0=\alpha_0(\gamma)>0$ from Lemma 3.1 and assume $\tilde\beta \leqslant \alpha_0$ then by Lemma 3.1, for each n,

$$\|M^{-1}u\eta_n^2\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}} \leqslant N(\|f_n\|^p_{\mathbb{H}^{\gamma}_{p,\theta}} + \|g_n\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}}) \tag{4.3}$$

and by (3.16),

$$\|(a^{ij}-a_n^{ij})\eta_n^2 M u_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}} \leqslant N\|\eta_n M u_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}} \sup_{[0,\infty)\times\mathbb{R}^d_+}|(a^{ij}-a_n^{ij})\eta_n|^s, \tag{4.4}$$

where $s>0$ is a constant depending only on γ and $|\gamma|+$.

By Lemma 3.3(ii), for each n and $x,y\in\operatorname{supp}\eta_n$ we have $|x-y|\leqslant N(\varepsilon)(x^1\wedge y^1)$, where $N(\varepsilon)=N(d,q,\varepsilon)$, and we can easily find not more than $N(\varepsilon)+2\leqslant 3N(\varepsilon)$ points x_i lying on the straight segment connecting x and y and including x and y, such that $|x_i-x_{i+1}|\leqslant x_i^1\wedge x_{i+1}^1$. It follows from our assumptions

$$\sup_{[0,\infty)\times\mathbb{R}^d_+}|(a^{ij}-a_n^{ij})\eta_n| \leqslant 3N(\varepsilon)\beta.$$

We substitute this to (4.4) and get

$$\|(a^{ij}-a_n^{ij})\eta_n^2 M u_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}} \leqslant NN(\varepsilon)\beta^s\|\eta_n M u_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}}.$$

Similarly,

$$\|\eta_n^2 M b^i u_{x^i}\|_{\mathbb{H}^{\gamma}_{p,\theta}} + \|\eta_n^2 M^2 c M^{-1} u\|_{\mathbb{H}^{\gamma}_{p,\theta}} + \|(\sigma^{ik} - \sigma_n^{ik})\eta_n^2 u_{x^i}\|_{\mathbb{H}^{\gamma+1}_{p,\theta}}$$

$$+\|\eta_n^2 M \nu M^{-1} u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}} \leqslant NN(\varepsilon)\beta^s(\|\eta_n u_x\|_{\mathbb{H}^{\gamma+1}_{p,\theta}} + \|\eta_n M^{-1} u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}}).$$

Coming back to (4.3) and (4.2) and using Lemma 3.2, we conclude

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}} \leqslant NN(\varepsilon)\beta^{ps}(\|Mu_{xx}\|^p_{\mathbb{H}^{\gamma}_{p,\theta}} + \|u_x\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}} + \|M^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}})$$

$$+NC^p(\|u_x\|^p_{\mathbb{H}^{\gamma}_{p,\theta}} + \|M^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}}) + N(\|Mf\|^p_{\mathbb{H}^{\gamma}_{p,\theta}} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}}), \quad (4.5)$$

where

$$C = \sup_{\mathbb{R}^d_+} \sup_{|\alpha| \leqslant q-2} \sum_{n=1}^{\infty} M^{|\alpha|}(|D^\alpha(M(\eta_n^2)_x)| + |D^\alpha(M^2(\eta_n^2)_{xx})|).$$

By construction, we have $C \leqslant N\varepsilon$. Furthermore (see, for instance, Krylov (1999b))

$$\|u_x\|_{H^{\gamma+1}_{p,\theta}} \leqslant N\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}}, \quad \|Mu_{xx}\|_{H^{\gamma}_{p,\theta}} \leqslant N\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}}. \quad (4.6)$$

Hence (4.5) yields

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}} \leqslant N_1(N(\varepsilon)\beta^{ps} + \varepsilon^p)\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}} + N(\|Mf\|^p_{\mathbb{H}^{\gamma}_{p,\theta}} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}}).$$

Finally, to get a priori estimate (2.13) it is enough to choose first ε and then β_0, so that $N_1(N(\varepsilon)\beta^{ps} + \varepsilon^p) \leqslant \frac{1}{2}$ for $\beta \leqslant \beta_0$.

Case 2: $\gamma \in \{0, 1, 2, \ldots\}$. First consider the case $\gamma = 0$. Assume $\tilde{\beta} \leqslant \alpha_0(-1) \wedge \alpha_0(0)$ and proceed as in case 1 with $\varepsilon = 1$ and arrive at (4.3) which is

$$\|M^{-1}u\eta_n^2\|^p_{\mathbb{H}^2_{p,\theta}} \leqslant N(\|f_n\|^p_{\mathbb{L}_{p,\theta}} + \|g_n\|^p_{\mathbb{H}^1_{p,\theta}}).$$

Notice that (4.4) holds with $s = 1$ (since $\gamma = 0$). Also by (3.15),

$$\|(\sigma^{ik} - \sigma_n^{ik})\eta_n^2 u_{x^i}\|_{\mathbb{H}^1_{p,\theta}} \leqslant N \sup_{[0,\infty)\times\mathbb{R}^d_+} |(\sigma^{ik} - \sigma_n^{ik})\eta_n| \|\eta_n u_x\|_{\mathbb{H}^1_{p,\theta}} + N\|\eta_n u_x\|_{\mathbb{L}_{p,\theta}}$$

$$\leqslant N\beta\|\eta_n u_x\|_{\mathbb{H}^1_{p,\theta}} + N\|\eta_n u_x\|_{\mathbb{L}_{p,\theta}}. \quad (4.7)$$

From this point by following the arguments in case 1, one gets

$$\|M^{-1}u\|_{\mathbb{H}^2_{p,\theta}} \leqslant N_1\beta\|M^{-1}u\|_{\mathbb{H}^2_{p,\theta}} + N\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}} + N\|Mf\|_{\mathbb{L}_{p,\theta}} + N\|g\|_{\mathbb{H}^1_{p,\theta}}.$$

Thus, if $N_1\beta_0 \leqslant \frac{1}{2}$ and $\beta \leqslant \beta_0$ then we have

$$\|M^{-1}u\|_{\mathbb{H}^2_{p,\theta}} \leqslant N\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}} + N\|Mf\|_{\mathbb{L}_{p,\theta}} + N\|g\|_{\mathbb{H}^1_{p,\theta}}. \quad (4.8)$$

Next by reducing β_0 (note that we are free to do this) we will estimate the norm $\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}}$. Take an $\varepsilon \in (0,1)$ to be specified later and proceed as in case 1 and write (4.2) and (4.3) for $\gamma = -1$. The latter is

$$\|M^{-1}u\eta_n^2\|^p_{\mathbb{H}^1_{p,\theta}} \leqslant N(\|f_n\|^p_{\mathbb{H}^{-1}_{p,\theta}} + \|g_n\|^p_{\mathbb{L}_{p,\theta}}).$$

Using the fact $\|f_n\|_{\mathbb{H}^{-1}_{p,\theta}} \leqslant \|f_n\|_{\mathbb{L}_{p,\theta}}$ and the previous arguments, one obtains

$$\|M^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}} \leqslant NN^p(\varepsilon)\beta^p(\|Mu_{xx}\|^p_{\mathbb{L}_{p,\theta}} + \|u_x\|^p_{\mathbb{L}_{p,\theta}} + \|M^{-1}u\|^p_{\mathbb{L}_{p,\theta}})$$

$$+ NC^p(\|u_x\|^p_{\mathbb{L}_{p,\theta}} + \|M^{-1}u\|^p_{\mathbb{L}_{p,\theta}}) + N(\|Mf\|^p_{\mathbb{L}_{p,\theta}} + \|g\|^p_{\mathbb{L}_{p,\theta}}),$$

where C is introduced after (4.5). By using (4.6) we get

$$\|M^{-1}u\|^p_{\mathbb{H}^1_{p,\theta}} \leqslant N(N^p(\varepsilon)\beta^p + \varepsilon^p)\|M^{-1}u\|^p_{\mathbb{H}^2_{p,\theta}} + N(\|Mf\|^p_{\mathbb{L}_{p,\theta}} + \|g\|^p_{\mathbb{L}_{p,\theta}}).$$

Finally by substituting this into (4.8) and then choosing ε and then β_0 properly, one gets the desired estimate.

To consider the case $\gamma = 1, 2, \ldots$, one uses the induction arguments based on (3.15) (just proceed as for (4.8)) and the result for $\gamma = 0$. We leave the details to the reader.

Case 3: $\gamma = -1$ and Assumption 2.2(1) is satisfied. In this case we prove the theorem directly without depending on an a priori estimate. Since Assumption 2.2(1) is stronger than Assumption 2.2(0), there exist $\beta_0, \tilde{\beta}_0$ such that the assertions of the theorem hold for $\gamma = 0$ if (2.15) is satisfied. Assume that (2.15) is satisfied with the above $\beta_0, \tilde{\beta}_0$, then the operator $\mathscr{R}$ which maps the couples $(f, g) \in M^{-1}\mathbb{L}_{p,\theta} \times \mathbb{H}^1_{p,\theta}$ into the solutions $u \in \mathfrak{H}^2_{p,\theta,0}$ of Eq. (2.1) is well-defined and bounded.

Now take $(f, g) \in M^{-1}\mathbb{H}^{-1}_{p,\theta} \times \mathbb{L}_{p,\theta}$. By Corollary 2.12 in Krylov (1999b) we have the following representations:

$$f = MD_\ell f^\ell, \quad g^k = MD_\ell g^{\ell k}, \tag{4.9}$$

where $f^\ell \in M^{-1}\mathbb{L}_{p,\theta},\ g^\ell \in \mathbb{H}^1_{p,\theta},\ \ell = 1, 2, \ldots, d$ and

$$\sum_{\ell=1}^d \|Mf^\ell\|_{\mathbb{L}_{p,\theta}} \leqslant N\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}, \quad \sum_{\ell=1}^d \|g^\ell\|_{\mathbb{H}^1_{p,\theta}} \leqslant N\|g\|_{\mathbb{L}_{p,\theta}}. \tag{4.10}$$

Next denote $v^\ell = \mathscr{R}(f^\ell, g^\ell)$ and $\bar{v} = \sum_{\ell=1}^d MD_\ell v^\ell$. Then by (4.6) $\bar{v} \in M\mathbb{H}^1_{p,\theta}$ and satisfies

$$\mathrm{d}\bar{v}_t = (a^{ij}\bar{v}_{x^ix^j} + b^i\bar{v}_{x^i} + c\bar{v} + f + \bar{f})\,\mathrm{d}t + (\sigma^{ik}\bar{v}_{x^i} + v^k\bar{v} + g^k + \bar{g}^k)\,\mathrm{d}w^k_t,$$

where

$$\bar{f} = v^\ell_{x^ix^j}MD_\ell a^{ij} + M^{-1}\bar{v}^\ell_{x^i}M^2D_\ell b^i + M^{-2}\bar{v}^\ell M^3D_\ell c - 2a^{i1}\bar{v}^\ell_{x^\ell x^i} - Mb^1M^{-1}\bar{v}^\ell_{x^\ell},$$

$$\bar{g}^k = (MD_\ell\sigma^{ik})v^\ell_{x^i} - \sigma^{1k}v^\ell_{x^\ell} + (M^2D_\ell v^k)M^{-1}v^\ell.$$

By assumptions one can easily check that $|\cdot|^{(0)}_0$-norm of $MD_\ell a^{ij}$, $M^2D_\ell b^i$, $M^3D_\ell c$ and $|\cdot|^{(0)}_1$-norm of $MD_\ell\sigma$, $M^2D_\ell v$ are finite. Therefore,

$$M\bar{f} \in \mathbb{L}_{p,\theta}, \quad \bar{g} \in \mathbb{H}^1_{p,\theta}.$$

Finally, we define $\bar{u} = \mathscr{R}(\bar{f}, \bar{g})$ and $u = \bar{v} - \bar{u}$. Then $u \in \mathfrak{H}^1_{p,\theta,0}$ satisfies (2.1) and (2.16) follows from the formulas defining u.

Next, we prove the uniqueness of solutions. Assume (2.15) holds with $\beta_0, \tilde{\beta}_0$ found above and assume $u \in \mathfrak{H}^1_{p,\theta,0}$ satisfies (2.1) with $f = g^k = 0$. Since we already have the uniqueness in the space $\mathfrak{H}^2_{p,\theta,0}$ under condition (2.15), to show $u = 0$ we only need to show $u \in \mathfrak{H}^2_{p,\theta}$. Take η_n from Lemma 3.3 corresponding to $\varepsilon = 1$. From (2.1) one can write equations for $\eta_n u$ for each n and get

$$\begin{aligned} \mathrm{d}(\eta_n u) = {} & (a^{ij}(\eta_n u)_{x^i x^j} + b^i(\eta_n u)_{x^i} + c(\eta_n u) + \tilde{f})\,\mathrm{d}t \\ & +(\sigma^{ik}(\eta_n u)_{x^i} + v^k(\eta_n u) + \tilde{g}^k)\,\mathrm{d}w^k_t, \end{aligned}$$

where

$$\tilde{f} = -2a^{ij}\eta_{nx^i}u_{x^j} - (a^{ij}\eta_{nx^ix^j} + b^i\eta_{nx^i})u, \quad \tilde{g}^k = -\sigma^{ik}\eta_{nx^i}u.$$

Since $u \in M\mathbb{H}^1_{p,\theta}$ and η_n has compact support, we easily have $(\tilde{f}, \tilde{g}) \in \mathbb{L}_p \times \mathbb{H}^1_p$ (see Krylov, 1999a for the notations). Also the above equation will not change if we change arbitrarily a, b, c, v outside of the support of η_n. Therefore using Remark 5.6 of Krylov (1999a), one easily concludes that $\eta_n u \in \mathbb{H}^2_p$ and therefore $M^{-1}\eta_n u \in \mathbb{H}^2_{p,\theta}, \eta_n u \in \mathfrak{H}^2_{p,\theta,0}$. Then finally by using (2.16) (which we have for $\gamma = 0$) and Lemma 3.2 one obtains $\|M^{-1}u\|_{\mathbb{H}^2_{p,\theta}} < \infty$, that is $u \in \mathfrak{H}^2_{p,\theta,0}$.

Case 4: $\gamma = -1$ with no additional assumptions. To prove an a priori estimate we use the results of case 3. Fix a nonnegative smooth function $\phi \in C^\infty_0(B_{1/2}(0))$ with a unit integral. Define

$$\bar{\sigma}(x) = \int \sigma(y)(x^1)^{-d}\phi\left(\frac{x-y}{x^1}\right)\,\mathrm{d}y.$$

Define $\bar{v}$ similarly. Observe that

$$|\bar{\sigma} - \sigma| \leqslant \beta, \quad |M\bar{v}| \leqslant 2\beta.$$

Also using the fact $x^1 \leqslant 2(x^1 - x^1z^1) \leqslant 4x^1$ for $|z^1| \leqslant \frac{1}{2}$, one can easily check that there is a constant $N(2) < \infty$ such that

$$|\bar{\sigma}|^{(0)}_2 + |\bar{v}|^{(1)}_2 < N(2).$$

For instance, let $i, j \geqslant 2$, and $\delta_{1\ell} = 1$ if $\ell = 1$ and $\delta_{1\ell} = 0$ otherwise, then

$$x^1\bar{\sigma}_{x^1}(x) = \int_{|z| \leqslant 1/2} \sigma(x - x^1z)[-\mathrm{d}\phi(z) + \phi_{x^\ell}(z)\cdot(\delta_{1\ell} - z^\ell)]\,\mathrm{d}z,$$

$$(x^1)^2\bar{v}_{x^1}(x) = \int_{|z| \leqslant 1/2} x^1 v(x - x^1z)[-\mathrm{d}\phi(z) + \phi_{x^\ell}(z)\cdot(\delta_{1\ell} - z^\ell)]\,\mathrm{d}z,$$

$$(x^1)^2\bar{\sigma}_{x^ix^j}(x) = \int_{|z| \leqslant 1/2} \sigma(x - x^1z)\phi_{x^ix^j}(z)\,\mathrm{d}z,$$

$$(x^1)^3\bar{v}_{x^ix^j}(x) = \int_{|z| \leqslant 1/2} x^1 v(x - x^1z)\phi_{x^ix^j}(z)\,\mathrm{d}z$$

and therefore it is obvious that the functions in the above are bounded. Also all other cases can be considered similarly.

Take $(f,g)\in M^1\mathbb{H}^1_{p,\theta}\times\mathbb{L}_{p,\theta}$ and let $u\in\mathfrak{H}^1_{p,\theta,0}$ be a solution of (2.1). Then

$$du=(a^{ij}u_{x^ix^j}+b^iu_{x^i}+cu+f)\,dt+(\bar\sigma^{ik}u_{x^i}+\bar\nu^ku+\bar g^k)\,dw^k_t,$$

where $\bar g=g+(\sigma-\bar\sigma)u_{x^i}+(\nu-\bar\nu)u$. Note

$$\|\bar g\|_{\mathbb{L}_{p,\theta}}\leqslant\|g\|_{\mathbb{L}_{p,\theta}}+\beta\|u_x\|_{\mathbb{L}_{p,\theta}}+3\beta\|M^{-1}u\|_{\mathbb{L}_{p,\theta}}. \tag{4.11}$$

Take $\tilde\beta_0$ from case 3. If $\max\{\tilde\beta,\beta\}\leqslant\frac12\tilde\beta_0$, then

$$|a-\tilde a|+|\bar\sigma-\tilde\sigma|\leqslant|a-\tilde a|+|\bar\sigma-\sigma|+|\sigma-\tilde\sigma|\leqslant\beta+\tfrac12\tilde\beta_0\leqslant\tilde\beta_0.$$

Thus, by the results of case 3, there exists β_0 such that, in addition to $\beta\leqslant\frac12\tilde\beta_0$, if $\beta\leqslant\beta_0$ then

$$\begin{aligned}\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}}&\leqslant N(\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}+\|\bar g\|_{\mathbb{L}_{p,\theta}})\\&\leqslant N_1(\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}+\|g\|_{\mathbb{L}_{p,\theta}})+N_2\beta\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}},\end{aligned} \tag{4.12}$$

where the second inequality comes from (4.6).

Finally, we assume

$$\beta\leqslant\beta_0\wedge(\tilde\beta_0/2)\wedge(2N_2)^{-1}.$$

Then (4.12) yields

$$\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}}\leqslant 2N_1(\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}+\|g\|_{\mathbb{L}_{p,\theta}}).$$

Thus we get the desired result for $\gamma=-1$.

Case 5: $\gamma=-2,-3,-4,\ldots$. Note that Assumption 2.2(γ) is stronger than Assumption 2.2($\gamma+1$). One can easily check that $|\cdot|^{(0)}_{\gamma+1}$-norm of $MD_\ell a^{ij}$, $M^2D_\ell b^i$, $M^3D_\ell c$ and $|\cdot|^{(0)}_{\gamma+2}$-norm of $MD_\ell\sigma$, $M^2D_\ell\nu$ are finite. Therefore it is enough just to proceed as in case 3. The theorem is proved. □

5. Proof of Theorem 2.9

First we introduce a lemma which is a modification of Theorem 2.10.

Lemma 5.1. *Let $G=\mathbb{R}^d_+$, $\bar\beta,\tilde\beta,\tilde K\in(0,\infty)$, and $\mathscr{A}$ be a set of $\mathscr{A}_{p,\theta}$-type. Let Assumption 2.2(γ) and (2.14) be satisfied, and let for each x, $(a_1(\omega,t,x),\sigma_1(\omega,t,x))$, $(a_2(\omega,t,x),\sigma_2(\omega,t,x))$ be bounded $\mathscr{M}_{d\times d}\times\mathscr{M}_{d\times\infty}$-valued predictable functions of (ω,t) such that*

$$(a_1(\omega,t,x),\sigma_1(\omega,t,x))\in\mathscr{A}\quad\forall\omega,t,x,$$

$$|a_2(\omega,t,x)|+|\sigma_2(\omega,t,x)|\leqslant\tilde\beta\quad\forall\omega,t,x.$$

Also assume there is a $\mathscr{M}_{d\times d}$-valued function $Q(x)$ which is Borel measurable in x such that

$$|Q(x)| + |Q^{-1}(x)| \leqslant \tilde{K} \quad \forall x,$$

$$\sup_{x,y} |Q(x) - Q(y)| \leqslant \bar{\beta},$$

$$a(t,x) = Q(x)(a_1(t,x) + a_2(t,x))Q^*(x) \quad \forall \omega, t, x,$$

$$\sigma(t,x) = Q(x)(\sigma_1(t,x) + \sigma_2(t,x)) \quad \forall \omega, t, x.$$

Then there exist $\tilde{\beta}_0 = \tilde{\beta}(\gamma, N_0)$, $\bar{\beta}_0 = \bar{\beta}_0(\gamma, \tilde{K}, N_0(\gamma)) > 0$ and $\beta_0 \in (0,1)$ depending only on $N_0, \delta_0, p, \theta, \gamma, \tilde{K}$ and K, such that, if

$$\beta \leqslant \beta_0, \quad \bar{\beta} \leqslant \bar{\beta}_0, \quad \tilde{\beta} \leqslant \tilde{\beta}_0,$$

then

(i) *for any $f \in M^{-1}\mathbb{H}^{\gamma}_{p,\theta}(\tau), g \in \mathbb{H}^{\gamma+1}_{p,\theta}(\tau)$ and $u_0 \in U^{\gamma+2}_{p,\theta}$ Eq.* (2.1) *with initial data u_0 admits a unique solution u in the class $\mathfrak{H}^{\gamma+2}_{p,\theta}(\tau)$,*

(ii) *for this solution*

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\tau)} \leqslant N(\|u_0\|_{U^{\gamma+2}_{p,\theta}} + \|Mf\|_{\mathbb{H}^{\gamma}_{p,\theta}(\tau)} + \|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(\tau)}), \tag{5.1}$$

where the constant N depends only on $N_0, p, \theta, \gamma, \delta_0, \tilde{K}$ and K.

Proof. Our proof is almost identical to the proof of Theorem 2.10. Fix $x_0 \in \mathbb{R}^d_+$, and define $\bar{Q}(x) = Q^{-1}(x)Q(x_0)$ and for $\lambda \in [0,1]$,

$$a_\lambda = \lambda a + (1-\lambda)Q(x_0)Q^*(x_0) = Q(\lambda a_1 + (1-\lambda)I + a_{2\lambda})Q^*,$$

$$\sigma_\lambda = Q(\lambda\sigma_1 + \lambda\sigma_2),$$

where $a_{2\lambda} := \lambda a_2 + (1-\lambda)(\bar{Q}\bar{Q}^* - I)$.

Observe

$$|a_{2\lambda}| + |\lambda\sigma_2| \leqslant \lambda(|a_2| + |\sigma_2|) + (1-\lambda)|\bar{Q}\bar{Q}^* - I|$$
$$\leqslant \lambda\tilde{\beta} + (1-\lambda)N(d,\tilde{K})|Q(x) - Q(x_0)| \leqslant \tilde{\beta}$$

if $N(d,\tilde{K})\bar{\beta} =: \bar{N}\bar{\beta} \leqslant \tilde{\beta}$. Thus if we assume this, then we get, instead of (4.1),

$$(a_\lambda, \sigma_\lambda) = (Q(\lambda a_1 + (1-\lambda)I + a_{2\lambda})Q^*, Q(\lambda\sigma_1 + \lambda\sigma_2)),$$

$$(\lambda a_1 + (1-\lambda)I, \lambda\sigma_1) \in \mathscr{A} \quad \forall \omega, t, x,$$

$$|a_{2\lambda}| + |\lambda\sigma_2| \leqslant \tilde{\beta}.$$

Now to finish the proof of the lemma it is enough to repeat the proof of Theorem 2.10 word for word. Indeed, as in the proof of Theorem 2.10 we again divide the proof into 5 cases. All the proofs of cases 1–3 and 5 go exactly the same way as before without changing a word, except that the constants N in the proof also depend on $\tilde{K}$

just because we are using Lemma 3.1, not only for $P=I$ but also for any invertible matrix P. Therefore, it is enough to consider case 4. Proceed as in the proof of case 4 of Theorem 2.10 up to (4.11) and then observe that

$$\bar{\sigma}(t,x)=Q(x)(\sigma_1(t,x)+\tilde{\sigma}_2(t,x))$$

with $\tilde{\sigma}_2 := Q^{-1}(\bar{\sigma}-\sigma)+\sigma_2$ and $|a_2|+|\tilde{\sigma}_2|=|Q^{-1}(\bar{\sigma}-\sigma)+\sigma_2|\leqslant \bar{N}\beta+\tilde{\beta}$.

Take $\tilde{\beta}_0$ from case 3. Assume $\max\{\tilde{\beta},\bar{N}\beta\}\leqslant\tilde{\beta}_0/2$, then

$$|a_2|+|\tilde{\sigma}_2|\leqslant\tilde{\beta}_0$$

and therefore by the results of case 3, there exists β_0 such that, additionally, if $\beta\leqslant\beta_0$ then

$$\begin{aligned}\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}} &\leqslant N(\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}+\|\bar{g}\|_{\mathbb{L}_{p,\theta}})\\ &\leqslant N_1(\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}+\|g\|_{\mathbb{L}_{p,\theta}})+N_2\beta\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}},\end{aligned} \tag{5.2}$$

where the second inequality comes from (4.6).

Finally, we assume

$$\beta\leqslant\beta_0\wedge(\bar{N}^{-1}\tilde{\beta}_0/2)\wedge(2N_2)^{-1}.$$

Then (5.2) yields

$$\|M^{-1}u\|_{\mathbb{H}^1_{p,\theta}}\leqslant 2N_1(\|Mf\|_{\mathbb{H}^{-1}_{p,\theta}}+\|g\|_{\mathbb{L}_{p,\theta}}).$$

Therefore we get the desired estimate. The lemma is proved. □

Now we turn our attention to the proof of Theorem 2.9. As in the proof of Theorem 2.14 in Kim and Krylov (2004a) we may assume $\tau\equiv T$. Using the results of Lemma 5.1 we first establish the a priori estimate (2.13) assuming that $u\in\mathfrak{H}^{\gamma+2}_{p,\theta}(G,T)$ satisfies (2.1) with initial data u_0. Let $x_0\in\partial G$ and Ψ be a function from Assumption 2.1.

Denote $P(x)=(\Psi^i_{x^j}(\Psi^{-1}(x))$ for each x in the closure of $\Psi(B_{r_0}(x_0)\cap G)$. Then we have $|P(x)|+|P^{-1}(x)|\leqslant 2dK_0$. Take $\tilde{\beta}_0,\bar{\beta}_0$ from Lemma 5.1 which correspond to $\gamma,2dK_0,N_0$. Also choose $\tilde{r}>0$ sufficiently small such that for each $x,y\in B^+_{\tilde{r}}(0)$,

$$|P^{-1}(x)P(0)P(0)^*(P^*)^{-1}(x)-I|\leqslant\tilde{\beta}_0, \tag{5.3}$$

$$|P(x)-P(y)|\leqslant\bar{\beta}_0. \tag{5.4}$$

Define $r=r_0/K_0\wedge\tilde{r}$ and fix smooth functions $\eta\in C^\infty_0(B_r),\varphi\in C^\infty(\mathbb{R})$ such that $0\leqslant\eta,\varphi\leqslant 1$, and $\eta=1$ in $B_{r/2}$, $\varphi(t)=1$ for $t\leqslant-3$, and $\varphi(t)=0$ for $t\geqslant-1$ and $0\geqslant\varphi'\geqslant-1$. Observe that $\Psi(B_{r_0}(x_0))$ contains B_r. For $m=1,2,\ldots$, $t>0$, $x\in\mathbb{R}^d_+$ introduce $\varphi_m(x)=\varphi(m^{-1}\ln x^1)$,

$$\hat{a}_m:=\tilde{a}\eta(x)\varphi_m+(1-\eta\varphi_m)P(0)P(0)^*,\quad \hat{b}_m:=\tilde{b}\eta\varphi_m,\quad \hat{c}_m:=\tilde{c}\eta\varphi_m,$$

$$\hat{\sigma}_m:=\tilde{\sigma}\eta\varphi_m,\quad \hat{\nu}_m:=\tilde{\nu}\eta\varphi_m,$$

where

$$\tilde{a}^{ij}(t,x)=\bar{a}^{ij}(t,\Psi^{-1}(x)),\qquad \tilde{b}^{i}(t,x)=\bar{b}^{i}(t,\Psi^{-1}(x)),$$

$$\tilde{\sigma}^{ik}(t,x)=\bar{\sigma}^{ik}(t,\Psi^{-1}(x)),\qquad \bar{a}^{ij}=a^{rs}\Psi^{i}_{x^r}\Psi^{j}_{x^s},$$

$$\bar{b}^{i}=a^{rs}\Psi^{i}_{x^rx^s}+b^{m}\Psi^{i}_{x^m},\qquad \bar{\sigma}^{ik}=\sigma^{rk}\Psi^{i}_{x^r},$$

$$\tilde{c}(t,x)=c(t,\Psi^{-1}(x)),\qquad \tilde{v}(t,x)=v(t,\Psi^{-1}(x)).$$

Using (2.6) and Lemma 3.4 of Kim and Krylov (2004b), one can easily check that $\hat{a}_m,\hat{b}_m,\hat{c}_m$, $\hat{\sigma}_m$, $\hat{v}$ satisfy Assumptions 2.2 with $G=\mathbb{R}^d_+$ and new constant $K'\in(0,\infty)$ independent of m and x_0.

Take β_0 from Lemma 5.1 corresponding to $\delta_0,p,\theta,\gamma,|\gamma|+$, $2dK_0$ and K'. Observe that $\varphi(m^{-1}\ln x^1)=0$ for $x^1\geqslant \mathrm{e}^{-m}$ and $|\varphi(m^{-1}\ln x^1)-\varphi(m^{-1}\ln y^1)|\leqslant m^{-1}$ if $|x^1-y^1|\leqslant x^1\wedge y^1$. Also we easily see that (2.7) implies $x^1\Psi_{xx}(\Psi^{-1}(x))\to 0$ as $x^1\to 0$. Using these facts, Assumption 2.1 and Assumption 2.3(ii), one can find $m>0$ independent of x_0 such that

$$\begin{aligned}&|\hat{a}_m(t,x)-\hat{a}_m(t,y)|+|\hat{\sigma}_m(t,x)-\hat{\sigma}_m(t,y)|+x^1|\hat{b}_m(t,x)|\\&\quad+(x^1)^2|\hat{c}_m(t,x)|+x^1|\hat{v}_m(t,x)|\leqslant\beta_0,\end{aligned}$$

whenever $t>0$, $x,y\in\mathbb{R}^d_+$ and $|x-y|\leqslant x^1\wedge y^1$. Now we fix a $\rho_0<r_0$ such that

$$\Psi(B_{\rho_0}(x_0))\subset B_{r/2}\cap\{x:x^1\leqslant \mathrm{e}^{-3m}\}.$$

Let ζ be a smooth function with support in $B_{\rho_0}(x_0)$ and denote $v:=(u\zeta)(\Psi^{-1})$ and continue v as zero in $\mathbb{R}^d_+\setminus\Psi(B_{\rho_0}(x_0))$. Since $\eta\varphi_m=1$ on $\Psi(B_{\rho_0}(x_0))$, the function v satisfies

$$\mathrm{d}v=(\hat{a}^{ij}_m v_{x^ix^j}+\hat{b}^i_m v_{x^i}+\hat{c}_m v+\hat{f})\,\mathrm{d}t+(\hat{\sigma}^{ik}_m v_{x^i}+\hat{v}^k_m v+\hat{g}^k)\,\mathrm{d}w^k_t,$$

where

$$\hat{f}=\tilde{f}(\Psi^{-1}),\qquad \tilde{f}=-2a^{ij}u_{x^i}\zeta_{x^j}-ua^{ij}\zeta_{x^ix^j}-ub^i\zeta_{x^i}+\zeta f,$$

$$\hat{g}=\tilde{g}(\Psi^{-1}),\qquad \tilde{g}=-\sigma^{ik}u\zeta_{x^i}+\zeta g.$$

Next we observe that by Lemma 2.5 and Theorem 3.2 in Lototsky (2000) (or see Kim and Krylov (2004b)) for any $v,\alpha\in\mathbb{R}$ and $h\in\psi^{-\alpha}H^{v}_{p,\theta}(G)$ with support in $B_{\rho_0}(x_0)$

$$\|\psi^{\alpha}h\|_{H^{v}_{p,\theta}(G)}\sim\|M^{\alpha}h(\Psi^{-1})\|_{H^{v}_{p,\theta}}.\tag{5.5}$$

Therefore, we conclude that $v\in\mathfrak{H}^{\gamma+2}_{p,\theta}(T)$. Here, to use Lemma 5.1 we define the following functions. For x such that $\eta(x)\varphi_m(x)=0$, define

$$Q(x)=P(0),\qquad a_1(t,x)=I,\qquad a_2(t,x)=0,\qquad \sigma_1=0$$

and for other x, we define

$$Q(x)=P(x),\quad a_1(t,x)=\eta(x)\varphi_m(x)a(t,\Psi^{-1}(x))+(1-\eta(x)\varphi_m(x))I,$$

$$a_2(t,x)=(P(x)^{-1}P(0)(P(x)^{-1}P(0))^*-I)(1-\eta(x)\varphi_m(x)),$$

$$\sigma_1(t,x)=\eta(x)\varphi_m(x)\sigma(t,\Psi^{-1}(x)).$$

Then

$$(\hat{a}_m(t,x),\hat{\sigma}_m(t,x))=(Q(x)(a_1(t,x)+a_2(t,x))Q^*(x),Q\sigma_1(t,x)).$$

Also by Definition 2.6(i), (5.3) and (5.4)

$$(a_1(t,x),\sigma_1(t,x))\in\mathscr{A}\quad\forall\omega,t,x,$$

$$|a_2(t,x)|\leqslant\tilde{\beta}_0,\quad\sup_{x,y}|Q(x)-Q(y)|\leqslant\bar{\beta}_0.$$

Therefore by Lemma 5.1 we have, for any $t\leqslant T$,

$$\|M^{-1}v\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(t)}\leqslant N(\|M\hat{f}\|_{\mathbb{H}^{\gamma}_{p,\theta}(t)}+\|\hat{g}\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(t)}+\|u_0(\Psi^{-1})\zeta(\Psi^{-1})\|_{U^{\gamma+2}_{p,\theta}}).$$

By using (5.5) again we obtain

$$\begin{aligned}\|\psi^{-1}u\zeta\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,t)}\leqslant{}& N\|a\zeta_x\psi\,u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}+N\|a\zeta_{xx}\psi\,u\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}\\&+N\|\zeta_x\psi\,bu\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}+N\|\sigma\zeta_x u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)}+N\|\zeta\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}\\&+\|\zeta g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)}+\|\zeta u_0\|_{U^{\gamma+2}_{p,\theta}(G)}.\end{aligned}$$

Next we use Theorem 3.1 in Lototsky (2000). Remembering that ρ and ψ are comparable in G, one can easily check that $|\psi\,b(t,\cdot)|^{(0)}_{|\gamma|+}$ is bounded on $[0,T]$. Also using Lemma 2.8 in Kim and Krylov (2004b) and Assumption 2.2(iii) one can easily estimate

$$|\zeta_x a(t,\cdot)|^{(0)}_{|\gamma|+},\quad|\zeta_{xx}\psi\,a(t,\cdot)|^{(0)}_{|\gamma|+},\quad|\zeta_x\psi\,b(t,\cdot)|^{(0)}_{|\gamma|+},\quad|\zeta_x\sigma(t,\cdot)|^{(0)}_{|\gamma+1|+}.$$

Then one concludes

$$\begin{aligned}\|\psi^{-1}u\zeta\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,t)}\leqslant{}& N\|\psi\,u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}+N\|u\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}\\&+N\|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}+\|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)}+N\|u_0\|_{U^{\gamma+2}_{p,\theta}(G)}.\end{aligned}$$

Note that the above constants ρ_0,m,K',N are independent of x_0. Therefore, to estimate the norm $\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,t)}$, one introduces a partition of unity $\zeta_{(i)}$, $i=0,1,2,\ldots,N$ such that $\zeta_{(0)}\in C_0^\infty(G)$ and $\zeta_{(i)}\in C_0^\infty(B_{\rho_0}(x_i))$, $x_i\in\partial G$ for $i\geqslant 1$. Then one estimates $\|\psi^{-1}u\zeta_{(0)}\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,t)}$ using Theorem 5.1 in Krylov (1999a) and the other norms as above. By summing up those estimates one gets

$$\begin{aligned}\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(G,t)}\leqslant{}& N\|\psi\,u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}+N\|u\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}\\&+N\|\psi f\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)}+N\|g\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)}+N\|u_0\|_{U^{\gamma+2}_{p,\theta}(G)}.\end{aligned}\tag{5.6}$$

Furthermore, we know from Theorem 4.1 of Lototsky (2000) that

$$\|\psi\, u_x\|_{H^{\gamma}_{p,\theta}(G)} \leqslant N\|u\|_{H^{\gamma+1}_{p,\theta}(G)}.$$

Therefore (5.6) yields

$$\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta}(G,t)} \leqslant N\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)} + N\|\psi f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(G,t)} + \|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(G,t)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta}(G)}.$$

Now (2.13) follows from inequality (2.21) of Lototsky (2001) and Gronwall's inequality.

Finally, considering the method of continuity, we finish the proof by showing that for any $u_0 \in U^{\gamma+2}_{p,\theta}(G)$ and $(f,g) \in \psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(G,T) \times \mathbb{H}^{\gamma+1}_{p,\theta}(G,T)$, there exists $u \in \mathfrak{H}^{\gamma+2}_{p,\theta}(G,T)$ such that $u(0,\cdot) = u_0$ and

$$\mathrm{d}u = (\Delta u + f)\,\mathrm{d}t + g^k\,\mathrm{d}w^k_t. \tag{5.7}$$

We can approximate $g = (g^1, g^2, \ldots)$ with functions having only finite nonzero entries and it is known (see Lototsky, 2000) that smooth functions with compact support are dense in $H^{\nu}_{p,\theta}(G)$. Therefore, it follows from a priori estimate (2.13) that we may assume that g has only finite nonzero entries and is bounded on $\Omega \times [0,T] \times G$ along with each derivative in x and vanishes if x is near ∂G. In that case it is well known that

$$v(t,x) := \int_0^t g^k(t,x)\,\mathrm{d}w^k_s$$

is infinitely differentiable in x and vanishes near ∂G. Therefore, we conclude $v \in \mathfrak{H}^{\nu}_{p,\theta}(G,T)$ for any $\nu \in \mathbb{R}$. Observe that Eq. (5.7) can be written as

$$\mathrm{d}\bar{u} = (\Delta\bar{u} + f + \Delta v)\,\mathrm{d}t,$$

where $\bar{u} := u - v$. Thus we reduced the case to the case in which $g \equiv 0$. So we may assume $g = 0$ in (5.7). By Theorem 2.10 in Kim and Krylov (2004b), for any nonrandom $z_1 \in U^{\gamma+2}_{p,\theta}(G,T)$ and $z_2 \in \psi^{-1}\mathbb{H}^{\gamma}_{p,\theta}(G,T)$ there exists a unique (nonrandom) solution $w \in \mathfrak{H}^{\gamma+2}_{p,\theta}(G,T)$ of equation

$$\mathrm{d}w = (\Delta w + z_2)\,\mathrm{d}t$$

with initial data z_1 and it satisfies

$$\|w\|_{\mathfrak{H}^{\gamma+2}_{p,\theta}(G,T)} \leqslant N(\|\psi\, z_2\|_{\mathbb{H}^{\gamma}_{p,\theta}(G,T)} + \|z_1\|_{U^{\gamma+2}_{p,\theta}(G)}). \tag{5.8}$$

Therefore, it follows from Theorem 2.10 in Kim and Krylov (2004b) and (5.8) that there exists a solution $v \in \mathfrak{H}^{\gamma+2}_{p,\theta}(G,T)$ of Eq. (5.7) with initial data u_0. The theorem is proved. □

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