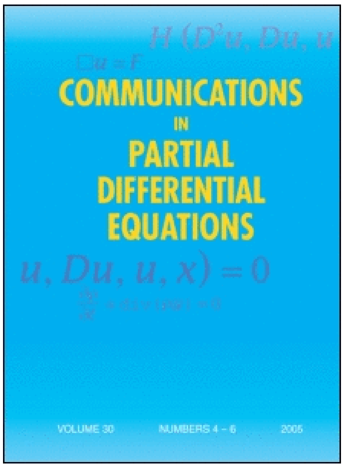

Communications in Partial Differential Equations
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597240

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To cite this Article Kim, Kyeong-Hun(2007) 'Sobolev Space Theory of Parabolic Equations Degenerating on the Boundary of C^1 Domains', Communications in Partial Differential Equations, 32: 8, 1261 — 1280
To link to this Article: DOI: 10.1080/03605300600910449
URL: http://dx.doi.org/10.1080/03605300600910449

Communications in Partial Differential Equations, 32: 1261–1280, 2007

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605300600910449

Sobolev Space Theory of Parabolic Equations Degenerating on the Boundary of C^1 Domains

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Second-order parabolic equations degenerating on the boundary of C^1 domains are considered. Existence and uniqueness results are given in weighted Sobolev spaces, and Hölder estimates of the solutions are presented.

Keywords C^1 domains; Degenerate parabolic equations; Sobolev spaces with weights.

1991 Mathematics Subject Classification 35K20; 35J15.

1. Introduction

The boundary value problems for parabolic and elliptic equations with different types of degeneracies have been studied widely by many authors. In this article we are dealing with the Sobolev space theory of second-order parabolic equations degenerating on the boundary of C^1 domains. Let Ω be a bounded domain in $\mathbb{R}^d$ and $\alpha \in [0, \infty)$. Denote $\rho(x) = \text{dist}(x, \partial\Omega)$. We consider the equation

$$\frac{du}{dt} = a^{ij}u_{x^ix^j} + b^iu_{x^i} + cu + f \tag{1.1}$$

with the "degeneracy of order α" near $\partial\Omega$: $\exists\delta_0, K > 0$ such that for any $\lambda \in \mathbb{R}^d$

$$\delta_0\rho^{2\alpha}(x)|\lambda|^2 \le a^{ij}(t, x)\lambda^i\lambda^j \le K\rho^{2\alpha}(x)|\lambda|^2. \tag{1.2}$$

Note that if $\alpha = 0$, then (1.1) becomes a uniformly parabolic equation. Needless to say, the unique solvability of such equation in L_p and Hölder spaces has been investigated quite completely. However even in this case since the boundary is not supposed to be regular enough, one has to look for solutions in function spaces with weights allowing derivatives of solutions to blow up near the boundary. In the framework of Hölder spaces such setting leads to investigating so-called interior Schauder estimates.

Received March 1, 2006; Accepted June 1, 2006
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The equation with the degeneracy of order $\alpha < 1/2$ was studied long time ago. See Aronson and Besala (1967), Pukal'skii (1977), and the references therein. In those articles, on the basis of the study of a Green function, interior Schauder estimates were established. We also refer to Lototsky (2001) for an L_p-theory of the equation with the degeneracy of order $\alpha = 1$ and for motivations of considering the case $\alpha > 0$.

In this article we present the unique solvability result of the equation with the degeneracy of order $\alpha \in [0, \infty)$ in weighted Sobolev spaces. Using embedding theorems we also give some Hölder estimates of the solution. Our approach is different from that of Lototsky (2001). If $\alpha = 1$ then scaling argument, on the basis of estimates for equations defined on $\mathbb{R}^d$, works perfectly well. This argument fails if $\alpha < 1$. Thus our main focus lies in establishing a priori estimate for equations defined on $\mathbb{R}^d_+$ (=half space). Another interesting feature of this article is that coefficients of the equation are allowed to be unbounded (see Remark 2.5), and the number of derivatives of the solution is allowed to be fractional.

We also refer to Oleinik and Radkevič (1973), Triebel (1995), and Višik and Grušin (1969) for elliptic equations degenerating on the boundary of domains.

Our main results are stated in Section 2 and consist of Theorem 2.6 and Corollary 2.7, on solvability of the equation in Sobolev spaces and Hölder estimates of the solution, respectively. Our results, when $\alpha = 0$, are not new and can be found in Kim and Krylov (2004). Notice that in Theorem 2.6 we only consider bounded domains, however actually, the result is also true for the domains Ω which are uniformly C^1 smooth in a natural sense.

In Section 3 we prove some auxiliary results, in particular we give some estimates for solutions of equations defined on $\mathbb{R}^d$ and $\mathbb{R}^d_+$. The final Section 4 is devoted to the proof of Theorem 2.6.

Here are notations used in the article. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \dots, x^d)$, $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x^1 > 0\}$ and $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$. For $i = 1, \dots, d$, multi-indices $\beta = (\beta_1, \dots, \beta_d)$, $\beta_i \in \{0, 1, 2, \dots\}$, and functions $u(x)$ we set

$$u_{x^i} = \partial u/\partial x^i = D_i u, \quad D^\beta u = D_1^{\beta_1} \cdot \dots \cdot D_d^{\beta_d} u, \quad |\beta| = \beta_1 + \dots + \beta_d.$$

We also use the notation D^m for a partial derivative of order m with respect to x.

2. Main Results

First, we state our assumption on Ω.

Assumption 2.1. The domain $\Omega \subset \mathbb{R}^d$ is of class C^1_u. In other words, there exist constants $r_0, K_0 > 0$ such that for any $x_0 \in \partial\Omega$ there exists a one-to-one continuously differentiable mapping Ψ from $B_{r_0}(x_0)$ onto a domain $J \subset \mathbb{R}^d$ such that

(i) $J_+ := \Psi(B_{r_0}(x_0) \cap \Omega) \subset \mathbb{R}^d_+ := \{x \in R^d : x^1 > 0\}$ and $\Psi(x_0) = 0$;
(ii) $\Psi(B_{r_0}(x_0) \cap \partial\Omega) = J \cap \{y \in \mathbb{R}^d : y^1 = 0\}$;
(iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))} \le K_0$ and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \le K_0|y_1 - y_2|$ for any $y_i \in J$;
(iv) Ψ_x is uniformly continuous in $B_{r_0}(x_0)$.

To proceed further we choose and fix a bounded smooth function ψ having properties in the following lemma (Gilbarg and Hörmander, 1980 or Kim and Krylov, 2004).

Lemma 2.2. *There is a bounded real-valued function ψ defined in $\bar{\Omega}$ such that*

(i) *ψ_x is uniformly continuous in $\overline{\Omega}$, $|\psi_x(x)| \geq 1$ on $\partial\Omega$, and the functions ψ and ρ are comparable near $\partial\Omega$ (in particular, if Ω is bounded then $\rho(x) \leq N\psi(x) \leq N\rho(x)$);*
(ii) *for any multi-index α we have*

$$\sup_{\Omega} \rho^{|\alpha|}(x)\, |D^\alpha \psi_x(x)| < \infty, \quad \lim_{\rho(x)\to 0} \rho^{|\alpha|}(x)|D^\alpha \psi_x(x)| = 0.$$

The functions ψ and ρ are comparable in Ω if Ω is bounded. Therefore, in many situations one can interchange $\psi(x)$ and $\rho(x)$. An advantage of using ψ on some occasions is that this function is infinitely differentiable.

We rewrite equation (1.1) in the following form:

$$\frac{du}{dt} = \psi^{2\alpha} a^{ij} u_{x^i x^j} + \psi^{2\alpha-1} b^i u_{x^i} + \psi^{2\alpha-2} cu + f. \tag{2.1}$$

Here i and j go from 1 to d, and the coefficients a^{ij}, b^i, c are Borel measurable functions of t, x.

To describe the assumptions on f we use the Banach spaces introduced in Kim and Krylov (2004), Krylov (1999a,b), and Lototsky (2001). If $\theta \in \mathbb{R}$, $n \in \{0, 1, 2, \dots\}$ and Ω is bounded, then

$$\begin{aligned} H_p^n &= H_p^n(\mathbb{R}^d) = \{u : u, Du, \dots, D^n u \in L_p\}, \\ L_{p,\theta}(\Omega) &:= H^0_{p,\theta}(\Omega) = L_p(\Omega, \rho^{\theta-d} dx), \\ H^n_{p,\theta}(\Omega) &:= \{u : u, \rho u_x, \dots, \rho^n D^n u \in L_{p,\theta}(\Omega)\}, \\ \|f\|^p_{H^n_{p,\theta}(\Omega)} &= \sum_{|\alpha|\leq n} \int_\Omega \left|\rho^{|\alpha|}(x) D^\alpha f(x)\right|^p \rho^{\theta-d}(x)\, dx. \end{aligned} \tag{2.2}$$

In general, for $p \in (1, \infty)$ and $\gamma \in \mathbb{R}$ define the space $H_p^\gamma = H_p^\gamma(\mathbb{R}^d) = (1-\Delta)^{-\gamma/2} L_p$ (called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the set of all distributions u such that $(1-\Delta)^{\gamma/2} u \in L_p$. We define

$$\|u\|_{H_p^\gamma} = \left\|(1-\Delta)^{\gamma/2} u\right\|_{L_p}.$$

The weighted Sobolev space $H^\gamma_{p,\theta}(\Omega)$ is defined as follows. Let $\zeta \in C_0^\infty(\mathbb{R}_+)$ be a function satisfying

$$\sum_{n=-\infty}^{\infty} \zeta(e^{n+t}) > 0, \quad \forall t \in \mathbb{R}.$$

For $x \in \Omega$ and $n \in \mathbb{Z} = \{0, \pm 1, \dots\}$ define

$$\zeta_n(x) = \zeta(e^n \psi(x)).$$

Then we have $\sum_n \zeta_n \geq \text{const} > 0$ in Ω and

$$\zeta_n \in C_0^\infty(\Omega), \quad |D^m \zeta_n(x)| \leq N(m) e^{mn}.$$

For $\theta, \gamma \in \mathbb{R}$, let $H^\gamma_{p,\theta}(\Omega)$ be the set of all distributions u on Ω such that

$$\|u\|^p_{H^\gamma_{p,\theta}(\Omega)} := \sum_{n\in\mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|^p_{H^\gamma_p} < \infty. \tag{2.3}$$

It is known (see, for instance, Lototsky, 2000) that the set $H^\gamma_{p,\theta}(\Omega)$ is independent of the choice of ζ and ψ, and the norms generated by different choices of ζ and ψ are all equivalent. In particular, if n is a nonnegative integer then

$$\|u\|^p_{H^n_{p,\theta}(\Omega)} \sim \sum_{|\alpha|\leq n} \int_\Omega |\psi^{|\alpha|} D^\alpha u|^p \psi^{\theta-d} dx. \tag{2.4}$$

The following results are taken from Krylov (1999b) and Lototsky (2000). For $v \in (0, 1]$, we denote

$$|u|_{C(X)} = \sup_X |u(x)|, \quad [u]_{C^v(X)} = \sup_{x\neq y} \frac{|u(x) - u(y)|}{|x - y|^v}.$$

Lemma 2.3. (i) *Assume that $\gamma - d/p = m + v$ for some $m = 0, 1, \ldots$ and $v \in (0, 1]$. Let i, j be multi-indices such that $|i| \leq m$, $|j| = m$. Then for any $u \in H^\gamma_{p,\theta}(\Omega)$, we have*

$$\psi^{|i|+\theta/p} D^i u \in C(\Omega), \quad \psi^{m+v+\theta/p} D^j u \in C^v_{loc}(\Omega),$$
$$|\psi^{|i|+\theta/p} D^i u|_{C(\Omega)} + [\psi^{m+v+\theta/p} D^j u]_{C^v(\Omega)} \leq N \|u\|_{H^\gamma_{p,\theta}(\Omega)}.$$

(ii) $\psi D, D\psi : H^\gamma_{p,\theta}(\Omega) \to H^{\gamma-1}_{p,\theta}(\Omega)$ *are bounded linear operators, and*

$$\|u\|_{H^\gamma_{p,\theta}(\Omega)} \leq N \|\psi u_x\|_{H^{\gamma-1}_{p,\theta}(\Omega)} + N \|u\|_{H^{\gamma-1}_{p,\theta}(\Omega)} \leq N \|u\|_{H^\gamma_{p,\theta}(\Omega)}, \tag{2.5}$$

$$\|u\|_{H^\gamma_{p,\theta}(\Omega)} \leq N \|(\psi u)_x\|_{H^{\gamma-1}_{p,\theta}(\Omega)} + N \|u\|_{H^{\gamma-1}_{p,\theta}(\Omega)} \leq N \|u\|_{H^\gamma_{p,\theta}(\Omega)}. \tag{2.6}$$

(iii) *For any $v, \gamma \in \mathbb{R}$, $\psi^v H^\gamma_{p,\theta}(\Omega) = H^\gamma_{p,\theta-pv}(\Omega)$, and*

$$\|u\|_{H^\gamma_{p,\theta-pv}(\Omega)} \leq N \|\psi^{-v} u\|_{H^\gamma_{p,\theta}(\Omega)} \leq N \|u\|_{H^\gamma_{p,\theta-pv}(\Omega)}. \tag{2.7}$$

Denote

$$\mathbb{H}^\gamma_p(T) = L_p((0, T], H^\gamma_p), \quad \mathbb{H}^\gamma_{p,\theta}(\Omega, T) = L_p((0, T], H^\gamma_{p,\theta}(\Omega)),$$
$$\mathbb{L}_{\ldots}(\ldots) = \mathbb{H}^0_{\ldots}(\ldots), \quad U^\gamma_p = H^{\gamma-2/p}_p,$$
$$U^\gamma_{p,\theta,\alpha}(\Omega) = \psi^{\frac{2}{p}(-1+\alpha)+1} H^{\gamma-2/p}_{p,\theta}(\Omega).$$

We write $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega, T)$ if $u \in \psi\, \mathbb{H}^{\gamma+2}_{p,\theta}(\Omega, T)$, $u(0, \cdot) \in U^{\gamma+2}_{p,\theta,\alpha}(\Omega)$ and for some $f \in \psi^{-1+2\alpha} \mathbb{H}^\gamma_{p,\theta}(\Omega, T)$

$$\frac{du}{dt} = f \tag{2.8}$$

in the sense of distribution. In other words, for any $\phi \in C_0^\infty(\Omega)$, the equality

$$(u(t,\cdot),\phi) = (u(0,\cdot),\phi) + \int_0^t (f(s,\cdot),\phi)ds$$

holds for all $t \le T$. In this situation we also write $f = u_t$.

The norm in $\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)$ is introduced by

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)} = [|u|]_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)} + \|u(0,\cdot)\|_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)},$$

where

$$\begin{aligned} [|u|]_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)} &:= \|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,T)} + \left\|\psi^{1-2\alpha}u_t\right\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,T)}, \\ \|u(0,\cdot)\|^p_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)} &= \left\|\psi^{\frac{2}{p}(1-\alpha)-1}u(0,\cdot)\right\|^p_{H^{\gamma+2-2/p}_{p,\theta}(\Omega)}. \end{aligned} \tag{2.9}$$

The set $\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)$ is independent of the choice of ψ, and for instance the norm $\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,T)}$ in (2.9) can be replaced by $\|u\|_{\mathbb{H}^{\gamma+2}_{p,\theta-p}(\Omega,T)}$. The simplest case of equation (2.1) is

$$\frac{du}{dt} = \psi^{2\alpha}\Delta u.$$

If $u \in \psi\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,T)$, then by Lemma 2.3, $\psi^{2\alpha}\Delta u = u_t \in \psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(\Omega,T)$. This shows that the semi-norm $[|u|]_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)}$ is introduced naturally.

Denote $\rho(x,y) = \rho(x)\wedge\rho(y)$. For $\sigma \in \mathbb{R}$, $\nu \in (0,1)$, and $k = 0,1,2,\ldots$, as in Gilbarg and Trudinger (1983), define

$$[f]^{(\sigma)}_k = [f]^{(\sigma)}_{k,\Omega} = \sup_{\substack{x\in\Omega\\|\beta|=k}} \rho^{k+\sigma}(x)|D^\beta f(x)|, \tag{2.10}$$

$$\begin{aligned} [f]^{(\sigma)}_{k+\nu} &= [f]^{(\sigma)}_{k+\nu,\Omega} = \sup_{\substack{x,y\in\Omega\\|\beta|=k}} \rho^{k+\nu+\sigma}(x,y)\frac{|D^\beta f(x) - D^\beta f(y)|}{|x-y|^\nu}, \\ |f|^{(\sigma)}_k &= |f|^{(\sigma)}_{k,\Omega} = \sum_{j=0}^k [f]^{(\sigma)}_{j,\Omega}, \quad |f|^{(\sigma)}_{k+\nu} = |f|^{(\sigma)}_{k+\nu,\Omega} = |f|^{(\sigma)}_{k,\Omega} + [f]^{(\sigma)}_{k+\nu,\Omega}. \end{aligned} \tag{2.11}$$

By $D^\beta f$ we mean either classical derivatives or Sobolev ones and in the latter case sup's in the above are understood as ess sup's.

Fix a function $\delta_0(\tau) \ge 0$ defined on $[0,\infty)$ such that $\delta_0(\tau) > 0$ unless $\tau \in \{0,1,2,\ldots\}$. For $\tau \ge 0$ define

$$\tau+ = \tau + \delta_0(\tau),$$

and fix constants

$$\delta_0, K \in (0,\infty), \quad p \in [2,\infty).$$

Assumption 2.4. (i) For any $\lambda \in \mathbb{R}^d$,

$$\delta_0|\lambda|^2 \leq a^{ij}(t,x)\lambda^i\lambda^j \leq K|\lambda|^2. \tag{2.12}$$

(ii) The function $a^{ij}(t,\cdot)$ is continuous in x, uniformly in t. In other words, for each $x \in \Omega$, $\varepsilon > 0$, there exists $\delta = \delta(x,\varepsilon) > 0$ such that

$$|a^{ij}(t,x) - a^{ij}(t,y)| < \epsilon$$

for all t and $y \in \Omega$ with $|x-y| < \delta$.

(iii) There is control on the behavior of a^{ij}, b^i, c near $\partial\Omega$. Namely,

$$\lim_{\substack{\rho(x)\to 0\\ x\in\Omega}} \sup_{\substack{y,t\\ |x-y|\leq\rho(x,y)}} |a^{ij}(t,x) - a^{ij}(t,y)| = 0. \tag{2.13}$$

$$\lim_{\substack{\rho(x)\to 0\\ x\in\Omega}} \sup_{t\leq T}(|b^i(t,x)| + |c(t,x)|) = 0. \tag{2.14}$$

(iv) For any $t > 0$,

$$|a^{ij}(t,\cdot)|^{(0)}_{\gamma+} + |b^i(t,\cdot)|^{(0)}_{\gamma+} + |c(t,\cdot)|^{(0)}_{\gamma+} \leq K.$$

Remark 2.5. It is easy to see that (2.13) is much weaker than uniform continuity of a. For instance, if $\delta \in (0,1)$, $d=1$ and $\Omega = \mathbb{R}_+$, then the function $a(x)$ equal to $2+\sin(|\ln x|^{\delta})$ for $0 < x \leq 1/2$ satisfies (2.13). Also note that the coefficients $\psi^{2\alpha-1}b^i$ and $\psi^{2\alpha-2}c$ are allowed to be unbounded if $\alpha < 1/2$ and $\alpha < 1$, respectively.

From this point on, we assume that

$$d-1 < \theta < d-1+p. \tag{2.15}$$

Here are our main results.

Theorem 2.6. *Let $\gamma \in [0,\infty)$ and Ω be bounded. Then under the above assumptions,*

(i) *for any $f \in \psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(\Omega,T)$ and $u_0 \in U^{\gamma+2}_{p,\theta,\alpha}(\Omega)$, equation (2.1) with initial data u_0 admits a unique solution u in the class $\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)$,*

(ii) *for this solution*

$$\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,T)} \leq N\Big(\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,T)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)}\Big), \tag{2.16}$$

where the constant N depends only on d, γ, p, θ, δ_0, α, K, T, and Ω.

The following results are immediate consequences of Lemma 2.3, Remark 3.7 in this article and Lemma 3.2 in Kim (2004).

Corollary 2.7. *Let $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega)$ be the solution of Theorem* 2.6.

(i) *If $\gamma + 2 - d/p = m + \nu$ for some $m = 0, 1, \ldots,$ $\nu \in (0, 1]$ and i, j are multi-indices such that $|i| \le m$, $|j| = m$, then for each t,*

$$\psi^{|i|-1+\theta/p} D^i u \in C(\Omega), \quad \psi^{m-1+\nu+\theta/p} D^j u \in C^{\nu}_{loc}(\Omega).$$

In particular,

$$\left|\psi^{|i|} D^i u(x)\right| \le N \psi^{1-\theta/p}(x).$$

(ii) *Let*

$$2/p < \mu < \beta \le 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon,$$

where $k \in \{0, 1, 2, \ldots\}$ and $\varepsilon \in (0, 1]$. Then for any $\delta \ge \beta - 1 + \theta/p$ and multi-indices i, j such that $|i| \le k$ and $|j| = k$,

$$\psi^{\delta+|i|} D^i u(t) - \psi^{\delta+|i|} D^i u(0) \in C^{\mu/2-1/p}([0, T], C(\Omega));$$

$$\sup_{s\neq t} \frac{|\psi^{\delta+|i|} D^i (u(t) - u(s))|^p_{C(\Omega)}}{|t-s|^{p\mu/2-1}} + \sup_{s\neq t} \frac{[\psi^{\delta+|j|+\varepsilon} D^j (u(t) - u(s))]^p_{C^{\varepsilon}(\Omega)}}{|t-s|^{p\mu/2-1}}$$

$$\le N\Big(\|\psi^{1-2\alpha} f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,T)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)} \Big).$$

3. Auxiliary Results

The main goal of this section is to give some estimates for solutions of equations defined on $\mathbb{R}^d$ and $\mathbb{R}^d_+$.

Lemma 3.1. *Suppose that a^{ij} are uniformly continuous in x, satisfy* (2.12) *and*

$$|a^{ij}(t, \cdot)|_{C^{|\gamma|+}} \le K.$$

Also assume that $f \in \mathbb{H}^{\gamma}_p(T)$, $u_0 \in U^{\gamma+2}_p$ and $u \in \mathbb{H}^{\gamma+1}_p(T)$ is a solution of

$$\frac{du}{dt} = a^{ij} u_{x^i x^j} + f \quad u(0, \cdot) = u_0.$$

Then $u \in \mathbb{H}^{\gamma+2}_p(T)$ and

$$\|u\|^p_{\mathbb{H}^{\gamma+2}_p(T)} \le N\left(\|u\|^p_{\mathbb{H}^{\gamma+1}_p(T)} + \|f\|^p_{\mathbb{H}^{\gamma}_p(T)} + \|u_0\|^p_{U^{\gamma+2}_p} \right), \tag{3.1}$$

where N depends only on d, p, δ_0, K and the continuity of a^{ij}.

Proof. One easily gets the assertion of this lemma by inspecting the proofs of Lemma 6.6 and Theorem 5.1 in Krylov (1999a). We only give few comments.

First, in Krylov (1999a), $\|u_{xx}\|_{\mathbb{H}^{\gamma}_{p}(T)}$ is estimated instead of $\|u\|_{\mathbb{H}^{\gamma+2}_{p}(T)}$. One gets the estimate $\|u\|_{\mathbb{H}^{\gamma+2}_{p}(T)}$ from

$$\|u\|_{\mathbb{H}^{\gamma+2}_{p}(T)} \leq N\|u\|_{\mathbb{H}^{\gamma}_{p}(T)} + N\|u_{xx}\|_{\mathbb{H}^{\gamma}_{p}(T)}.$$

Second, actually the constant N in Lemma 6.6 in Krylov (1999a) depends also on T, but this is used just to drop the term $\|u\|^{p}_{\mathbb{H}^{\gamma+1}_{p}(T)}$ in (3.1). □

If $\Omega = \mathbb{R}^{d}_{+}$, we will use the spaces $H^{\gamma}_{p,\theta}$ and $\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)$ introduced in Krylov (1999b). They are defined by (formally) taking $\psi(x) = x^{1}$, so that $\zeta_{-n}(e^{n}x) = \zeta(x)$ and (2.3) becomes

$$\|u\|^{p}_{H^{\gamma}_{p,\theta}} := \sum_{n\in\mathbb{Z}} e^{n\theta}\|\zeta(\cdot)u(e^{n}\cdot)\|^{p}_{H^{\gamma}_{p}}.$$

For any smooth function $\eta \in C^{\infty}_{0}(\mathbb{R}_{+})$,

$$\sum_{n\in\mathbb{Z}} e^{n\theta}\|\eta(\cdot)u(e^{n}\cdot)\|^{p}_{H^{\gamma}_{p}} \leq N\|u\|^{p}_{H^{\gamma}_{p,\theta}} \tag{3.2}$$

As in Krylov (1999b), by M^{α} we denote the operator of multiplying by $(x^{1})^{\alpha}$ and $M = M^{1}$. Thus,

$$\|u\|_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)} = [|u|]_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)} + \|u(0,\cdot)\|_{U^{\gamma+2}_{p,\theta,\alpha}},$$

where

$$[|u|]_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)} := \|M^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} + \left\|M^{1-2\alpha}u_{t}\right\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)},$$

$$\|u(0,\cdot)\|_{U^{\gamma+2}_{p,\theta,\alpha}} = \left\|M^{\frac{2}{p}(1-\alpha)-1}u(0,\cdot)\right\|_{H^{\gamma+2-2/p}_{p,\theta}}.$$

Lemma 3.2 (Krylov, 1999b). *All the claims of Lemma 2.3 hold true if ψ and $H^{\gamma}_{p,\theta}(\Omega)$ are (formally) replaced by M and $H^{\gamma}_{p,\theta}$, respectively. Moreover for any fixed p and γ, the constants N (corresponding to those in (2.5)–(2.7)) and the constant N in (3.2) are bounded on any closed interval $I \subset (d-1, d-1+p)$.*

Lemma 3.3. *For any nonnegative integer $n \geq \gamma$, the set*

$$\mathfrak{H}^{n}_{p,\theta,\alpha}(T) \cap \bigcup_{k=1}^{\infty} C([0,T], C^{n}_{0}(G_{k})), \tag{3.3}$$

where $G_{k} = (1/k, k) \times \{|x'| < k\}$, is everywhere dense in $\mathfrak{H}^{\gamma}_{p,\theta,\alpha}(T)$.

Proof. See the proof of Theorem 2.9 in Krylov and Lototsky (1999). Actually in Krylov and Lototsky (1999), the lemma is proved when $\alpha = 0$. One can easily check that the same arguments hold true for any $\alpha \geq 0$. □

The following three lemmas are taken from Kim and Krylov (2004).

Lemma 3.4. *Let constants $C, \delta \in (0, \infty)$, a function $u \in H^{\gamma}_{p,\theta}$, and q be the smallest integer such that $|\gamma| + 2 \le q$.*

(i) *Let $\eta_n \in C^{\infty}(\mathbb{R}^d_+)$, $n = 1, 2, \ldots$, satisfy*

$$\sum_n M^{|\alpha|}|D^{\alpha}\eta_n| \le C \quad \text{in } \Omega \tag{3.4}$$

for any multi-index α such that $0 \le |\alpha| \le q$. Then

$$\sum_n \|\eta_n u\|^p_{H^{\gamma}_{p,\theta}} \le NC^p \|u\|^p_{H^{\gamma}_{p,\theta}},$$

where the constant N is independent of u, θ, and C.

(ii) *If in addition to the condition in* (i)

$$\sum_n \eta_n^2 \ge \delta \quad \text{on } \mathbb{R}^d_+, \tag{3.5}$$

then

$$\|u\|^p_{H^{\gamma}_{p,\theta}} \le N \sum_n \|\eta_n u\|^p_{H^{\gamma}_{p,\theta}}, \tag{3.6}$$

where the constant N is independent of u and θ.

The reason the first inequality in (3.7) below is written for η_n^4 (not for η_n^2) as in the above lemma is to have the possibility to apply Lemma 3.4 to η_n^2. Also observe that obviously $\sum a^2 \le (\sum |a|)^2$.

Lemma 3.5. *For each $\varepsilon > 0$ and $q = 1, 2, \ldots$ there exist non-negative functions $\eta_n \in C^{\infty}_0(\mathbb{R}^d_+)$, $n = 1, 2, \ldots$ such that* (i) *on $\mathbb{R}^d_+$ for each multi-index α with $1 \le |\alpha| \le q$ we have*

$$\sum_n \eta_n^4 \ge 1, \quad \sum_n \eta_n \le N(d), \quad \sum_n M^{|\alpha|}|D^{\alpha}\eta_n| \le \varepsilon; \tag{3.7}$$

(ii) *for any n and $x, y \in \operatorname{supp}\eta_n$ we have $|x - y| \le N(x^1 \wedge y^1)$, where $N = N(d, q, \varepsilon) \in [1, \infty)$.*

Lemma 3.6. *Let $p \in (1, \infty)$, $\gamma, \theta \in \mathbb{R}$. Then there exists a constant $N = N(\gamma, |\gamma|+, p, d)$ such that*

(i)

$$\|af\|_{H^{\gamma}_{p,\theta}(\Omega)} \le N|a|^{(0)}_{|\gamma|+}\|f\|_{H^{\gamma}_{p,\theta}(\Omega)}, \tag{3.8}$$

(ii) *if $\gamma = 1, 2, \ldots$ then*

$$\|af\|_{H^{\gamma}_{p,\theta}} \le N \sup_{\mathbb{R}^d_+}|a|\,\|f\|_{H^{\gamma}_{p,\theta}} + N\|f\|_{H^{\gamma-1}_{p,\theta}} \sup_{\mathbb{R}^d_+} \sup_{1\le|\alpha|\le\gamma} |M^{|\alpha|}D^{\alpha}a|, \tag{3.9}$$

(iii) *if* $|\gamma| \notin \{1, 2, \dots\}$ *then*

$$\|af\|_{H^{\gamma}_{p,\theta}} \leq N\Big(\sup_{\mathbb{R}^d_+}|a|\Big)^s \Big(|a|^{(0)}_{|\gamma|+}\Big)^{1-s}\|f\|_{H^{\gamma}_{p,\theta}}, \tag{3.10}$$

where $s = 1$ *if* $\gamma = 0$, *and* $s = 1 - \frac{|\gamma|}{|\gamma|+}$ *otherwise.*

Remark 3.7. It follows that for any $\nu \geq 0$, ψ^{ν} is a point-wise multiplier in $H^{\gamma}_{p,\theta}(\Omega)$. Thus, by Lemma 2.3(iii), if $\theta_1 \leq \theta_2$ then

$$\|u\|_{H^{\gamma}_{p,\theta_2}(\Omega)} \leq N\|\psi^{(\theta_2-\theta_1)/p}u\|_{H^{\gamma}_{p,\theta_1}(\Omega)} \leq N\|u\|_{H^{\gamma}_{p,\theta_1}(\Omega)}.$$

Consequently if $0 \leq \alpha_1 \leq \alpha_2$ then

$$\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta,0}(\Omega,T)} \leq N\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta,\alpha_1}(\Omega,T)} \leq N\|u\|_{\mathfrak{H}^{\gamma}_{p,\theta,\alpha_2}(\Omega,T)}.$$

Lemma 3.8. *Let* a^{ij} *be independent of* x. *If* $f \in M^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(T)$, $u_0 \in U^{\gamma+2}_{p,\theta,\alpha}$ *and* $u \in \mathfrak{H}^{\gamma+1}_{p,\theta,\alpha}(T)$ *is a solution of*

$$\frac{du}{dt} = M^{2\alpha}a^{ij}u_{x^ix^j} + f, \quad u(0) = u_0 \tag{3.11}$$

then $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)$ *and*

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \leq N\left(\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(T)} + \|M^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}}\right), \tag{3.12}$$

where N *depends only on* $\delta_0, \alpha, p, d, \theta, \gamma$ *and* K.

Proof. Denote $c_n := e^{2n(1-\alpha)}$. Obviously

$$\begin{aligned}\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} &\leq N\sum_n e^{n(\theta-p)}\|u(t,e^nx)\zeta\|^p_{\mathbb{H}^{\gamma+2}_p(T)} \\ &= N\sum_n e^{n(\theta-p+2-2\alpha)}\|u(c_nt,e^nx)\zeta\|^p_{\mathbb{H}^{\gamma+2}_p(c_n^{-1}T)}.\end{aligned} \tag{3.13}$$

Choose a function $\xi \in C_0^{\infty}(\mathbb{R}_+)$ such that $\xi = 1$ on the support of ζ, and define

$$a^{ij}_n(t,x) := (x^1)^{2\alpha}a^{ij}(c_nt)\xi(x) + (1-\xi(x))\delta^{ij},$$

where $\delta^{ij} = 1$ if $i = j$, and $\delta^{ij} = 0$ otherwise. It is easy to see that for each $k > 0$,

$$\sup_n\sup_t|a^{ij}_n|_{C^k} < \infty \tag{3.14}$$

and a^{ij}_n satisfy (2.12) (uniformly in n). Also note that $v_n(t,x) := u(c_nt, e^nx)\zeta(x)$ satisfies

$$\frac{dv_n}{dt} = a^{ij}_n v_{nx^ix^j} + f_n$$

where

$$f_n = -2a_n^{ij}e^n u_{x^i}(c_n t, e^n x)\zeta_{x^j} - a_n^{ij} u(c_n t, e^n x)\zeta_{x^i x^j} + c_n f(c_n t, e^n x)\zeta.$$

By Lemma 3.1 it follows that $v_n \in \mathbb{H}_p^{\gamma+2}(c_n^{-1}\tau)$ and

$$\|v_n\|^p_{\mathbb{H}_p^{\gamma+2}(c_n^{-1}\tau)} \le N\left(\|v_n\|^p_{\mathbb{H}_p^{\gamma+1}(c_n^{-1}\tau)} + \|f_n\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}\tau)} + \|u_0(e^n x)\zeta\|^p_{U_p^{\gamma+2}}\right),$$

where the constant N is independent of n.

Coming back to (3.13) one gets (3.12). Indeed, by (3.14) (also see Lemma 5.2 in Krylov, 1999a),

$$\|a_n e^n u_x(c_n t, e^n x)\zeta_x\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}T)} \le N(d,p,\gamma)\|e^n u_x(c_n t, e^n x)\zeta_x\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}T)},$$
$$\|a_n u(c_n t, e^x)\zeta_{xx}\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}\tau)} \le N(d,p,\gamma)\|u(c_n t, e^x)\zeta_{xx}\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}\tau)}.$$

Also

$$\sum_n e^{n(\theta-p+2-2\alpha)}\|e^n u_x(c_n t, e^n x)\zeta_x\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}T)}$$
$$= \sum_n e^{n\theta}\|u_x(t, e^n x)\zeta_x\|^p_{\mathbb{H}_p^{\gamma}(T)} \le N\|u_x\|^p_{\mathbb{H}_{p,\theta}^{\gamma}(T)} \le N\|M^{-1}u\|^p_{\mathbb{H}_{p,\theta}^{\gamma+1}(T)},$$
$$\sum_n e^{n(\theta-p+2-2\alpha)}\|u(c_n t, e^x)\zeta_{xx}\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}T)}$$
$$= \sum_n e^{n(\theta-p)}\|u(t, e^n x)\zeta_{xx}\|^p_{\mathbb{H}_p^{\gamma}(T)} \le N\|M^{-1}u\|^p_{\mathbb{H}_{p,\theta}^{\gamma}(T)},$$
$$\sum_n e^{n(\theta-p+2-2\alpha)}\|c_n f(c_n t, e^n x)\zeta\|^p_{\mathbb{H}_p^{\gamma}(c_n^{-1}T)}$$
$$= \sum_n e^{n(\theta+p(1-2\alpha))}\|f(t, e^n x)\zeta\|^p_{\mathbb{H}_p^{\gamma}(T)} \le N\left\|M^{1-2\alpha}f\right\|^p_{\mathbb{H}_{p,\theta}^{\gamma}(T)},$$
$$\sum_n e^{n(\theta-p+2-2\alpha)}\|u_0(e^n x)\zeta\|^p_{U_p^{\gamma+2}} \le N\|u_0\|^p_{U_{p,\theta,\alpha}^{\gamma+2}}.$$

The lemma is proved. □

Lemma 3.9. *Assume that a^{ij} are independent of x. Then for any $u \in \mathfrak{H}^2_{p,\theta,\alpha}(T)$,*

$$\|M^{-1}u\|^p_{\mathbb{L}_{p,\theta}(T)} \le N\|M^{1-2\alpha}(u_t - M^{2\alpha}a^{ij}u_{x^i x^j})\|^p_{\mathbb{L}_{p,\theta}(T)} + N\|u(0,\cdot)\|^p_{U^1_{p,\theta,\alpha}} \tag{3.15}$$

where N depends only on δ_0, α, θ, p, d and K.

Proof. If $\alpha = 0$, then this lemma is already proved in Krylov and Lototsky (1999). Here we modify the proof. The operator

$$MD_iD_j : \mathfrak{H}^{\gamma}_{p,\theta,\alpha}(T) \to \mathbb{H}^{\gamma-2}_{p,\theta}(T)$$

is bounded. By Lemma 3.3, it follows that we need to prove the lemma only for functions u belonging to the set (3.3) with sufficiently large n.

Denote $c = 2 + \theta - d - p$, $\bar{c} = c - 2\alpha$ and

$$f := u_t - M^{2\alpha} a^{ij} u_{x^i x^j}$$

Then,

$$\begin{aligned}(x^1)^{\bar{c}}|u(T,x)|^p &= (x^1)^{\bar{c}}|u_0|^p + (x^1)^{\bar{c}} \int_0^T (|u|^p)_t dt \\ &= (x^1)^{\bar{c}}|u_0|^p + \int_0^T [p(x^1)^c |u|^{p-2} u a^{ij} u_{x^i x^j} + p(x^1)^{c-1}|u|^{p-2} u M^{1-2\alpha} f]\, dt\end{aligned}$$

Integrate over $\mathbb{R}^d_+$ and get

$$\begin{aligned}0 \le \int_{\mathbb{R}^d_+} (x^1)^{\bar{c}}|u_0|^p\, dx + \int_0^T \int_{\mathbb{R}^d_+} [&-p(p-1)(x^1)^c|u|^{p-2} a^{ij} u_{x^i} u_{x^j} \\ &- c(x^1)^{c-1} a^{i1}(|u|^p)_{x^i} + p(x^1)^{c-1}|u|^{p-1}|M^{1-2\alpha} f|] dx\, dt. \qquad (3.16)\end{aligned}$$

By Young's inequality and Corollary 6.2 in Krylov (1999b),

$$p(x^1)^{c-1}|u|^{p-1}|M^{1-2\alpha} f| \le \delta a^{11}|M^{-1}u|^p (x^1)^{\theta-d} + N|M^{1-2\alpha} f|^p (x^1)^{\theta-d},$$

$$\int_{\mathbb{R}^d_+} (x^1)^c |u|^{p-2} a^{ij} u_{x^i} u_{x^j}\, dx \ge a^{11}(1-c)^2 p^{-2} \int_{\mathbb{R}^d_+} |M^{-1}u|^p (x^1)^{\theta-d}\, dx.$$

Also observe

$$-c\int_{\mathbb{R}^d_+} (x^1)^{c-1} a^{i1}(|u|^p)_{x^i}\, dx = a^{11} c(c-1) \int_{\mathbb{R}^d_+} |M^{-1}u|^p (x^1)^{\theta-d}\, dx,$$

$$\int_{\mathbb{R}^d_+} (x^1)^{\bar{c}} |u_0|^p\, dx = \int_{\mathbb{R}^d_+} \left|M^{\frac{2}{p}(1-\alpha)-1} u_0\right|^p (x^1)^{\theta-d} dx \le \|u_0\|^p_{U^{1,\alpha}_{p,\theta}}.$$

Coming back to (3.16) we get

$$\begin{aligned}&a^{11}[p(p-1)(1-c)^2 p^{-2} + c(1-c) - \delta]\|M^{-1}u\|^p_{\mathbb{L}_{p,\theta}(T)} \\ &\le N\left(\|M^{1-2\alpha} f\|^p_{\mathbb{L}_{p,\theta}(T)} + \|u_0\|^p_{U^{1,\alpha}_{p,\theta}}\right).\end{aligned}$$

Now it is enough to check that, if $d - 1 < \theta < d - 1 + p$, for sufficiently small δ

$$p(p-1)(1-c)^2 p^{-2} + c(1-c) - \delta = p^{-1}(d + p - \theta - 1)(\theta + 1 - d) - \delta > 0.$$

This leads the proof of lemma to the end. □

Lemmas 3.8 and 3.9 yield the following result.

Corollary 3.10. *Let $\gamma \in [0, \infty)$ and a^{ij} be independent of x. If $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)$ is a solution of equation* (3.11), *then*

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \leq N(\theta)\left(\|M^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + N\|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}}\right), \tag{3.17}$$

where $N(\theta)$ depends only on δ_0, p, d, α, θ, γ and K.

Theorem 3.11. *Let $\gamma \in [0, \infty)$,*

$$|a^{ij}(t, \cdot)|^{(0)}_{\gamma+} + |b^i(t, \cdot)|^{(0)}_{\gamma+} + |c(t, \cdot)|^{(0)}_{\gamma+} \leq K_1$$

and

$$|a^{ij}(t, x) - a^{ij}(t, y)| + |b^i(t, x)| + |c(t, x)| \leq \beta \tag{3.18}$$

whenever $t > 0$, $x, y \in \mathbb{R}^d_+$, and $|x - y| \leq x^1 \wedge y^1$. Then there exists $\beta_0 = \beta_0(\theta)$ depending only on d, p, θ, α, δ_0, K, γ and K_1 such that if

$$\beta \leq \beta_0 \tag{3.19}$$

and $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)$ is a solution of

$$\frac{du}{dt} = M^{2\alpha}a^{ij}u_{x^ix^j} + M^{2\alpha-1}b^iu_{x^i} + M^{2\alpha-2}cu + f, \quad u(0) = u_0 \tag{3.20}$$

then

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \leq N\left(\|M^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}}\right), \tag{3.21}$$

where the constant N depends only on d, p, θ, α, γ, δ_0, K and K_1.

Proof. Case 1: $\gamma \notin \{1, 2, \ldots\}$. Take the least integer $q \geq \gamma + 4$. Also take an $\varepsilon \in (0, 1)$ to be specified later and take a sequence of functions η_n, $n = 1, 2, \ldots$, from Lemma 3.5 corresponding to ε, q. Then by Lemma 3.4, we have

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \leq N\sum_{n=1}^{\infty}\left\|M^{-1}u\eta_n^2\right\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)}. \tag{3.22}$$

For any n let x_n be a point in $\operatorname{supp}\eta_n$ and $a_n(t) = a(t, x_n)$. From (3.20), we easily have

$$\frac{d(u\eta_n^2)}{dt} = M^{2\alpha}a^{ij}_n(u\eta_n^2)_{x^ix^j} + M^{-1+2\alpha}f_n,$$

where

$$\begin{aligned} f_n = {} & \left(a^{ij} - a^{ij}_n\right)\eta_n^2Mu_{x^ix^j} - 2a^{ij}_nM(\eta_n^2)_{x^i}u_{x^j} - a^{ij}_nM^{-1}uM^2(\eta_n^2)_{x^ix^j} \\ & + \eta_n^2b^iu_{x^i} + \eta_n^2cM^{-1}u + M^{1-2\alpha}f\eta_n^2. \end{aligned}$$

By Corollary 3.10, for each n,

$$\|M^{-1}u\eta_n^2\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \le N\Big(\|f_n\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|u_0\eta_n^2\|_{U^{\gamma+2}_{p,\theta,\alpha}}\Big) \tag{3.23}$$

and by (3.10),

$$\|(a^{ij}-a^{ij}_n)\eta_n^2 Mu_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} \le N\|\eta_n Mu_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} \sup_{[0,T)\times\mathbb{R}^d_+} |(a^{ij}-a^{ij}_n)\eta_n|^s, \tag{3.24}$$

where $s=1$ if $\gamma=0$, and $s=1-\gamma/\gamma+$ otherwise.

By Lemma 3.5(ii), for each n and $x, y \in \operatorname{supp}\eta_n$ we have $|x-y| \le N(\varepsilon)(x^1\wedge y^1)$, where $N(\varepsilon)=N(d,q,\varepsilon)$, and we can easily find not more than $N(\varepsilon)+2\le 3N(\varepsilon)$ points x_i lying on the straight segment connecting x and y and including x and y, such that $|x_i-x_{i+1}|\le x^1_i\wedge x^1_{i+1}$. It follows from our assumptions

$$\sup_{[0,T)\times\mathbb{R}^d_+} |(a^{ij}-a^{ij}_n)\eta_n| \le 3N(\varepsilon)\beta.$$

We substitute this to (3.24) and get

$$\|(a^{ij}-a^{ij}_n)\eta_n^2 Mu_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} \le NN(\varepsilon)\beta^s\|\eta_n Mu_{x^ix^j}\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)}.$$

Similarly,

$$\|\eta_n^2 b^i u_{x^i}\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|\eta_n^2 cM^{-1}u\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} \le N\beta^s\Big(\|\eta_n u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|\eta_n M^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(T)}\Big).$$

Furthermore

$$\|u_x\|_{H^{\gamma+1}_{p,\theta}} \le N\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}}, \quad \|Mu_{xx}\|_{H^{\gamma}_{p,\theta}} \le N\|M^{-1}u\|_{H^{\gamma+2}_{p,\theta}}. \tag{3.25}$$

Coming back to (3.23) and (3.22) and using Lemma 3.4, we conclude

$$\begin{aligned}\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \le{}& NN(\varepsilon)\beta^{ps}\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} + N\|M^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)}\\ &+ N\|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}} + NC^p\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(T)},\end{aligned} \tag{3.26}$$

where

$$C = \sup_{|\alpha|\le q-2}\sum_{n=1}^{\infty} M^{|\alpha|}\big(|D^\alpha(M(\eta_n^2)_x)| + |D^\alpha(M^2(\eta_n^2)_{xx})|\big).$$

By construction, we have $C\le N\varepsilon$.

Hence (3.26) yields

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \le N_1(N(\varepsilon)\beta^{ps}+\varepsilon^p)\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} + N\Big(\|M^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}}\Big).$$

Finally to get the estimate (3.21) it's enough to choose first ε and then β_0, so that $N_1(N(\varepsilon)\beta^{ps}+\varepsilon^p)\le 1/2$ for $\beta\le\beta_0$.

Case 2: $\gamma \in \{1, 2, 3, \dots\}$. Take ε, η_n as in case 1 and arrive at (3.23) which is

$$\|M^{-1}u\eta_n^2\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \le N\Big(\|f_n\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|u_0\eta_n^2\|_{U^{\gamma+2}_{p,\theta,\alpha}}\Big).$$

By (3.9),

$$\big\|(a^{ij} - a^{ij}_n)\eta_n^2 M u_{x^ix^j}\big\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} \le N \sup_{t,x}\big\|(a^{ij} - a^{ij}_n)\eta_n\big|\|Mu_{xx}\eta_n\|_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + N\|Mu_{xx}\eta_n\|_{\mathbb{H}^{\gamma-1}_{p,\theta}(T)}$$

After this, one proceeds as in case 1, and gets (by choosing ε and β_0 properly)

$$\|M^{-1}u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} \le N\Big(\|M^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(T)} + \|M^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(T)} + \|u_0\|_{U^{\gamma+2}_{p,\theta,\alpha}}\Big).$$

Here we use

$$\|M^{-1}u\|_{\mathbb{H}^{\gamma+1}_{p,\theta}(T)} \le \delta\|M^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(T)} + N(\delta)\|M^{-1}u\|_{\mathbb{H}^{2}_{p,\theta}(T)}.$$

Now it is enough to use the result of case 1 (by reducing β_0 if necessary). □

Remark 3.12. By inspection the proofs of previous lemmas (based on Lemma 3.2) one can easily check that for any fixed p, γ and closed interval $I \subset (d-1, d-1+p)$,

$$N_I := \sup_{\theta\in I} N(\theta) < \infty,$$

where $N(\theta)$ is the constant in (3.17). Hence, from the proof of Theorem 3.11 it also follows that

$$\beta_{0I} := \inf_{\theta\in I} \beta_0(\theta) > 0.$$

4. Proof of Theorem 2.6

First we prove a priori estimate.

Lemma 4.1. *The estimate* (2.16) *holds true given that a solution* $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega, T)$ *already exists.*

Proof. Let $x_0 \in \partial\Omega$. Without loss of generality we assume that $x_0 = 0$ and $e_1 := \langle 1, 0, 0, \dots\rangle$ is the unit normal vector of $\partial\Omega$ at x_0 (otherwise do rotation and translation properly). For the function Ψ in Assumption 2.1 we define $\Psi := (\psi(x), x')$, so that for $n = 0, 1, 2, \dots$

$$|\Psi_x|^{(0)}_{n,B_{r_0}(x_0)\cap\Omega} + |\Psi^{-1}_x|^{(0)}_{n,J_+} < N(n) < \infty, \tag{4.1}$$

$$\rho(x)\Psi_{xx}(x) \to 0 \quad \text{as } \rho(x) \to 0, \tag{4.2}$$

$$\psi(\Psi^{-1}) = x^1 \quad x \in J_+.$$

Denote $r = r_0/K_0$ and fix smooth functions $\eta \in C_0^\infty(B_r)$, $\varphi \in C^\infty(\mathbb{R})$ such that $0 \le \eta, \varphi \le 1$, and $\eta = 1$ in $B_{r/2}$, $\varphi(t) = 1$ for $t \le -3$, and $\varphi(t) = 0$ for $t \ge -1$ and $0 \ge \varphi' \ge -1$.

For each sufficiently large integer m, $t > 0$, $x \in \mathbb{R}^d_+$ introduce $\varphi_m(x) = \varphi(m^{-1} \ln x^1)$,

$$a_m := \tilde{a}\varphi_m \eta + (1 - \varphi_m \eta)I, \quad b_m := \tilde{b}\varphi_m \eta, \quad c_m := \tilde{c}\varphi_m \eta,$$

where I is the $d \times d$ identity matrix,

$$\tilde{a}^{ij}(t, x) = \hat{a}^{ij}(t, \Psi^{-1}(x)), \quad \tilde{b}^i(t, x) = \hat{b}^i(t, \Psi^{-1}(x)),$$
$$\hat{a}^{ij} = a^{rs}\Psi^i_{x^r}\Psi^j_{x^s}, \qquad \hat{b}^i = \psi a^{rs}\Psi^i_{x^r x^s} + b^r \Psi^i_{x^r}, \quad \tilde{c}(t, x) = c(t, \Psi^{-1}(x)).$$

Using Lemma 3.7 of Kim and Krylov (2004), one can easily check that there exists a constant K' (independent of m) such that

$$|a^{ij}_m|^{(0)}_{\gamma+} + |b^i_m|^{(0)}_{\gamma+} + |c_m|^{(0)}_{\gamma+} \leq K'.$$

Take β_0 from Theorem 3.11 corresponding to $d, p, \theta, \alpha, \delta_0, \gamma, K$ and K'. Observe that $\varphi(m^{-1} \ln x^1) = 0$ for $x^1 \geq e^{-m}$ and $|\varphi(m^{-1} \ln x^1) - \varphi(m^{-1} \ln y^1)| \leq m^{-1}$ if $|x^1 - y^1| \leq x^1 \wedge y^1$. Also note that

$$x^1 \Psi_{xx}(\Psi^{-1}(x)) \to 0 \quad \text{as } x^1 \to 0.$$

Thus one easily finds $m > 0$ such that

$$|a_m(t, x) - a_m(t, y)| + |b_m(t, x)| + |c_m(t, x)| \leq \beta_0,$$

whenever $t > 0, x, y \in \mathbb{R}^d_+$ and $|x - y| \leq x^1 \wedge y^1$.

Now we fix a $\rho_0 < r_0$ such that

$$\Psi(B_{\rho_0}(x_0)) \subset B_{r/2} \cap \{x : x^1 \leq e^{-3m}\}.$$

Let ξ be a smooth function with support in $B_{\rho_0}(x_0)$ and denote $v := (u\xi)(\Psi^{-1})$ and continue v as zero in $\mathbb{R}^d_+ \setminus \Psi(B_{\rho_0}(x_0))$. Since $\eta\varphi_m = 1$ on $\Psi(B_{\rho_0}(x_0))$, the function v satisfies

$$\frac{dv}{dt} = M^{2\alpha} a^{ij}_m v_{x^i x^j} + M^{2\alpha-1} b^i_m v_{x^i} + M^{2\alpha-2} c_m v + \hat{f}$$

where

$$\hat{f} = \tilde{f}(\Psi^{-1}), \quad \tilde{f} = -2\psi^{2\alpha} a^{ij} u_{x^i} \xi_{x^j} - \psi^{2\alpha} u a^{ij} \xi_{x^i x^j} - \psi^{2\alpha-1} u b^i \xi_{x^i} + \xi f.$$

Next we observe that by Theorem 3.2 in Lototsky (2000) (or see Kim and Krylov, 2004) for any $\nu, \alpha \in \mathbb{R}$ and $h \in \psi^{-\alpha} H^\nu_{p,\theta}(\Omega)$ with support in $B_{\rho_0}(x_0)$

$$\|\psi^\alpha h\|_{H^\nu_{p,\theta}(\Omega)} \sim \left\|M^\alpha h(\Psi^{-1})\right\|_{H^\nu_{p,\theta}}. \tag{4.3}$$

Therefore we conclude that $v \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(T)$, and thus by Theorem 3.11 for each $t \leq T$,

$$\|M^{-1} v\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(t)} \leq N\|M^{1-2\alpha} \hat{f}\|_{\mathbb{H}^\gamma_{p,\theta}(t)} + N\left\|u_0(\Psi^{-1})\xi(\Psi^{-1})\right\|_{U^{\gamma+2}_{p,\theta,\alpha}}.$$

By using (4.3) again we obtain

$$\|\psi^{-1}u\xi\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,t)} \leq N\|a\xi_x\psi u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|a\xi_{xx}\psi u\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|\xi_x bu\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|\xi\psi^{1-2\alpha}f\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|\xi u_0\|_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)}.$$

Next, since

$$|\xi_x a(t,\cdot)|^{(0)}_{\gamma+}, \quad |\xi_{xx}\psi a(t,\cdot)|^{(0)}_{\gamma+}, \quad |\xi_x b(t,\cdot)|^{(0)}_{\gamma+}$$

are bounded on $[0, T]$,

$$\|\psi^{-1}u\xi\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,t)} \leq N\|\psi u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|u\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|\psi^{1-2\alpha}f\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|u_0\|_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)}.$$

Note that the above constants ρ_0, m, K', N are independent of x_0. Therefore, to estimate the norm $\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,t)}$, one introduces a partition of unity $\zeta_{(i)}$, $i = 0, 1, 2, \ldots, N$ such that $\zeta_{(0)} \in C^{\infty}_0(\Omega)$ and $\zeta_{(i)} \in C^{\infty}_0(B_{\rho_0}(x_i))$, $x_i \in \partial\Omega$ for $i \geq 1$. Then one estimates $\|\psi^{-1}u\zeta_{(0)}\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,t)}$ using Theorem 5.1 in Krylov (1999a) and the other norms as above. By summing up those estimates one gets

$$\|\psi^{-1}u\|_{\mathbb{H}^{\gamma+2}_{p,\theta}(\Omega,t)} \leq N\|\psi u_x\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|u\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|\psi^{1-2\alpha}f\|_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|u_0\|_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)}. \tag{4.4}$$

Furthermore, we know

$$\|\psi u_x\|_{H^{\gamma}_{p,\theta}(\Omega)} \leq N\|u\|_{H^{\gamma+1}_{p,\theta}(\Omega)}.$$

Therefore (4.4) yields

$$\|u\|^p_{\mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega,t)} \leq N\|u\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta}(\Omega,t)} + N\|\psi^{1-2\alpha}f\|^p_{\mathbb{H}^{\gamma}_{p,\theta}(\Omega,t)} + N\|u_0\|^p_{U^{\gamma+2}_{p,\theta,\alpha}(\Omega)}.$$

Now the estimate (2.16) follows from Theorem 2.7 in Lototsky (1999) (or see Kim, 2004, p. 16) and Gronwall's inequality. The lemma is proved. □

Remark 4.2. Instead of (2.14), actually we are using

$$\lim_{\substack{\rho(x)\to 0\\ x\in\Omega}} \sup_t (|b^i(t,x)| + |c(t,x)|) < \bar{\beta}_0(\theta), \tag{4.5}$$

for some $\bar{\beta}_0(\theta) = \bar{\beta}_0(\theta, d, p, \gamma) > 0$, and by Remark 3.12 one can take $\bar{\beta}_0(\theta)$ such that for any closed interval $I \subset (d-1, d-1+p)$

$$\bar{\beta}_{0I} := \inf_{\theta\in I} \bar{\beta}_0(\theta) > 0.$$

Finally considering the method of continuity, we finish the proof of theorem by showing that for any $f \in \psi^{-1+2\alpha}\mathbb{H}^{\gamma}_{p,\theta}(\Omega, T)$ and $u_0 \in U^{\gamma+2}_{p,\theta,\alpha}(\Omega)$, there exists $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega, T)$ such that

$$\frac{du}{dt} = \psi^{2\alpha}\Delta u + f, \quad u(0,\cdot) = u_0. \tag{4.6}$$

We can approximate f with smooth functions having compact support. Therefore it follows from the estimate (2.16) that we may assume that f is bounded along with each derivatives in x and vanishes if x is near the boundary of the domain. The same arguments show that we may assume that u_0 is bounded along with each derivatives and has compact support in Ω. Thus we only need to prove the following lemma.

Lemma 4.3. *Let $d-1 < \theta < d-1+p$. Suppose that f and u_0 are bounded along with each derivatives and have compact supports in Ω. Then the equation*

$$\frac{du}{dt} = \psi^{2\alpha}\Delta u + f, \quad u(0,\cdot) = u_0 \tag{4.7}$$

has a solution $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega, T)$.

Proof. By modifying the arguments in the proof of Lemma 3.8, one can easily check that if $u \in \psi\mathbb{L}_{p,\theta}(\Omega, T)$ is a solution of (4.7), then $u \in \mathfrak{H}^{\gamma+2}_{p,\theta,\alpha}(\Omega, T)$ for any γ. By Theorem 2.10 in Kim and Krylov (2004), for each $\varepsilon \in (0, 1]$ the equation

$$\frac{du}{dt} = (\varepsilon + \psi^{2\alpha})\Delta u + f, \quad u(0,\cdot) = u_0 \tag{4.8}$$

has a unique solution u^{ε} such that $u^{\varepsilon} \in \mathfrak{H}^{\gamma+2}_{p,\theta,0}(\Omega, T)$ for any $\gamma \in \mathbb{R}$ and $\theta \in (d-1, d-1+p)$. Using Lemma 2.3 (also see Lemma 3.2 in Kim, 2004) one can check that $u^{\varepsilon} \in C(\bar{\Omega})$, $u|_{\partial\Omega} = 0$, and u^{ε} is a classical solution. Now we use estimates for solutions of degenerate parabolic equations (see, for instance, Theorem I.2.9 in Ladyzhenskaya et al., 1968) and get

$$\sup_{\varepsilon}\sup_{\omega,t,x} |u^{\varepsilon}(t,x)| < \infty.$$

Consequently, if $\theta > p$ then

$$\sup_{\varepsilon} \|\psi^{-1}u^{\epsilon}\|_{\mathbb{L}_{p,\theta}(\Omega,T)} < \infty.$$

If $\theta > p$, then there exists a subsequence ε_k and $u \in \psi\mathbb{L}_{p,\theta}(\Omega, T)$ such that $u^{\varepsilon_k} \to u$ weakly in $\psi\mathbb{L}_{p,\theta}(\Omega, T)$. It is obvious that u is a solution (4.7). Thus the lemma is proved for $\theta > p$, and the theorem holds true if $\theta > p$. If $\theta \le p$, we proceed as follows. Fix $\theta_1 \in (p, d-1+p)$ and closed interval I such that $\theta, \theta_1 \in I$. Choose $\bar{\beta} > 0$ such that

$$2\bar{\beta}|\psi_x| + \bar{\beta}(3\bar{\beta}+1)|\psi_x|^2 + \bar{\beta}|\psi\psi_{xx}| \le \bar{\beta}_{0I}.$$

Let $\beta \le \bar{\beta}$. By Remark 4.2 and the previous results for $\theta > p$, the equation

$$\frac{du}{dt} = \psi^{2\alpha}\Delta u + \psi^{2\alpha-1}b^i u_{x^i} + \psi^{2\alpha-2}cu + \psi^{-\beta}f$$

has a solution $u \in \psi\mathbb{L}_{p,\theta_1}(\Omega, T)$, where

$$b^i := -2\beta\psi_{x^i},$$
$$c := -[\beta(3\beta+1)\psi_{x^i}^2 + \beta\psi\Delta\psi].$$

Observe that $v := \psi^{-\beta}u \in \psi\mathbb{L}_{p,\theta_1-\beta p}(\Omega, T)$ satisfies

$$\frac{dv}{dt} = \psi^{2\alpha}\Delta v + f.$$

Thus the lemma is proved if $\theta \ge \theta_1 - \bar{\beta}p$. One can repeat this process until $\theta \ge \theta_1 - n\bar{\beta}p$ for some integer n. This is possible since $\bar{\beta}$ is independent of each step. The lemma is proved. □

Acknowledgment

The author is sincerely grateful to the referee for many useful comments.

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