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A refined Green's function estimate of the time measurable parabolic operators with conic domains .

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Abstract We present a refined Green's function estimate of the time measurable parabolic operators on conic domains that involves mixed weights consisting of appropriate powers of the distance to the vertex and of the distance to the boundary.

Keywords Green's function estimate · conic domains · stochastic parabolic equation

Mathematics Subject Classification (2010) 34B27 · 35K08 · 35K20 · 60H15

1 Introduction

In recent years we have been interested in the stochastic heat diffusion occurring in wedge shaped subdomains of $\mathbb{R}^2$, which are probably simplest non-smooth Lipschitz domains. In the literature there exist almost fully developed regularity results for the stochastic heat diffusion on C^1 domains, but when it comes to Lipschitz domains the results are quite unsatisfactory and very little is known. To fill in the

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gap between the theory for C^1 domains and the theory for Lipschitz domains, the wedge domains are what we decided to pay attention first.

Along the way, we set the theme that the angle around the vertex affects regularity of the temperature when the boundary temperature is controlled. We believe that our previous work [4] captured such relation in a certain way. Based on this work, in [3] we proceeded to construct a theory on the stochastic diffusion in polygonal domains. The main tool of our results was an estimate on Green's function for the heat operator with the wedge domains obtained in [5]. Looking back, what we feel sorry about is that the estimate only involves the weight of powers of the distance to the vertex. "only" means that it could be better or much better if the estimate also involves weight of the distance to the boundary. Having weight depending only on the distance to the vertex in the estimate did not yield satisfactory boundary regularity of the solution and caused quite a bit of trouble when we constructed a global regularity theory for polygonal domains.

Aiming more natural and hopefully complete theory for polygonal domains, we imagined a refined Green's function estimate that involves both the distance to the vertex and the distance to the boundary. This paper is about this improvement task.

The main contents of this paper are as follows. In Section 2, we introduce a Green's function estimate of the time measurable parabolic operator $\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i,j=1}^{d} a_{ij}(t)D_{ij}$ defined on a conic domain $\mathcal{D} \subset \mathbb{R}^d$ with a vertex at the origin. We prove an estimate of the type

$$
\begin{aligned}
G(t,s,x,y) \le N(\beta_1,\beta_2)\, \frac{e^{-\sigma\frac{|x-y|^2}{t-s}}}{(t-s)^{d/2}} \left(\frac{|x|}{\sqrt{t-s}} \wedge 1\right)^{\beta_1} \left(\frac{|y|}{\sqrt{t-s}} \wedge 1\right)^{\beta_2} \\
\times \left(\frac{\rho(x)}{\sqrt{t-s}} \wedge 1\right) \left(\frac{\rho(y)}{\sqrt{t-s}} \wedge 1\right), \quad \beta_1,\ \beta_2 \ge 0,
\end{aligned} \tag{1.1}
$$

where $\rho(x) := \mathrm{dist}(x, \partial\mathcal{D})$. The ranges of β_1 and β_2 are determined by $\mathcal{D}$ and $\mathcal{L}$ and described in Remark 2.2. Note that estimate (1.1) involves both the distance to the vertex and the distance to the boundary, and gives a subtle decay rate as x, y approach the boundary or the origin. In Sections 3 and 4, we obtain some critical upper bounds of β_1, β_2 for the operator $\mathcal{L}$.

In this paper we use the following notations:

- $\alpha \wedge \beta = \min\{\alpha,\beta\}$, $\alpha \vee \beta = \max\{\alpha,\beta\}$
- $N(\cdots)$ means a constant depending only on what are indicated
- $D_{ij}u = \frac{\partial^2 u}{\partial x_j \partial x_i}$

and

- $B_R(x) = \{y \in \mathbb{R}^d \mid |y-x| < R\}$
- $B_R^{\mathcal{D}}(x) = B_R(x) \cap \mathcal{D}$
- $Q_R(t,x) = (t-R^2, t] \times B_R(x)$
- $Q_R^{\mathcal{D}}(t,x) = (t-R^2, t] \times (B_R(x) \cap \mathcal{D})$.

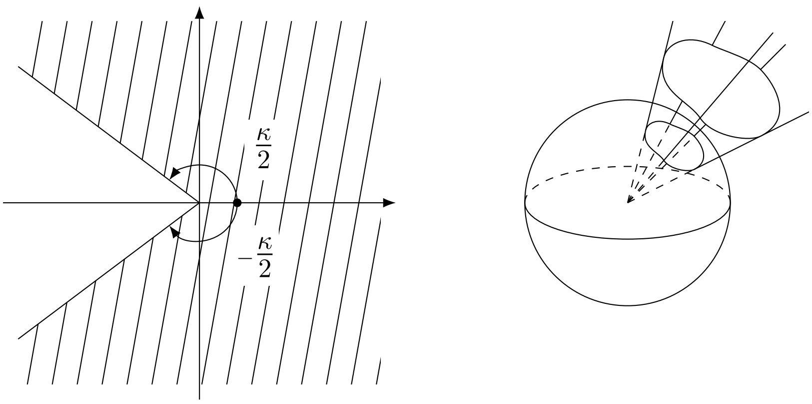
Also, we will frequently use the following sets of functions (see [6]).

- $\mathcal{V}(Q_R(t_0,x_0))$: the set of functions u defined at least on $Q_R(t_0,x_0)$ and satisfying
$$\sup_{t\in(t_0-R^2,t_0]} \|u(t,\cdot)\|_{L_2(B_R(x_0))} + \|\nabla u\|_{L_2(Q_R(t_0,x_0))} < \infty.$$

- $\mathcal{V}_{loc}(Q_R(t_0,x_0))$: the set of functions u defined at least on $Q_R(t_0,x_0)$ and satisfying
$$u \in \mathcal{V}(Q_r(t_0,x_0)), \quad \forall r \in (0,R).$$
- $\mathcal{V}(Q_R^{\mathcal{D}}(t_0,x_0))$: the set of functions u defined at least on $Q_R^{\mathcal{D}}(t_0,x_0)$ and satisfying
$$\sup_{t\in(t_0-R^2,t_0]} \|u(t,\cdot)\|_{L_2(B_R^{\mathcal{D}}(x_0))} + \|\nabla u\|_{L_2(Q_R^{\mathcal{D}}(t_0,x_0))} < \infty.$$
- $\mathcal{V}_{loc}(Q_R^{\mathcal{D}}(t_0,x_0))$: the set of functions u defined at least on $Q_R^{\mathcal{D}}(t_0,x_0)$ and satisfying
$$u \in \mathcal{V}(Q_r^{\mathcal{D}}(t_0,x_0)), \quad \forall r \in (0,R).$$

2 Main result

We define our conic domain in $\mathbb{R}^d$ by
$$\mathcal{D} = \Big\{ x \in \mathbb{R}^d \setminus \{0\} \ \Big| \ \frac{x}{|x|} \in \mathcal{M} \Big\},$$
where $\mathcal{M}$ is a connected open subset in the sphere $S^{d-1} = \{\xi \in \mathbb{R}^d \mid |\xi| = 1\}$ which has C^2 boundary. Here, C^2 boundary means that for any fixed point $p \in S^{d-1} \setminus \overline{\mathcal{D}}$ and the stereographic projection of $S^{d-1} \setminus \{p\}$ onto the tangent hyperplane at $-p$, the antipode of p, the image of $\mathcal{D}$ has C^2 boundary in the hyperplane.



$d = 2$ $d = 3$

Fig. 2.1 Cases of $d = 2$ and $d = 3$

For example, when $d = 2$, for each fixed angle $\kappa \in (0, 2\pi)$ we can consider
$$\mathcal{D} = \mathcal{D}_\kappa = \Big\{ (r\cos\theta,\ r\sin\theta) \in \mathbb{R}^2 \mid r \in (0,\ \infty),\ -\frac{\kappa}{2} < \theta < \frac{\kappa}{2} \Big\}. \tag{2.1}$$
In this paper we consider the Green's function of the operator
$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i,j} a_{ij}(t) D_{ij} \tag{2.2}$$

with the domain $\mathcal{D}$. We assume that the diffusion coefficients a_{ij}, $i,j=1,\dots,d$, are real valued measurable functions of t, $a_{ij}=a_{ji}$, $i,j=1,\dots,d$, and satisfy the uniform parabolicity condition, i.e. there exists a constant $\nu\in(0,1]$ such that for any $t\in\mathbb{R}$ and $\xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d$,

$$\nu|\xi|^2\le\sum_{i,j}a_{ij}(t)\xi_i\xi_j\le\nu^{-1}|\xi|^2. \tag{2.3}$$

We denote the Green's function by $G(t,s,x,y)$. By the definition of Green's function G is nonnegative and, for any fixed $s\in\mathbb{R}$ and $y\in\mathcal{D}$, the function $v=G(\,\cdot,s,\,\cdot,y)$ satisfies

$$\mathcal{L}v=0\quad\text{in}\quad(s,\infty)\times\mathcal{D}\,;\quad v=0\quad\text{on}\quad(s,\infty)\times\partial\mathcal{D}\,;\quad v(t,\cdot)=0\quad\text{for}\quad t<s.$$

Also, in this paper we use the notations $\rho_0(x)=|x|$, $\rho(x)=\mathrm{dist}(x,\partial\mathcal{D})$ and

$$R_{t,x}:=\frac{|x|}{\sqrt{t}}\wedge 1=\frac{\rho_0(x)}{\sqrt{t}}\wedge 1,\quad J_{t,x}:=\frac{\rho(x)}{\sqrt{t}}\wedge 1.$$

Remark 2.1 Since $\frac{a}{a+1}\le a\wedge 1\le 2\cdot\frac{a}{a+1}$ for any $a\ge 0$, we can also define $R_{t,x}$ and $J_{t,x}$ by

$$R_{t,x}:=\frac{\rho_0(x)}{\rho_0(x)+\sqrt{t}},\quad J_{t,x}:=\frac{\rho(x)}{\rho(x)+\sqrt{t}}.$$

From the probabilitstic point of view related to a Brownian motion killed at the boundary of $\partial\mathcal{D}$, G is essentially a transition probability and bounded by a constant multiple of Gaussian density function:

$$0\le G(t,s,x,y)\le N\frac{1}{(t-s)^{d/2}}e^{-\sigma\frac{|x-y|^2}{t-s}},\quad t>s,\quad x,y\in\mathcal{D}, \tag{2.4}$$

where the constants N, $\sigma>0$ depend only on space dimension d and ν in the assumption (2.3).

Having further information of the domain, the right hand side of (2.4) can be refined. Especially, for our conic domains $\mathcal{D}$, one can pursue the following type of estimate

$$G(t,s,x,y)\le N\frac{1}{(t-s)^{d/2}}\,R^{\lambda^+}_{t-s,x}\,R^{\lambda^-}_{t-s,y}\,e^{-\sigma\frac{|x-y|^2}{t-s}},\quad t>s,\quad x,y\in\mathcal{D}$$

for some positive constants λ^+,λ^-. Since $R_{t,x}$ is less than equal to 1, this estimate is sharper as we find bigger λ^+,λ^- satisfying the estimate.

Remark 2.2 As in [6], the critical upper bound $\lambda^+_c>0$ of λ^+ can be characterized by the supremum of all λ such that for some constant $\epsilon=\epsilon(\lambda)\in(1/2,1)$ it holds that

$$|u(t,x)|\le N(\lambda,\epsilon)\left(\frac{|x|}{R}\right)^{\lambda}\sup_{Q^{\mathcal{D}}_{\epsilon R}(t_0,0)}|u|,\quad\forall\,(t,x)\in Q^{\mathcal{D}}_{R/2}(t_0,0)$$

for any $t_0>0$, $R>0$, and u belonging to $\mathcal{V}_{loc}(Q^{\mathcal{D}}_R(t_0,0))$ and satisfying

$$\mathcal{L}u=0\quad\text{in } Q^{\mathcal{D}}_R(t_0,0)\quad;\quad u(t,x)=0\quad\text{for }\ x\in\partial\mathcal{D}.$$

Moreover, the critical upper bound $\lambda_c^- > 0$ of λ^- is characterized by the supremum of λ with above property for the operator

$$\hat{\mathcal{L}} = \frac{\partial}{\partial t} - \sum_{i,j} a_{ij}(-t)D_{ij}. \tag{2.5}$$

Both λ_c^+ and λ_c^- will definitely depend on $\mathcal{M} = \mathcal{D} \cap S^{d-1}$. Especially when $\mathcal{D} = \mathcal{D}_\kappa$ in (2.1), λ_c^+ and λ_c^- will depend on the opening angle κ. If in addition $\mathcal{L}$ is the heat opeartor, $\mathcal{L} = \frac{\partial}{\partial t} - \Delta_x$, then

$$\lambda_c^+ = \lambda_c^- = \frac{\pi}{\kappa}.$$

See Section 2 of [6] and Section 3 of this paper for details.

The following lemma is, we think, the most updated estimate of G among the ones involving $R_{t,x}$ only.

Lemma 2.1 ([6]) *Fix $\lambda^+ \in (0, \lambda_c^+)$, $\lambda^- \in (0, \lambda_c^-)$. Then there exist constants N, $\sigma > 0$ depending only on $\mathcal{M}, \nu$, λ^+, λ^- such that*

$$G(t,s,x,y) \leq N \, \frac{1}{(t-s)^{d/2}} R_{t-s,x}^{\lambda^+} \, R_{t-s,y}^{\lambda^-} \, e^{-\sigma \frac{|x-y|^2}{t-s}} \tag{2.6}$$

and

$$|\nabla_x G(t,s,x,y)| \leq N \, \frac{1}{(t-s)^{(d+1)/2}} R_{t-s,x}^{\lambda^+ - 1} \, R_{t-s,y}^{\lambda^-} \, e^{-\sigma \frac{|x-y|^2}{t-s}}$$

for any $t > s$, $x, y \in \mathcal{D}$.

Remark 2.3 In fact, [6] has the estimates of the derivatives of G up to the second order that contain Lemma 2.1 as a part. We refer to Theorem 3.10 of [6]. Yet, the estimates involve $R_{t,x}$ only.

Remark 2.4 Despite the beauty in estimate (2.6), we note that the right hand side of (2.6) does not go to zero as x or y approaches boundary of $\mathcal{D}$, meaning that the estimate is not sharp enough in terms of the boundary behavior of the Green's function.

On the other hand, for any domain satisfying, for instance, the uniform exterior ball condition, the corresponding Green's function of $\mathcal{L}$ is bounded by the constant multiple of

$$\frac{1}{(t-s)^{d/2}} J_{t-s,x} \; J_{t-s,y} \; e^{-\sigma \frac{|x-y|^2}{t-s}},$$

which is now forcing the degeneracy of the Green's function at the boundary (see e.g. [2]).

Of course, our domains, for instance, like $\mathcal{D}_\kappa$ in (2.1) does not satisfy the uniform exterior ball condition if $\kappa > \pi$. However, for any κ, $\mathcal{D}_\kappa$ is mostly flat except a samll neighborhood of the vertex and we hoped a refined estimate that involves both $R_{t,x}$ and $J_{t,x}$ together. After all, we settled down with the following theorem, which is the refined estimate we mentioned in the introduction and is the main result of this paper.

Theorem 2.1 *Take λ_c^+, λ_c^- from Remark 2.2. Then for any $\lambda^+ \in (0, \lambda_c^+)$, $\lambda^- \in (0, \lambda_c^-)$, there exist constants N, $\sigma > 0$ depending only on $\mathcal{M}$, ν, λ^+, λ^- such that*

$$G(t,s,x,y) \leq \frac{N}{(t-s)^{d/2}} R_{t-s,x}^{\lambda^+-1} R_{t-s,y}^{\lambda^--1} J_{t-s,x} J_{t-s,y} e^{-\sigma \frac{|x-y|^2}{t-s}} \tag{2.7}$$

for any $t > s$, $x, y \in \mathcal{D}$.

Remark 2.5 Obviously estimate (2.7) is sharper than estimate (2.6) since $J_{t,x} \leq R_{t,x}$. Moreover, estimate (2.7) gives delicate boundary behavior of Green's funciton.

Remark 2.6 The strategy of our proof of Theorem 2.7 is inspired by [2] and [7] although the details are quite different.

In the proof of Theorem 2.1, we will use the following two lemmas from [6].

Lemma 2.2 (Proposition 3.2 of [6]) *Let u belong to $\mathcal{V}(Q_R(t_0,x_0))$ and satisfy $\mathcal{L}u = 0$ in $Q_R(t_0,x_0)$, then*

$$|\nabla u(t,x)| \leq \frac{N}{R} \sup_{Q_R(t_0,x_0)} |u|, \quad \forall (t,x) \in Q_{R/2}(t_0,x_0),$$

where the constant N depends only on ν and d.

Lemma 2.3 (Proposition 3.4 of [6]) *There exists a sufficently samll δ_0 such that the following holds for any $\delta \in (0,\delta_0)$: Let $x_0 \in \mathcal{D}$, $\rho(x_0) < \delta|x_0|$, and $R \leq \frac{|x_0|}{2}$. Then if u belongs to $\mathcal{V}(Q_R^{\mathcal{D}}(t_0,x_0))$ and satisfies $\mathcal{L}u = 0$ in $Q_R^{\mathcal{D}}(t_0,x_0)$ and $u(t,x) = 0$ for $x \in \partial\mathcal{D}$, then*

$$|\nabla u(t,x)| \leq \frac{N}{R} \sup_{Q_R^{\mathcal{D}}(t_0,x_0)} |u|, \quad \forall (t,x) \in Q_{R/8}^{\mathcal{D}}(t_0,x_0),$$

where the constant N depends only on $\mathcal{M}, \nu, \delta$.

Proof (Proof of Theorem 2.1)

1. First, we fix $s \in \mathbb{R}$, $y \in \mathcal{D}$. We show that there exist constants N, σ depending only on $\mathcal{M}, \nu, \lambda^+, \lambda^-$ such that for any $t \in (s,\infty)$ and $x \in \mathcal{D}$,

$$G(t,s,x,y) \leq \frac{N}{(t-s)^{d/2}} J_{t-s,x} R_{t-s,x}^{\lambda^+-1} R_{t-s,y}^{\lambda^-} e^{-\sigma \frac{|x-y|^2}{t-s}}. \tag{2.8}$$

For given $t \in (s,\infty)$, we consider the following two cases of $x \in \mathcal{D}$.
- **Case** $\rho(x) \geq \frac{1}{2}\left(|x| \wedge \sqrt{t-s}\right)$.
In this case, by assumption we have

$$2\frac{\rho(x)}{\sqrt{t-s}} \geq \left(\frac{|x|}{\sqrt{t-s}} \wedge 1\right).$$

Therefore,

$$R_{t-s,x} = \frac{|x|}{\sqrt{t-s}} \wedge 1 \leq 2\frac{\rho(x)}{\sqrt{t-s}} \wedge 2 = 2\left(\frac{\rho(x)}{\sqrt{t-s}} \wedge 1\right). \tag{2.9}$$

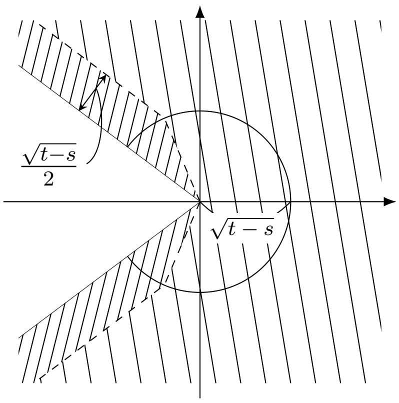


Fig. 2.2 Two cases of x

Then, using Lemma 2.1, we immediately get (2.8).

- Case $\rho(x) < \frac{1}{2}\left(|x| \wedge \sqrt{t-s}\right)$; the point close to the boundary.

For such point $x \in \mathcal{D}$, there exists $x_0 \in \partial\mathcal{D}$ such that $|x - x_0| = \rho(x)$. For this $x_0 \in \partial\mathcal{D}$, $G(t,s,x_0,y) = 0$ and there exists $\theta \in (0,1)$ such that

$$\begin{aligned} G(t,s,x,y) &= G(t,s,x,y) - G(t,s,x_0,y) \\ &\le |x - x_0||\nabla_x G(t,s,\bar{x},y)| \\ &= \rho(x)|\nabla_x G(t,s,\bar{x},y)|, \end{aligned} \tag{2.10}$$

where $\bar{x} = (1-\theta)x + \theta x_0 \in \mathcal{D}$.

To estimate the gradient part, we make use of Lemma 2.1. Now, since

$$|\bar{x}| \ge |x| - \theta|x - x_0| \ge |x| - \rho(x) > \frac{1}{2}|x|, \qquad |\bar{x}| \le |x| + \theta|x - x_0| \le |x| + \rho(x) < 2|x|,$$

we note that

$$\frac{1}{2}R_{t-s,x} \le R_{t-s,\bar{x}} \le 2R_{t-s,x}.$$

In addition, the inequalities

$$|x - y| \le |\bar{x} - y| + |\bar{x} - x| \le |\bar{x} - y| + |x - x_0| \le |\bar{x} - y| + \sqrt{t-s}$$

give

$$-|\bar{x} - y|^2 \le -\frac{1}{2}|x - y|^2 + t - s.$$

Hence, $|\nabla_x G(t,s,\bar{x},y)|$ is bounded by

$$N' \frac{1}{(t-s)^{(d+1)/2}} R_{t-s,x}^{\lambda^+ - 1} R_{t-s,y}^{\lambda^-} e^{-\sigma' \frac{|x-y|^2}{t-s}},$$

where the constants N', $\sigma' > 0$ still depend only on $\mathcal{M}$, ν, λ^+, and λ^-. This, (2.10), and $\rho(x) \le \sqrt{t-s}$ lead us to (2.8) again.

2. Now, we consider the operator $\hat{\mathcal{L}}$ defined in (2.5). Let $\hat{G}$ denote the Green's function for $\hat{\mathcal{L}}$ with the same domain $\mathcal{D}$. Note that the diffusion coefficients $a_{ij}(-t)$, $i,j = 1,\ldots,d$, also satisfy the uniform parabolicity condition (2.3) with the same

ν. Since for any $s \in \mathbb{R}$ and $y \in \mathcal{D}$, $\hat{\mathcal{L}}\hat{G}(\cdot, s, \cdot, y) = 0$ on $(s, \infty) \times \mathcal{D}$ and $\hat{G}(\cdot, s, \cdot, y) = 0$ on $(s, \infty) \times \partial\mathcal{D}$, we can repeat the argument in Step 1 literally line by line. Hence, denoting the critical upper bounds of λ for the operator $\hat{\mathcal{L}}$ by $\hat{\lambda}_c^+$, $\hat{\lambda}_c^-$ and noting that $\hat{\lambda}_c^+ = \lambda_c^-$, $\hat{\lambda}_c^- = \lambda_c^+$ by Remark 2.2, with the same constants N, σ in (2.8) which depending only on $\mathcal{M}, \nu, \lambda^+, \lambda^-$, we obtain that

$$\hat{G}(t, s, x, y) \leq \frac{N}{(t-s)^{d/2}} \; J_{t-s,x} \; R_{t-s,x}^{\lambda^- -1} \; R_{t-s,y}^{\lambda^+} \; e^{-\sigma \frac{|x-y|^2}{t-s}} \tag{2.11}$$

for any $t > s$ and $x, y \in \mathcal{D}$. Note that the locations of λ^+, λ^- in (2.11) in comparison with the locations of them in (2.8). This is simply because $\lambda^- \in (0, \hat{\lambda}_c^+)$ and $\lambda^+ \in (0, \hat{\lambda}_c^-)$.

3. Next, using the result of Step 2 and the following identity

$$G(-s, -t, y, x) = \hat{G}(t, s, x, y) \quad \text{or} \quad G(t, s, x, y) = \hat{G}(-s, -t, y, x), \quad t > s$$

which is due to a duality argument (see (3.12) of [6] for the detail), we observe that with the same constants N, σ in (2.8) we have

$$\begin{aligned} G(t, s, x, y) &\leq \frac{N}{(t-s)^{d/2}} \; J_{t-s,y} \; R_{t-s,y}^{\lambda^- -1} \; R_{t-s,x}^{\lambda^+} \; e^{-\sigma \frac{|x-y|^2}{t-s}} \\ &= \frac{N}{(t-s)^{d/2}} \; R_{t-s,x}^{\lambda^+} \; J_{t-s,y} \; R_{t-s,y}^{\lambda^- -1} \; e^{-\sigma \frac{|x-y|^2}{t-s}} \end{aligned} \tag{2.12}$$

for any $t > s$ and $x, y \in \mathcal{D}$.

4. Finally to finish the proof of (2.7) we repeat the argument in Step 1.

For the points x away from the boundary the argument is the same. Indeed, if $\rho(x) \geq \frac{1}{2}\left(|x| \wedge \sqrt{t-s}\right)$, then (2.9) and (2.12) certainly give (2.7).

Therefore, for the rest of the proof, we may assume

$$\rho(x) < \frac{1}{2}\left(|x| \wedge \sqrt{t-s}\right).$$

In this case we first show

$$|\nabla_x G(t, s, x, y)| \leq N \frac{1}{(t-s)^{(d+1)/2}} J_{t-s,y} R_{t-s,x}^{\lambda^+ -1} R_{t-s,y}^{\lambda^- -1} e^{-\sigma \frac{|x-y|^2}{t-s}}. \tag{2.13}$$

For this, we fix (s, y) and set

$$u(t, x) = G(t, s, x, y).$$

Take $\delta \in (0, \delta_0 \wedge 1/2)$, where δ_0 is from Lemma 2.3 which depends only on $\mathcal{M}$. We consider the following two cases.

- **Case** $\rho(x) \geq \delta|x|$. Put $R = \frac{\delta}{2}(|x| \wedge \sqrt{t-s})$ which is less than $\frac{1}{2}\rho(x)$ so that $\bar{B}_R(x) \subset \mathcal{D}$. Since u belongs to $\mathcal{V}(Q_R(t, x))$ and satisfies $\mathcal{L}u = 0$ in $Q_R(t, x)$, by Lemma 2.2, we get

$$|\nabla_x u(t, x)| \leq \frac{N}{R} \sup_{Q_R(t,x)} |u|.$$

We note that for $(r, z) \in Q_R(t, x)$,

$$0 \leq t - r \leq \frac{t-s}{4}, \quad \frac{3}{4}(t-s) \leq r - s \leq t - s,$$

$$|z| \le |x| + R \le 2|x|, \quad |z| \ge |x| - R \ge \frac{1}{2}|x|$$

and

$$\begin{aligned}
&|z-y| \ge |x-y| - R \ge |x-y| - \sqrt{t-s}, \\
&-|z-y|^2 \le -\frac{1}{2}|x-y|^2 + (t-s), \\
&-\frac{|z-y|^2}{r-s} \le -\frac{1}{2}\frac{|x-y|^2}{t-s} + \frac{4}{3}.
\end{aligned}$$

Hence, using (2.12) we get

$$\begin{aligned}
|u(r,z)| &\le \frac{N}{(r-s)^{d/2}} \; R^{\lambda^+}_{r-s,z} \; J_{r-s,y} \; R^{\lambda^- -1}_{r-s,y} \; e^{-\sigma \frac{|z-y|^2}{r-s}} \\
&\le \frac{N}{(t-s)^{d/2}} R^{\lambda^+}_{t-s,x} J_{t-s,y} R^{\lambda^- -1}_{t-s,y} e^{-\sigma' \frac{|x-y|^2}{t-s}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
|\nabla_x u(t,x)| &\le \frac{N}{R} \sup_{Q_R(t,x)} |u| \\
&\le \frac{N}{|x| \wedge \sqrt{t-s}} \frac{1}{(t-s)^{d/2}} R_{t-s,x} R^{\lambda^+ -1}_{t-s,x} J_{t-s,y} R^{\lambda^- -1}_{t-s,y} e^{-\sigma' \frac{|x-y|^2}{t-s}} \\
&= \frac{N}{(t-s)^{(d+1)/2}} J_{t-s,y} R^{\lambda^+ -1}_{t-s,x} R^{\lambda^- -1}_{t-s,y} e^{-\sigma' \frac{|x-y|^2}{t-s}},
\end{aligned}$$

and thus (2.13) is proved.

- **Case** $\rho(x) \le \delta|x|$. In this case, we put $R = \frac{1}{2}(|x| \wedge \sqrt{t-s})$. Since u belongs to $\mathcal{V}(Q^{\mathcal{D}}_R(t,x))$ and satisfies $\mathcal{L}u = 0$ in $Q^{\mathcal{D}}_R(t,x)$, and $u(t,x) = 0$ for $x \in \partial\mathcal{D}$, we can apply Lemma 2.3, and have

$$|\nabla_x u(t,x)| \le \frac{N}{R} \sup_{Q^{\mathcal{D}}_R(t,x)} |u|.$$

Similarly as before, we again obtain (2.13).

Finally, by (2.10), the computations below (2.10), and (2.13), we obtain (2.7). This ends the proof.

3 On the critical upper bounds $\lambda^{\pm}_c$

In this section we discuss some detailed informations of the critical upper bounds λ^+_c and λ^-_c, whose characterizations are given in Remark 2.2.

We first introduce some known results on $\lambda^{\pm}_c$. The following statements are the 3rd, the 8th, and the 7th in Theorem 2.4 of [6]:

– If $\mathcal{L} = \mathcal{L}_0 := \frac{\partial}{\partial t} - \Delta_x$, then

$$\lambda_c^{\pm}(\mathcal{L}_0, \mathcal{D}) = -\frac{d-2}{2} + \sqrt{\Lambda + \frac{(d-2)^2}{4}}, \tag{3.1}$$

where Λ is the first eigenvalue of Laplace-Beltrami operator with the Dirichlet condition on domain $\mathcal{M} = \mathcal{D} \cap S^{d-1}$, where S^{d-1} is the sphere with radius 1 in $\mathbb{R}^d$.

– Suppose that $(a_{ij})_{d\times d}$ is a constant matrix. Then

$$\lambda_c^{\pm}(\mathcal{L}, \mathcal{D}) = \lambda_c^{\pm}(\mathcal{L}_0, \widetilde{\mathcal{D}}) = -\frac{d-2}{2} + \sqrt{\widetilde{\Lambda} + \frac{(d-2)^2}{4}}, \tag{3.2}$$

where $\widetilde{\Lambda}$ is the first eigenvalue of the Dirichlet boundary value problem to Beltrami-Laplacian in the domain $\widetilde{\mathcal{M}} = \widetilde{\mathcal{D}} \cap S^{d-1}$ while cone $\widetilde{\mathcal{D}}$ is the image of $\mathcal{D}$ under the change of variables $x \to y$ that reduces $(a_{ij})_{d\times d}$ to the canonical form $(\delta_{ij})_{d\times d}$ with the Kronecker delta δ_{ij}, $i, j = 1, \ldots, d$.

– For the general operator $\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i,j=1}^{d} a_{ij}(t) D_{ij}$ in (2.2), we have

$$\lambda_c^{\pm} \geq -\frac{d}{2} + \nu\sqrt{\Lambda + \frac{(d-2)^2}{4}}, \tag{3.3}$$

where ν is the uniform parabolicity constant in (2.3).

Remark 3.1 One big difference between (3.2) and (3.3) is that "d" appears in (3.3) in place of "$d-2$". This actually causes a big gap between (3.2) and (3.3). To demonstrate this, let $d = 2$, $\mathcal{D} = \mathcal{D}_\kappa$ in (2.1), and $\mathcal{L} = \mathcal{L}_0 = \frac{\partial}{\partial t} - (D_{x_1x_1} + D_{x_2x_2})$. Then we can easily find Λ in (3.1), which is the same as $\tilde{\Lambda}$ in (3.2). To find Λ, we just need to find the smallest eigenvalue $\lambda > 0$ and its eigenfunction $\phi = \phi(\theta)$ satisfying

$$-\phi'' = \lambda\phi, \quad -\frac{\kappa}{2} < \theta < \frac{\kappa}{2}, \quad ; \quad \phi\left(\frac{\kappa}{2}\right) = \phi\left(-\frac{\kappa}{2}\right) = 0,$$

which yields $\phi(\theta) = \cos(\sqrt{\lambda}\theta)$ and $\cos\left(\sqrt{\lambda}\,\kappa/2\right) = 0$. Hence, the eigenvalues satisfy $\sqrt{\lambda}\,\kappa/2 = \pi/2 + k\pi$, $k = 0, 1, 2, \ldots$, and thus $\Lambda = \pi^2/\kappa^2$.

In this example, if for instance $\kappa = \pi$, then (3.3) yields, as we can take $\nu = 1$, a trivial information $\lambda_c^{\pm} \geq 0$, whereas (3.2) gives $\lambda_c^{\pm} = 1$.

In this section we improve (3.3). In particular, we will replace d in (3.3) by $d-2$. We assume that the coefficients $a_{ij}(t)$, $i, j = 1, \cdots, d$, satisfy $a_{ij}(t) = a_{ji}(t)$, and there exist constants $\nu_1, \nu_2 > 0$ such that for any $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$,

$$\nu_1|\xi|^2 \leq \sum_{i,j} a_{ij}(t)\xi_i\xi_j \leq \nu_2|\xi|^2. \tag{3.4}$$

The condition (2.3) is a special case of this condition: $\nu_1 = \nu, \nu_2 = \nu^{-1}$.

Theorem 3.1 *For the operator $\mathcal{L}$ in (2.2), we have*

$$\lambda_c^{\pm} \geq -\frac{d-2}{2} + \sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda + \frac{(d-2)^2}{4}}, \tag{3.5}$$

where ν_1, ν_2 are the uniform parabolicity constants in (3.4).

Note that if $\nu \le \nu_1 \le \nu_2 \le \nu^{-1}$, the right hand side of (3.5) is quite bigger that that of (3.5). Indeed,

$$\left(-\frac{d-2}{2}+\sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda+\frac{(d-2)^2}{4}}\right)-\left(-\frac{d}{2}+\nu\sqrt{\Lambda+\frac{(d-2)^2}{4}}\right)$$
$$=1+(\sqrt{\frac{\nu_1}{\nu_2}}-\nu)\sqrt{\Lambda+\frac{(d-2)^2}{4}}\ge 1.$$

To prove the above theorem, we start with the following lemma which is a slight modificaiton of Lemma A.1 of [6].

Lemma 3.1 *Let* $\mu^2<\frac{\nu_1}{\nu_2}\left(\Lambda+\frac{(d-2)^2}{4}\right)$ *and* $0<\epsilon_1<\epsilon_2\le 1$. *Then there exists a constant* N *depending only on* $\mu,\epsilon_1,\epsilon_2$ *such that*

$$\int_{Q^{\mathcal{D}}_{\epsilon_1 R}(t_0,0)}|x|^{2\mu}|\nabla u|^2dxdt+\int_{Q^{\mathcal{D}}_{\epsilon_1 R}(t_0,0)}|x|^{2\mu-2}|u|^2dxdt\le NR^{2\mu-2}\int_{Q^{\mathcal{D}}_{\epsilon_2 R}(t_0,0)}|u|^2dxdt$$

for any $R>0$ *and any function* u *belonging to* $\mathcal{V}_{loc}(Q^{\mathcal{D}}_R(t_0,0))$ *and satisfying* $\mathcal{L}u=0$ *in* $Q^{\mathcal{D}}_R(t_0,0)$, $u=0$ *on* $\mathbb{R}\times\partial\mathcal{D}$.

Proof The proof of this lemma is almost the same as that of Lemma A.1 of [6]. The only difference is that we use conditon (3.4) instead of condition (2.3).

Proof (Proof of Theorem 3.1)

1. Refering to Remark 2.2, we note that it is enough to show that for any $\mu\in\mathbb{R}$ satisfying $\mu^2<\frac{\nu_1}{\nu_2}\left(\Lambda+\frac{(d-2)^2}{4}\right)$, there exists a constant N depending only on $\mathcal{M},\mu,d$ such that

$$|u(t,x)|\le N\left(\frac{|x|}{R}\right)^{-\frac{d-2}{2}-\mu}\sup_{Q^{\mathcal{D}}_{\frac{7}{8}R}(t_0,0)}|u|,\quad\forall\,(t,x)\in Q^{\mathcal{D}}_{R/2}(t_0,0)$$

for any $t_0>0$, $R>0$, and u belonging to $\mathcal{V}_{loc}(Q^{\mathcal{D}}_R(t_0,0))$ and satisfying

$$\mathcal{L}u=0\quad\text{in }Q^{\mathcal{D}}_R(t_0,0)\quad;\quad u(t,x)=0\quad\text{for }\ x\in\partial\mathcal{D}.$$

Also, we note that we may assume $t_0=0$, $R=1$.

2. Take any function u satisfying the conditions in Step 1 with $t_0=0,\ R=1$ and take any $(t,x)\in Q^{\mathcal{D}}_{1/2}(0,0)$. Let us denote

$$r=|x|\,(<\frac{1}{2}),\qquad D_r=(t-r^2/4,t]\times(B_{\frac{3}{2}r}(0)\setminus B_{\frac{1}{2}r}(0)).$$

Then as in the proof of statement 7 of Theorem 2.4 in [6], we have

$$|u(t,x)|^2\le Nr^{-d-2}\int_{D_r}|u(\tau,y)|^2dyd\tau$$
$$\le Nr^{-d-2\mu}\int_{D_r}|y|^{2\mu-2}|u(\tau,y)|^2dyd\tau.\tag{3.6}$$

The last inequality in (3.6) holds since for the points y in D_r, $|y|$ are comparable with r.

Now, we define a time-changed function of u:

$$v(s,y) := u(t + r^2 s, y).$$

This function is well defined at least on $Q_1^{\mathcal{D}}(0,0)$ due to $t + r^2 s \in (-1, 0]$ for $s \in (-1, 0]$. Moreover, v belongs to $\mathcal{V}_{loc}(Q_1^{\mathcal{D}}(0,0))$ and satisfies

$$\tilde{\mathcal{L}} v = 0 \quad \text{in } Q_1^{\mathcal{D}}(0,0) \quad ; \quad v = 0 \quad \text{on } \mathbb{R} \times \partial\mathcal{D},$$

where $\tilde{\mathcal{L}} = \frac{\partial}{\partial s} - \sum_{i,j} r^2 a_{ij}(s) D_{ij}$. We note that

$$r^2 \nu_1 |\xi|^2 \le \sum_{i,j} r^2 a_{ij}(s) \xi_i \xi_j \le r^2 \nu_2 |\xi|^2$$

is the uniform parabolicity condition for $\tilde{\mathcal{L}}$ and the ratio $\frac{r^2\nu_1}{r^2\nu_2}$ is the same as $\frac{\nu_1}{\nu_2}$ and hence we can apply Lemma 3.1 for $\tilde{\mathcal{L}}$ and v. Having this in mind, we continue with (3.6) as below.

Since

$$(t + r^2 s, y) \in D_r \quad \Rightarrow \quad (s, y) \in (-1/4, 0] \times B_{\frac{3}{2}r}(0)$$

and $(-1/4, 0] \times B_{\frac{3}{2}r}(0) \subset Q^{\mathcal{D}}_{\frac{3}{4}}(0,0)$, the last quantity in (3.6) is bounded by

$$N r^{-d+2-2\mu} \int_{Q^{\mathcal{D}}_{\frac{3}{4}}(0,0)} |y|^{2\mu-2} |v(s,y)|^2 dyds. \tag{3.7}$$

Then we apply Lemma 3.1 with $\epsilon_1 = \frac{3}{4}, \epsilon_2 = \frac{7}{8}$ and see

$$\begin{aligned} \int_{Q^{\mathcal{D}}_{\frac{3}{4}}(0,0)} |y|^{2\mu-2} |v(s,y)|^2 dyds &\le N \int_{Q^{\mathcal{D}}_{\frac{7}{8}}(0,0)} |v(s,y)|^2 dyds \\ &\le N \sup_{Q^{\mathcal{D}}_{\frac{7}{8}}(0,0)} |v|^2 \\ &\le N \sup_{Q^{\mathcal{D}}_{\frac{7}{8}}(0,0)} |u|^2, \end{aligned} \tag{3.8}$$

where the last quantity in (3.8) follows the observation $t + r^2 s \in (-\left(\frac{7}{8}\right)^2, 0]$ for any $s \in (-\left(\frac{7}{8}\right)^2, 0]$.

All the constants N in this Step 2 depend only on $\mathcal{M}$, μ, and d. Hence, (3.6), (3.7), and (3.8) give the claim in Step 1.

Remark 3.2 For instance, let $d = 3$ and for any fixed $\kappa \in (0, 2\pi)$ take

$$\begin{aligned} \mathcal{D} = \mathcal{D}_\kappa = \Big\{ &(r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta) \in \mathbb{R}^3 \mid \\ &r \in (0,\ \infty),\ 0 \le \theta < \frac{\kappa}{2},\ 0 < \phi \le 2\pi \Big\}. \end{aligned}$$

Then the first eigenvalue Λ of Laplace-Beltrami operator with the Dirichlet condition on domain $\mathcal{D}_\kappa \cap S^2$ satisfies

$$\frac{1}{2|\log(\cos(\kappa/4))|} \le \Lambda \le \frac{4 j_0^2}{\kappa^2} \tag{3.9}$$

where $j_0 \approx 2.4048$ is the first zero of the Bessel function J_0 (see [1]). Hence, using (3.9) and Theorem 3.1 we can obtain rough lower bounds of $\lambda_c^{\pm}$.

4 Evaluation of $\lambda_c^{\pm}$ when $d = 2$

Finding the exact values of $\lambda_c^{\pm}$ are very difficult in general. In Section 3 we presented a decent estimation of them from below. In this section we attempt to evaluate $\lambda_c^{\pm}$ when $d = 2$ and the diffusion coefficients a_{ij}, $i, j = 1, 2$, in our operator $\mathcal{L}$ are constants.

As $a_{12} = a_{21}$, we can set

$$A := (a_{ij})_{2\times 2} := \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

By (2.3) matrix A is positive-definite and the eigenvalues are greater than equal to ν and in particular there is a symmetric matrix B such that $A = B^2$.

For any fixed $\kappa \in (0, 2\pi)$ and $\alpha \in [0, 2\pi)$ we denote

$$\mathcal{D}_{\kappa,\alpha} := \left\{ x = (r\cos\theta,\ r\sin\theta) \in \mathbb{R}^2 \,|\, r \in (0,\ \infty),\ -\frac{\kappa}{2} + \alpha < \theta < \frac{\kappa}{2} + \alpha \right\},$$

calling κ the central angle of the domain $\mathcal{D}_{\kappa,\alpha}$.

We consider the operator

$$\mathcal{L} = \frac{\partial}{\partial t} - \big(aD_{x_1x_1} + b(D_{x_1x_2} + D_{x_2x_1}) + cD_{x_2x_2}\big)$$

with the conic (angular) domain $\mathcal{D}_{\kappa,\alpha}$.

Below arctan is a map from $\mathbb{R} \to (-\pi/2, \pi/2)$.

Proposition 4.1 *For $\mathcal{L}$ and $\mathcal{D}_{\kappa,\alpha}$ defined above, we have*

$$\lambda_c^{\pm}(\mathcal{L}, \mathcal{D}_{\kappa,\alpha}) = \frac{\pi}{\widetilde{\kappa}},$$

where

$$\widetilde{\kappa} = \pi - \arctan\Big(\frac{\bar{c}\,\cot(\kappa/2) + \bar{b}}{\sqrt{\det(A)}}\Big) - \arctan\Big(\frac{\bar{c}\,\cot(\kappa/2) - \bar{b}}{\sqrt{\det(A)}}\Big) \tag{4.1}$$

with constants $\bar{a}, \bar{b}$ from the relation

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{4.2}$$

In particular,

(i) if $\kappa = \pi$, then $\tilde{\kappa} = \pi$;

(ii) if $\alpha = 0$ and $b = 0$, then $\tilde{\kappa}$ is determined by the relation

$$\tan\Big(\frac{\widetilde{\kappa}}{2}\Big) = \sqrt{\frac{a}{c}}\tan\Big(\frac{\kappa}{2}\Big) \tag{4.3}$$

for $\kappa \in (0, 2\pi) \setminus \{\pi\}$.

Proof **1**. We first consider the operator

$$\mathcal{L}_0 := \frac{\partial}{\partial t} - \Delta_x$$

with domain $\mathcal{D}_{\kappa,\alpha}$. In this case we note $\tilde{\kappa} = \kappa$ and, as in Remark 3.1, we again have

$$\lambda_c^+ = \lambda_c^- = \sqrt{\Lambda} = \frac{\pi}{\kappa}.$$

Indeed, the eigenvalue/eigenfunction problem

$$-\phi''(\theta) = \lambda\phi(\theta), \quad \theta \in \Big(-\frac{\kappa}{2} + \alpha, -\frac{\kappa}{2} + \alpha \Big); \qquad \phi\left(-\frac{\kappa}{2} + \alpha\right) = \phi\left(\frac{\kappa}{2} + \alpha\right) = 0$$

leads us to have $\phi(\theta) = \cos\big(\sqrt{\lambda}(\theta - \alpha)\big)$ and $\cos\big(\sqrt{\lambda}\ \kappa/2\big) = 0$. Hence, the first eigenvlaue Λ again satisfies $\sqrt{\Lambda}\ \kappa/2 = \pi/2$. Thus we have

$$\lambda_c^{\pm}(\mathcal{L}_0, \mathcal{D}_{\kappa,\alpha}) = \sqrt{\Lambda} = \frac{\pi}{\tilde{\kappa}}.$$

2. General case. Having (3.2) and the accompanied explanation in mind, we take a symmetric matrix B satisfying $A = B^2$. The change of variables $x = By$ transforms the operator $aD_{x_1x_1}+bD_{x_1x_2}+bD_{x_2x_1}+cD_{x_2x_2}$ into $\Delta_y = D_{y_1y_1}+D_{y_2y_2}$ in y-coordinates, that is, putting $v(t,y) = u(t,By)$, we obtain

$$\big(aD_{11}u + bD_{12}u + bD_{21}u + cD_{22}u\big)(t, By) = \Delta_y v(t,y), \quad (t,y) \in \mathbb{R} \times \widetilde{\mathcal{D}},$$

where $\widetilde{\mathcal{D}}$ is the image of $\mathcal{D}_{\kappa,\alpha}$ under a linear transformation defined by

$$\widetilde{\mathcal{D}} := B^{-1}\mathcal{D}_{\kappa,\alpha} := \Big\{ B^{-1}x \,:\, x \in \mathcal{D}_{\kappa,\alpha} \Big\}.$$

We note that $\widetilde{\mathcal{D}}$ is also a conic (angular) domain with a certain central angle $\widetilde{\kappa}$. In fact, we can use (3.2) and Step 1 to have

$$\lambda_c^{\pm}(\mathcal{L}, \mathcal{D}_{\kappa,\alpha}) = \lambda_c^{\pm}(\mathcal{L}_0, \widetilde{\mathcal{D}}) = \frac{\pi}{\widetilde{\kappa}}.$$

Let us verify the formula for $\widetilde{\kappa}$. We first note

$$\frac{\widetilde{\kappa}}{2\pi} = \frac{|\widetilde{\mathcal{D}} \cap B_1(0)|_\ell}{|B_1(0)|_\ell} \quad \text{and hence} \quad \widetilde{\kappa} = 2 \cdot |\widetilde{\mathcal{D}} \cap B_1(0)|_\ell,$$

where $|E|_\ell$ denotes the Lebesgue measure of $E \subset \mathbb{R}^2$. By the relation $y = B^{-1}x$, we then have

$$\begin{aligned}
|\widetilde{\mathcal{D}} \cap B_1(0)|_\ell &= \int_{\{y \in \widetilde{\mathcal{D}} \,:\, |y| \leq 1\}} dy \\
&= \frac{1}{|\det(B)|} \int_{\{x \in \mathcal{D} \,:\, |B^{-1}x| \leq 1\}} dx \\
&= \frac{1}{\sqrt{\det(A)}} \int_{-\kappa/2+\alpha}^{\kappa/2+\alpha} \int_0^{|B^{-1}v_\theta|^{-1}} r\, dr\, d\theta \\
&= \frac{1}{2\sqrt{\det(A)}} \int_{-\kappa/2+\alpha}^{\kappa/2+\alpha} \frac{1}{|B^{-1}v_\theta|^2}\, d\theta \\
&= \frac{1}{2\sqrt{\det(A)}} \int_{-\kappa/2+\alpha}^{\kappa/2+\alpha} \frac{1}{v_\theta^T A^{-1} v_\theta}\, d\theta,
\end{aligned}$$

where $v_\theta := \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$. Now, a direct calculation based on translation, symmetry, and change of variable gives

$$
\begin{aligned}
&|\widetilde{\mathcal{D}} \cap B_1(0)|_\ell \\
&= \frac{1}{2\sqrt{\det(A)}} \left(\int_0^{\kappa/2} \frac{1}{v_\theta^T\, \overline{A}^{-1}\, v_\theta}\, d\theta + \int_{-\kappa/2}^{0} \frac{1}{v_\theta^T\, \overline{A}^{-1}\, v_\theta}\, d\theta \right) \\
&= \frac{\sqrt{\det(A)}}{2} \int_0^{\kappa/2} \left(\frac{1}{\bar c \cot^2\theta - 2\bar b \cot\theta + \bar a} + \frac{1}{\bar c \cot^2\theta + 2\bar b \cot\theta + \bar a} \right) \cdot \frac{1}{\sin^2\theta}\, d\theta \\
&= \frac{\sqrt{\det(A)}}{2} \int_{\cot(\kappa/2)}^{\infty} \frac{1}{\bar c\, t^2 - 2\bar b\, t + \bar a} + \frac{1}{\bar c\, t^2 + 2\bar b\, t + \bar a} dt \\
&= \frac{1}{2} \left(\pi - \arctan\left(\frac{\bar c\, \cot(\kappa/2) - \bar b}{\sqrt{\det(A)}} \right) - \arctan\left(\frac{\bar c\, \cot(\kappa/2) + \bar b}{\sqrt{\det(A)}} \right) \right),
\end{aligned}
$$

where

$$\overline{A} = \begin{pmatrix}\bar a & \bar b\\ \bar b & \bar c\end{pmatrix}$$

with $\bar a$, $\bar b$, and $\bar c$ defined in (4.2). Hence, we obtain (4.1) for $\tilde\kappa$ and the proof is done.

Remark 4.1 Let us consider the simple but essential case of $b = 0$ and $\alpha = 0$, i.e., $\mathcal{L}$ with $A = \begin{pmatrix}a & 0\\ 0 & c\end{pmatrix}$ and domain $\mathcal{D}_\kappa$. Then, from (4.3), we observe that the ratio $r := \frac{a}{c}$ of the diffusion constants, rather than the exact values of a and c, along with κ decides $\tilde\kappa$ and hence the values $\lambda_c^\pm$. We also note that for $\kappa \in (0,\pi)$

$$\tilde\kappa \to \pi^- \quad \text{as} \quad r \to \infty\,; \quad \tilde\kappa \to 0^+ \quad \text{as} \quad r \to 0^+$$

and for $\kappa \in (\pi, 2\pi)$

$$\tilde\kappa \to \pi^+ \quad \text{as} \quad r \to \infty\,; \quad \tilde\kappa \to 2\pi^- \quad \text{as} \quad r \to 0^+.$$

In particular, if $\kappa \in (0,\pi)$, or domain $\mathcal{D}_\kappa$ is convex, and the diffusion constant to x_2 direction is relatively much lager than the the diffusion constant to x_1 direction, then $\lambda_c^\pm$ are much bigger than 1 and hence Green's function estimate (2.7) gives better decay near the vertex since $R_{t,x} \le 1$.

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