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Sobolev space theory and Hölder estimates for the stochastic partial differential equations on conic and polygonal domains

Kyeong-Hun Kim [a,1], Kijung Lee [b,2], Jinsol Seo [a,*,1]

[a] Department of Mathematics, Korea University, Anam-ro 145, Sungbuk-gu, Seoul, 02841, Republic of Korea
[b] Department of Mathematics, Ajou University, Worldcup-ro 206, Yeongtong-gu, Suwon, 16499, Republic of Korea

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Abstract

We establish existence, uniqueness, and Sobolev and Hölder regularity results for the stochastic partial differential equation

$$du = \Big(\sum_{i,j=1}^{d} a^{ij} u_{x^i x^j} + f^0 + \sum_{i=1}^{d} f^i_{x^i} \Big) dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t > 0,\ x \in \mathcal{D}$$

given with non-zero initial data. Here $\{w^k_t : k = 1, 2, \cdots\}$ is a family of independent Wiener processes defined on a probability space $(\Omega, \mathbb{P})$, $a^{ij} = a^{ij}(\omega, t)$ are merely measurable functions on $\Omega \times (0, \infty)$, and $\mathcal{D}$ is either a polygonal domain in $\mathbb{R}^2$ or an arbitrary dimensional conic domain of the type

$$\mathcal{D}(\mathcal{M}) := \left\{ x \in \mathbb{R}^d : \frac{x}{|x|} \in \mathcal{M} \right\}, \qquad \mathcal{M} \subsetneq S^{d-1}, \quad (d \geq 2) \tag{0.1}$$

* Corresponding author.
E-mail addresses: kyeonghun@korea.ac.kr (K.-H. Kim), kijung@ajou.ac.kr (K. Lee), seo9401@korea.ac.kr (J. Seo).

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where $\mathcal{M}$ is an open subset of S^{d-1} with C^2 boundary. We measure the Sobolev and Hölder regularities of arbitrary order derivatives of the solution using a system of mixed weights consisting of appropriate powers of the distance to the vertices and of the distance to the boundary. The ranges of admissible powers of the distance to the vertices and to the boundary are sharp.

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1. Introduction

The goal of this article is to present a Sobolev space theory and Hölder regularity results for the stochastic partial differential equation (SPDE)

$$du = \left(\sum_{i,j}^{d} a^{ij} u_{x^i x^j} + f^0 + \sum_{i=1}^{d} f^i_{x^i} \right) dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t > 0; \quad u(0, \cdot) = u_0 \tag{1.1}$$

defined on either multi-dimensional conic domains $\mathcal{D}(\mathcal{M})$ (see (0.1)) or two dimensional polygonal domains. Here, $\mathcal{M}$ is an open subset of S^{d-1} with $\mathcal{C}^2$ boundary, $\{w^k_t : k = 1, 2, \cdots\}$ is an infinite sequence of independent one dimensional Wiener processes, and the coefficients a^{ij} are merely measurable functions of (ω, t) with the uniform parabolicity condition; see Assumption 2.2 below.

To give the reader a flavor of our results in this article we state a particular one, an estimate, below: Let $\mathcal{D} = \mathcal{D}(\mathcal{M})$ be a conic domain in $\mathbb{R}^d$, $\rho(x) := dist(x, \partial\mathcal{D})$, and $\rho_\circ(x) := |x|$. Then for the solution u of (1.1) with zero boundary and zero initial conditions, the following holds for any $p \geq 2$:

$$\begin{aligned} &\mathbb{E} \int_0^T \int_{\mathcal{D}} \left(|\rho^{-1} u|^p + |u_x|^p \right) \rho_\circ^{\theta-\Theta} \rho^{\Theta-d} \, dx \, dt \\ \leq \; & C \, \mathbb{E} \int_0^T \int_{\mathcal{D}} \left(|\rho f^0|^p + \sum_{i=1}^{d} |f^i|^p + |g|^p_{l_2} \right) \rho_\circ^{\theta-\Theta} \rho^{\Theta-d} \, dx \, dt \end{aligned} \tag{1.2}$$

with $d - 1 < \Theta < d - 1 + p$ accompanied with the sharp admissible range of θ; see (1.8) below. Also see (1.7) for higher order derivative estimates. Unlike the range of Θ, the range of θ is affected by the shape of domain $\mathcal{D}$, which is determined by $\mathcal{M}$. Estimate (1.2), if $\rho_\circ$ is replaced by the distance to the set of vertices, also holds when $\mathcal{D}$ is a (bounded) polygonal domain in $\mathbb{R}^2$. Regarding Hölder regularity, we have for instance, if $1 - \frac{d}{p} = \delta > 0$,

$$|\rho^{-1+\frac{\Theta}{p}} \rho_\circ^{(\theta-\Theta)/p} u(\omega, t, \cdot)|_{\mathcal{C}(\mathcal{D})} + [\rho^{-1+\delta+\frac{\Theta}{p}} \rho_\circ^{(\theta-\Theta)/p} u(\omega, t, \cdot)]_{\mathcal{C}^\delta(\mathcal{D})} < \infty,$$

for a.e. (ω, t). In particular,

$$|u(\omega,t,x)| \le C(\omega,t)\rho^{1-\frac{\Theta}{p}}(x)\rho_{\circ}^{(-\theta+\Theta)/p}(x) \quad \text{for all } x \in \mathcal{D}. \tag{1.3}$$

Estimate (1.3) shows how θ and Θ are involved in measuring the boundary behavior of the solution with respect to ρ and $\rho_\circ$. See Theorem 2.25 and Theorem 5.6 for the full Hölder regularity results with respect to both space and time variables.

To position our results in the context of regularity theory of stochastic parabolic equations, let us provide a stream of historical remarks.

The L_p-theory ($p \ge 2$) of equation (1.1) defined on the entire space $\mathbb{R}^d$ was first introduced by N.V. Krylov [17,21]. In these articles the author used an analytic approach and proved the maximal regularity estimate

$$\|u_x\|_{\mathbb{L}_p(T)} \le C\Big(\|f^0\|_{\mathbb{L}_p(T)} + \sum_{i=1}^{d}\|f^i\|_{\mathbb{L}_p(T)} + \||g|_{\ell_2}\|_{\mathbb{L}_p(T)}\Big), \qquad p \ge 2, \tag{1.4}$$

provided that $u(0,\cdot) \equiv 0$, where $\mathbb{L}_p(T) := L_p(\Omega \times (0,T); L_p(\mathbb{R}^d))$.

As for other approaches on Sobolev regularity theory, the method based on H^∞-calculus is also available in the literature. This approach was introduced in [5], in which the maximal regularity of $\sqrt{-A}u$ is obtained for the stochastic convolution

$$u(t) := \int_0^t e^{(t-s)A} g(s)dW_H(s).$$

Here, $W_H(t)$ is a cylindrical Brownian motion on a Hilbert space H, and the operator $-A$ is assumed to admit a bounded H^∞-calculus of angle less than $\pi/2$ on $L^q(\mathcal{O})$, where $q \ge 2$ and $\mathcal{O}$ is a domain in $\mathbb{R}^d$. The result of [5] generalizes (1.4) with $f^i = 0$, $i = 1, \dots, d$ as one can take $A = \Delta$ and $\mathcal{O} = \mathbb{R}^d$.

One advantage of the approach based on H^∞-calculus is that it provides a unified way of handling a class of differential operators satisfying the above mentioned condition. However this approach is not applicable for SPDEs with operators depending on (ω, t), and even the simplest case $A = \Delta$, it is needed that $\partial\mathcal{O}$ is regular enough, that is $\partial\mathcal{O} \in \mathcal{C}^2$. Compared to the approach based on H^∞-calculus, Krylov's analytic approach works well for SPDEs with operators depending also on (ω, t), and it also provides the arbitrary order regularity of solutions without much extra efforts even under weaker smoothness condition on domains.

Since the work of [17,21] on $\mathbb{R}^d$, the analytic approach has been further used for the regularity theory of SPDEs on half space [18,19,14] and on $\mathcal{C}^1$-domains [13,11,10]. The major obstacle of studying SPDEs on domains is that, unless certain compatibility conditions (cf. [4]) are fulfilled, the second and higher order derivatives of solutions to SPDEs blow up near the boundary, and such blow-ups are inevitable even on $\mathcal{C}^\infty$-domains. Hence, one needs appropriate weight system to understand the behavior of solutions near the boundary.

It is shown in [18,13,11] that if domains satisfy $\mathcal{C}^1$ boundary condition, then blow-ups of derivatives of solutions can be described very accurately by a weight system introduced in [20,13,23]. This weight system is based solely on the distance to the boundary. Surprisingly enough, under this weight system it is irrelevant whether domains have $\mathcal{C}^\infty$-boundary or $\mathcal{C}^1$-boundary, that is, the regularity of solutions is not affected by the smoothness of the boundary provided that the boundary is at least of class $\mathcal{C}^1$. To be more specific, let $\mathcal{O}$ be a $\mathcal{C}^1$-domain, $\rho(x) = dist(x, \partial\mathcal{O})$, then it holds that (see [11,13]) for any $d-1 < \Theta < d-1+p$,

$$\mathbb{E}\int_0^T\int_{\mathcal{O}} (|\rho^{-1}u| + |u_x|)^p \rho^{\Theta-d} dt$$
$$\leq C\mathbb{E}\int_0^T\int_{\mathcal{O}} \left(|\rho f^0|^p + \sum_{i=1}^d |f^i|^p + |g|_{\ell_2}^p\right)^p \rho^{\Theta-d} dt. \tag{1.5}$$

The condition $\Theta \in (d-1, d-1+p)$ is sharp and is not affected by further smoothness of $\partial\mathcal{O}$ as long as $\partial\mathcal{O} \in \mathcal{C}^1$. Note that estimate (1.5) with smaller Θ gives better decay of solutions near the boundary than that with larger Θ. In particular, we have $u(\omega, t, \cdot) \in W_0^{1,p}(\mathcal{O})$ from (1.5) if $\Theta \leq d$.

As for results on non-smooth domains, that is $\partial\mathcal{O} \notin \mathcal{C}^1$, very few fragmentary results are known. It turns out that (1.5) holds true on general Lipschitz domains if $\Theta \approx d-2+p$ (see [9]), and hence the case $\Theta = d$ is not included in general if $p > 2$. An example in [9] also shows that if $\Theta < p/2$, then estimate (1.5) fails to hold even on simple wedge domains of the type

$$\mathcal{D}^{(\kappa)} = \left\{(r\cos\eta, r\sin\eta) \in \mathbb{R}^2 : r > 0, \eta \in (-\kappa/2, \kappa/2)\right\}, \quad \kappa < 2\pi. \tag{1.6}$$

The vertex 0 makes the boundary non-smooth and changes the game.

Our interest on conic and polygonal domains arises from such question which, in particular, ask if estimates similar to (1.5) hold on such simple Lipschitz domains. We got the clue of the problem from a PDE result on conic domains [15] (also see [24,26]) which is similar to (1.5), without the term $g = (g^1, g^2, \cdots)$ of course. It uses the weight based only on the distance to the vertex. A work on SPDE using a weight system based only on the distance to the vertex is introduced in [3] (also see [2]), in which we studied the model case of $d = 2$ and $a^{ij} = \delta_{ij}$ for a starter of the program.

Even for the model case considered in [2,3] we struggled to have higher order derivative estimate and left the problem as the future work. The main issue is to include the distance to the boundary in our weight system to have a satisfactory regularity relation between solutions and the inputs. In fact, there was an omen of aforementioned difficulty that is implied in the Green's function estimate used in [3] and [2]. The estimate dominating Green's function does not vanish at the boundary although it does at the vertex. We need more refined Green's function estimate for the starter of a satisfactory regularity result.

We then set a program of three steps: (i) preparing a refined d-dimensional Green's function estimate for operators with measurable coefficients (ii) preparing PDE result (iii) establishing SPDE result addressing the higher order derivative estimates. First two steps are done in [7] and [8], and this article fulfills the last step. In [7] the refined Green's function estimate involves both the distance to the vertex and the distance to the boundary and it now vanishes at all the points on the boundary with informative decay rate near the boundary. The work [8] fully makes use of what we prepared in [7] and it is designed to serve this article well.

Now let us explain our L_p-regularity result in more detail. Recall $\rho_\circ(x) := |x|$ and $\rho(x) := d(x, \partial\mathcal{D})$, which denote the distance from x to vertex and to the boundary of the conic domain $\mathcal{D} = \mathcal{D}(\mathcal{M})$, respectively. We prove that for any $p \geq 2$ and $n = 0, 1, 2, \cdots$, the estimate

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{\mathcal{D}}\left(|\rho^{-1}u|^p+|u_x|^p+\cdots+|\rho^n D^{n+1}u|^p\right)\rho_\circ^{\theta-\Theta}\rho^{\Theta-d}\,dx\,dt\\
\leq &C\mathbb{E}\int_0^T\int_{\mathcal{D}}\Big(|\rho f^0|^p+\cdots+|\rho^{n+1}D^n f^0|^p\\
&\qquad+\sum_{i=1}^d|f^i|^p+\cdots+\sum_{i=1}^d|\rho^n D^n f^i|^p\\
&\qquad+|g|_{\ell_2}^p+\cdots+|\rho^n D^n g|_{\ell_2}^p\Big)\rho_\circ^{\theta-\Theta}\rho^{\Theta-d}\,dx\,dt \qquad (1.7)
\end{aligned}
$$

holds for the solution $u = u(\omega, t, x)$ to equation (1.1) with zero initial condition, provided that

$$
d-1<\Theta<d-1+p, \qquad p(1-\lambda_c^+)<\theta<p(d-1+\lambda_c^-). \qquad (1.8)
$$

Here, λ_c^+ and λ_c^- are positive constants which depend on $\mathcal{M}$ and are defined in Definition 2.14 below (also see Proposition 2.17 and Remark 2.18). The same estimate holds for polygonal domains in $\mathbb{R}^2$. Estimate (1.7) with condition (1.8) is indeed an (seamless) extension of [8] to SPDEs, and what is satisfactory is that the ranges of Θ and θ in (1.8) are not shrunken smaller than the ranges for the deterministic parabolic equation. For this however very delicate computation is required and providing the work done successfully is one of main purposes of this article.

Finally, we want to summarize the improvement in this article over the results in [3] and [2]. Our domains $\mathcal{D}(\mathcal{M})$ in $\mathbb{R}^d$, $d \geq 2$, generalize two dimensional angular domains (1.6); the choice of $\mathcal{M}$ is much richer when $d > 2$. Our operator $\sum_{i,j} a^{ij}(\omega, t)D_{ij}$ far generalizes Laplacian operator Δ in [3] and [2]. These generalizations make computation much more involved, especially, for the stochastic part of the solution. Also, thanks to the mixed weight system, we can now study the higher order derivatives in an appropriate manner and implementing it requires quite a work. Moreover, in this article we do not pose zero initial condition and hence we propose right function spaces for the initial condition in terms of regularity relations between inputs and output, where the initial condition is one of inputs. This result is new even for deterministic PDEs on conic domains. Hölder regularity results based on aforementioned improvements are also new even for PDEs on conic domains.

This article is organized as follows. In Section 2 we introduce some properties of weighted Sobolev spaces and present our main results on conic domains, including Hölder regularity results. In Section 3 we estimate weighed L_p norm of the zero-th order derivative of the solution on conic domains based on the solution representation via Green's function and elementary but highly involved computations. The estimates of the derivatives of the solution on conic domains are obtained in Section 4 and the proof of the main results on conic domains are posed there, too. In section 5 we establish a regularity theory on polygonal domains in $\mathbb{R}^2$.

Notations.

- We use := to denote a definition.

- For a measure space $(A, \mathcal{A}, \mu)$, a Banach space B and $p \in [1, \infty)$, we write $L_p(A, \mathcal{A}, \mu; B)$ for the collection of all B-valued $\bar{\mathcal{A}}$-measurable functions f such that

$$\|f\|^p_{L_p(A,\mathcal{A},\mu;B)} := \int_A \|f\|^p_B \, d\mu < \infty.$$

Here, $\bar{\mathcal{A}}$ is the completion of $\mathcal{A}$ with respect to μ. We will drop $\mathcal{A}$ or μ or even B in $L_p(A, \mathcal{A}, \mu; B)$ when they are obvious from the context.
- $\mathbb{R}^d$ stands for the d-dimensional Euclidean space of points $x = (x^1, \cdots, x^d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$, $\mathbb{R}^d_+ := \{x = (x^1, \dots, x^d) : x^1 > 0\}$, and $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$.
- For a domain $\mathcal{O} \subset \mathbb{R}^d$, $B^{\mathcal{O}}_R(x) := B_R(x) \cap \mathcal{O}$ and $Q^{\mathcal{O}}_R(t, x) := (t - R^2, t] \times B^{\mathcal{O}}_R(x)$.
- $\mathbb{N}$ denotes the natural number system, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\mathbb{Z}$ denotes the set of integers.
- For x, y in $\mathbb{R}^d$, $x \cdot y := \sum_{i=1}^d x^i y^i$ denotes the standard inner product.
- For a domain $\mathcal{O}$ in $\mathbb{R}^d$, $\partial\mathcal{O}$ denotes the boundary of $\mathcal{O}$.
- For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0\} \cup \mathbb{N}$,

$$f_t = \frac{\partial f}{\partial t}, \quad f_{x^i} = D_i f := \frac{\partial f}{\partial x^i}, \quad D^\alpha f(x) := D_d^{\alpha_d} \cdots D_1^{\alpha_1} f(x).$$

We denote $|\alpha| := \sum_{i=1}^d \alpha_i$. For the second order derivatives we denote $D_j D_i f$ by $D_{ij} f$. We often use the notation $|gf_x|^p$ for $|g|^p \sum_i |D_i f|^p$ and $|gf_{xx}|^p$ for $|g|^p \sum_{i,j} |D_{ij} f|^p$. We also use $D^m f$ to denote arbitrary partial derivatives of order m with respect to the space variable.
- $\Delta_x f := \sum_i D_{ii} f$, the Laplacian for f.
- For $n \in \{0\} \cup \mathbb{N}$, $W^n_p(\mathcal{O}) := \{f : \sum_{|\alpha| \le n} \int_{\mathcal{O}} |D^\alpha f|^p dx < \infty\}$, the Sobolev space.
- For a domain $\mathcal{O} \subseteq \mathbb{R}^d$ and a Banach space X with the norm $|\cdot|_X$, $\mathcal{C}(\mathcal{O}; X)$ denotes the set of X-valued continuous functions f in $\mathcal{O}$ such that $|f|_{\mathcal{C}(\mathcal{O};X)} := \sup_{x \in \mathcal{O}} |f(x)|_X < \infty$. Also, for $\alpha \in (0, 1]$, we define the Hölder space $\mathcal{C}^\alpha(\mathcal{O}; X)$ as the set of all X-valued functions f such that

$$|f|_{\mathcal{C}^\alpha(\mathcal{O};X)} := |f|_{\mathcal{C}(\mathcal{O};X)} + [f]_{\mathcal{C}^\alpha(\mathcal{O};X)} < \infty$$

with the semi-norm $[f]_{\mathcal{C}^\alpha(\mathcal{O};X)}$ defined by

$$[f]_{\mathcal{C}^\alpha(\mathcal{O};X)} = \sup_{x \neq y \in \mathcal{O}} \frac{|f(x) - f(y)|_X}{|x - y|^\alpha}.$$

In particular, $\mathcal{O}$ can be an interval in $\mathbb{R}$.
- For a domain $\mathcal{O} \subseteq \mathbb{R}^d$, $\mathcal{C}^\infty_c(\mathcal{O})$ is the space of infinitely differentiable functions with compact support in $\mathcal{O}$. $supp(f)$ denotes the support of the function f. Also, $\mathcal{C}^\infty(\mathcal{O})$ denotes the space of infinitely differentiable functions in $\mathcal{O}$.
- For a distribution f on $\mathcal{O}$ and $\varphi \in \mathcal{C}^\infty_c(\mathcal{O})$, the expression (f, φ) denote the evaluation of f with the test function φ.
- For functions $f = f(\omega, t, x)$ depending on $\omega \in \Omega$, $t \ge 0$ and $x \in \mathbb{R}^d$, we usually drop the argument ω and just write $f(t, x)$ when there is no confusion.

- Throughout the article, the letter C denotes a finite positive constant which may have different values along the argument while the dependence will be informed; $C = C(a, b, \cdots)$, meaning that C depends only on the parameters inside the parentheses.
- $A \sim B$ means that there exist constants $C_1, C_2 > 0$ independent of A and B such that $A \leq C_1 B \leq C_2 A$.
- $d(x, \mathcal{O})$ stands for the distance between a point x and a set $\mathcal{O} \in \mathbb{R}^d$.
- $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$.
- 1_U the indicator function on U.
- We will use the following sets of functions (see [15]).
  - $\mathcal{V}(Q_R^{\mathcal{O}}(t_0, x_0))$: the set of functions u defined at least on $Q_R^{\mathcal{O}}(t_0, x_0)$ and satisfying

$$\sup_{t \in (t_0 - R^2, t_0]} \|u(t, \cdot)\|_{L_2(B_R^{\mathcal{O}}(x_0))} + \|\nabla u\|_{L_2(Q_R^{\mathcal{O}}(t_0, x_0))} < \infty.$$

  - $\mathcal{V}_{loc}(Q_R^{\mathcal{O}}(t_0, x_0))$: the set of functions u defined at least on $Q_R^{\mathcal{O}}(t_0, x_0)$ and satisfying

$$u \in \mathcal{V}(Q_r^{\mathcal{O}}(t_0, x_0)), \quad \forall r \in (0, R).$$

2. SPDE on d-dimensional conic domains

Throughout this article we assume $d \geq 2$. Let $\mathcal{M}$ be a nonempty open set in $S^{d-1} := \left\{x \in \mathbb{R}^d : |x| = 1\right\}$ and $\overline{\mathcal{M}}$ denotes the closure of $\mathcal{M}$. We assume $\overline{\mathcal{M}} \neq S^{d-1}$, and define the d-dimensional conic domain $\mathcal{D}$ by

$$\mathcal{D} = \mathcal{D}(\mathcal{M}) := \left\{x \in \mathbb{R}^d \setminus \{0\} \;\Big|\; \frac{x}{|x|} \in \mathcal{M}\right\}.$$

When $d = 2$, the shapes of conic domains are quite simple (Fig. 1). For instance, with a fixed angle κ in the range of $(0, 2\pi)$ we can consider

$$\mathcal{D} = \mathcal{D}^{(\kappa)} := \left\{(r\cos\eta,\ r\sin\eta) \in \mathbb{R}^2 \mid r \in (0,\ \infty),\ -\frac{\kappa}{2} < \eta < \frac{\kappa}{2}\right\}. \tag{2.1}$$

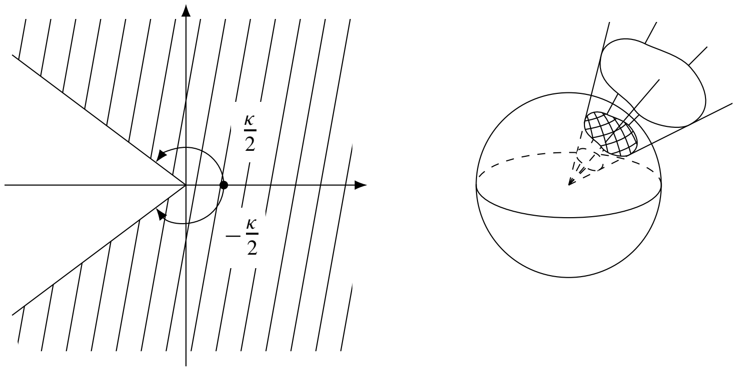
Let $\{w_t^k\}_{k \in \mathbb{N}}$ be a family of independent one-dimensional Wiener processes defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with an increasing filtration of σ-fields $\mathscr{F}_t \subset \mathscr{F}$, each of which contains all $(\mathscr{F}, \mathbb{P})$-null sets. By $\mathcal{P}$ we denote the predictable σ-field on $\Omega \times (0, \infty)$ generated by $\mathscr{F}_t$.

In this article we study the regularity theory of the stochastic partial differential equation

$$du = \left(\mathcal{L}u + f^0 + \sum_{i=1}^d f^i_{x^i}\right)dt + \sum_{k=1}^\infty g^k dw_t^k, \quad t > 0,\ x \in \mathcal{D}(\mathcal{M}) \tag{2.2}$$

under the zero Dirichlet boundary condition. Here

$$\mathcal{L} := \sum_{i,j=1}^d a^{ij}(\omega, t) D_{ij}.$$



$d=2$ $d=3$

Fig. 1. Cases of $d=2$ and $d=3$.

- Each of the stochastic integrals in (2.2) is understood as an Itô stochastic integral against the given Wiener process.
- The infinite sum of stochastic integrals is understood as the limit in probability (uniformly in t) of the finite sums of stochastic integrals. See Remark 2.9.

Here are our assumptions on $\mathcal{M}$ and the diffusion coefficients.

Assumption 2.1. The boundary $\partial\mathcal{M}$ of $\mathcal{M}$ in S^{d-1} is of class $\mathcal{C}^2$.

Assumption 2.2. The diffusion coefficients a^{ij}, $i, j = 1, \cdots, d$, are real-valued $\mathcal{P}$-measurable functions of (ω, t), symmetric; $a^{ij} = a^{ji}$, and satisfy the uniform parabolicity condition, i.e. there exist constants $\nu_1, \nu_2 > 0$ such that for any $t \in \mathbb{R}$, $\omega \in \Omega$ and $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$,

$$\nu_1|\xi|^2 \le \sum_{i,j} a^{ij}(\omega, t)\xi_i\xi_j \le \nu_2|\xi|^2. \tag{2.3}$$

To explain our main result in the frame of weighted Sobolev regularity, we introduce some function spaces (cf. [2,8]). These spaces collect the functions whose weak derivatives can be measured by the help of appropriate weights consisting of powers of the distance to the vertex and of the distance to the boundary. Let us define

$$\rho_\circ(x) = \rho_{\circ,\mathcal{D}} := |x|, \qquad \rho(x) = \rho_{\mathcal{D}}(x) := d(x, \partial\mathcal{D}).$$

For $p \in (1, \infty)$, $\theta \in \mathbb{R}$ and $\Theta \in \mathbb{R}$, we define

$$L_{p,\theta,\Theta}(\mathcal{D}) := L_p(\mathcal{D}, \rho_\circ^{\theta-\Theta}\rho^{\Theta-d}dx),$$

and for $m \in \mathbb{N}_0$ define

$$K^m_{p,\theta,\Theta}(\mathcal{D}) := \{f : \rho^{|\alpha|}D^\alpha f \in L_{p,\theta,\Theta}(\mathcal{D}),\ |\alpha| \le m\}.$$

The norm in $K^m_{p,\theta,\Theta}(\mathcal{D})$ is defined by

$$\|f\|_{K^m_{p,\theta,\Theta}(\mathcal{D})} = \sum_{|\alpha|\le m} \left(\int_{\mathcal{D}} |\rho^{|\alpha|} D^\alpha f|^p \rho_\circ^{\theta-\Theta} \rho^{\Theta-d}\, dx \right)^{1/p}. \tag{2.4}$$

The space $K^m_{p,\theta,\Theta}(\mathcal{D})$ is related to the weighted Sobolev space $H^m_{p,\Theta}(\mathcal{D})$ introduced in [13,20,23] as follows:

$$H^m_{p,\Theta}(\mathcal{D}) = K^m_{p,\Theta,\Theta}(\mathcal{D}),$$

whose norm is given by

$$\|f\|_{H^m_{p,\Theta}(\mathcal{D})} := \sum_{|\alpha|\le m} \left(\int_{\mathcal{D}} |\rho^{|\alpha|} D^\alpha f|^p \rho^{\Theta-d}\, dx \right)^{1/p}, \quad m \in \mathbb{N}_0. \tag{2.5}$$

Note that the weight of $H^m_{p,\Theta}(\mathcal{D})$ is based only on the distance to the boundary. Using the fact that for any $\mu \in \mathbb{R}$ and multi-index α

$$\sup_{x\in\mathcal{D}} \rho_\circ^{|\alpha|-\mu} |D^\alpha \rho_\circ^\mu(x)| \le C(\mu,\alpha) < \infty, \tag{2.6}$$

one can easily check

$$f \in K^m_{p,\theta,\Theta}(\mathcal{D}) \quad \text{if and only if} \quad \rho_\circ^{(\theta-\Theta)/p} f \in H^m_{p,\Theta}(\mathcal{D}),$$

and the norms in their corresponding spaces are equivalents, that is,

$$\|f\|_{K^m_{p,\theta,\Theta}(\mathcal{D})} \sim \|\rho_\circ^{(\theta-\Theta)/p} f\|_{H^m_{p,\Theta}(\mathcal{D})}, \quad n \in \mathbb{N}_0. \tag{2.7}$$

Below we use relation (2.7) to define $K^\gamma_{p,\theta,\Theta}(\mathcal{D})$ for all $\gamma \in \mathbb{R}$. Let $\psi = \psi_{\mathcal{D}}$ be a smooth function in $\mathcal{D}$ (see e.g. [22, Lemma 4.13]) such that for any $m \in \mathbb{N}_0$,

$$\psi_{\mathcal{D}}(x) \sim \rho_{\mathcal{D}}(x), \quad \rho^m_{\mathcal{D}} |D^{m+1} \psi_{\mathcal{D}}| \le N(m) < \infty. \tag{2.8}$$

Actually, such ψ exists on any domains. Indeed, let $\mathcal{O}$ be an arbitrary domain, and put $\rho_{\mathcal{O}}(x) = d(x, \partial\mathcal{O})$, and

$$\mathcal{O}_{n,k} := \{x \in \mathcal{O} : e^{-n-k} < \rho_{\mathcal{O}}(x) < e^{-n+k}\}. \tag{2.9}$$

Then mollifying $1_{\mathcal{O}_{n,2}}$ one can easily construct ξ_n such that

$$\xi_n \in C_c^\infty(\mathcal{O}_{n,3}), \quad |D^m \xi_n| \le C(m) e^{mn}, \quad \sum_{n\in\mathbb{Z}} \xi_n(x) \sim 1,$$

and then one can take

$$\psi=\psi_{\mathcal{O}}=\sum_{n\in\mathbb{Z}}e^{-n}\xi_n(x). \tag{2.10}$$

It is easy to check that $\psi=\psi_{\mathcal{O}}$ satisfies (2.8) with $\rho_{\mathcal{O}}$ in place of $\rho_{\mathcal{D}}$.

Next we choose a nonnegative function $\zeta\in\mathcal{C}_c^\infty(\mathbb{R}_+)$ such that $\zeta>0$ on $[e^{-1},e]$. Then, by the periodicity,

$$\sum_{n=-\infty}^{\infty}\zeta(e^{n+t})>c>0,\quad \forall t\in\mathbb{R}. \tag{2.11}$$

For $p\in(1,\infty)$ and $\gamma\in\mathbb{R}$, by $H_p^\gamma=H_p^\gamma(\mathbb{R}^d)$ we denote the space of Bessel potential with the norm

$$\|u\|_{H_p^\gamma}:=\|(1-\Delta)^{\gamma/2}u\|_{L_p(\mathbb{R}^d)}:=\|\mathcal{F}^{-1}[(1+|\xi|^2)^{\gamma/2}\mathcal{F}(u)(\xi)]\|_{L_p(\mathbb{R}^d)}.$$

In case $\gamma\in\mathbb{N}_0$, $H_p^\gamma(\mathbb{R}^d)$ coincides with $W_p^\gamma(\mathbb{R}^d)$. The spaces of Bessel potentials enjoy the property

$$\|u\|_{H_p^{\gamma_1}}\le\|u\|_{H_p^{\gamma_2}},\quad \gamma_1\le\gamma_2.$$

Especially, we have $\|u\|_{L_p}\le\|u\|_{H_p^\gamma}$ for any $\gamma\ge0$. For ℓ_2-valued functions g we also define

$$\|g\|_{H_p^\gamma(\ell_2)}:=\||(1-\Delta)^{\gamma/2}g|_{\ell_2}\|_{L_p(\mathbb{R}^d)}.$$

Moreover, for $\mathbb{R}^d$-valued functions $\mathbf{f}=(f^1,\dots,f^d)$ we define

$$\|\mathbf{f}\|_{H_p^\gamma(d)}:=\|\,|(1-\Delta)^{\gamma/2}\mathbf{f}|\,\|_{L_p(\mathbb{R}^d)}.$$

From now on, if a function defined on a domain $\mathcal{O}$ vanishes near the boundary of $\mathcal{O}$, then by a trivial extension we consider it as a function defined on $\mathbb{R}^d$. In particular, for any $k\in\mathbb{Z}$ and a function f on $\mathcal{O}$, the function $\zeta(e^{-k}\psi_{\mathcal{O}}(x))f(x)$ has a compact support in $\mathcal{O}$ and can be considered as a function on $\mathbb{R}^d$.

Definition 2.3. Let $p\in(1,\infty)$, $\Theta,\gamma\in\mathbb{R}$, and $\mathcal{O}$ be a domain in $\mathbb{R}^d$. By $H_{p,\theta}^\gamma(\mathcal{O})$ we denote the class of all distributions f on $\mathcal{O}$ such that

$$\|f\|^p_{H^\gamma_{p,\Theta}(\mathcal{O})}:=\sum_{n\in\mathbb{Z}}e^{n\Theta}\|\zeta(e^{-n}\psi(e^n\cdot))f(e^n\cdot)\|^p_{H_p^\gamma(\mathbb{R}^d)}<\infty, \tag{2.12}$$

where $\psi=\psi_{\mathcal{O}}$ is taken from (2.10). Similarly, $H_{p,\theta}^\gamma(\mathcal{O};\ell_2)$ is the set of ℓ_2-valued functions g such that

$$\|g\|^p_{H^\gamma_{p,\Theta}(\mathcal{O};\ell_2)}:=\sum_{n\in\mathbb{Z}}e^{n\Theta}\|\zeta(e^{-n}\psi(e^n\cdot))g(e^n\cdot)\|^p_{H_p^\gamma(\mathbb{R}^d;\ell_2)}<\infty.$$

It turns out (see [23, Proposition 2.2] or [8, Lemma 4.3]) that the new norm in (2.12) is equivalent to the norm in (2.5) if $\gamma \in \mathbb{N}_0$. In other words, for $\gamma \in \mathbb{N}_0$,

$$\sum_{n\in\mathbb{Z}} e^{n\Theta} \|\zeta(e^{-n}\psi_{\mathcal{O}}(e^n\cdot)) f(e^n\cdot)\|^p_{H^\gamma_p} \sim \sum_{|\alpha|\le\gamma} \int_{\mathcal{O}} |\rho^{|\alpha|} D^\alpha f|^p \rho^{\Theta-d}\, dx, \tag{2.13}$$

and the equivalence relation depends only on $p, \gamma, \Theta, d, n, \zeta, \psi$ and $\mathcal{O}$.

Now we use equivalence relations (2.7) and (2.13), and define $K^\gamma_{p,\theta,\Theta}(\mathcal{D})$ for any chosen $\gamma \in \mathbb{R}$.

Definition 2.4. Let $p \in (1,\infty), \theta, \Theta, \gamma \in \mathbb{R}$, and $\mathcal{D}$ be a conic domain in $\mathbb{R}^d$. We write $f \in K^\gamma_{p,\theta,\Theta}(\mathcal{D})$ if and only if $\rho_\circ^{(\theta-\Theta)/p} f \in H^\gamma_{p,\Theta}(\mathcal{D})$, and define

$$\|f\|_{K^\gamma_{p,\theta,\Theta}(\mathcal{D})} := \|\rho_\circ^{(\theta-\Theta)/p} f\|_{H^\gamma_{p,\Theta}(\mathcal{D})}. \tag{2.14}$$

The space $K^\gamma_{p,\theta,\Theta}(\mathcal{D};\ell_2)$ and its norm are defined similarly. Also we write $\mathbf{f}=(f^1, f^2, \cdots, f^d) \in K^\gamma_{p,\theta,\Theta}(\mathcal{D};\mathbb{R}^d)$ if

$$\|\mathbf{f}\|_{K^\gamma_{p,\theta,\Theta}(\mathcal{D};\mathbb{R}^d)} := \sum_{i=1}^d \|f^i\|_{K^\gamma_{p,\theta,\Theta}(\mathcal{D})} < \infty.$$

Note that the new norm of the space $K^\gamma_{p,\theta,\Theta}(\mathcal{D})$ is equivalent to the previous one if $\gamma \in \mathbb{N}_0$. Below we collect some basic properties of the space $K^\gamma_{p,\theta,\Theta}(\mathcal{D})$.

Lemma 2.5. *Let $p \in (1,\infty)$ and $\theta, \Theta, \gamma \in \mathbb{R}$.*

(i) *For a domain $\mathcal{O}$ and $\eta \in C^\infty_c(\mathbb{R}_+)$,*

$$\sum_{n\in\mathbb{Z}} e^{n\Theta} \|\eta(e^{-n}\psi_{\mathcal{O}}(e^n\cdot)) f(e^n\cdot)\|^p_{H^\gamma_p} \le C(p,\Theta,d,\gamma,\eta,\mathcal{O}) \|f\|^p_{H^\gamma_{p,\Theta}(\mathcal{O})}. \tag{2.15}$$

The reverse inequality also holds if η satisfies (2.11). Moreover, the same statements hold for ℓ_2-valued functions.

(ii) *$C^\infty_c(\mathcal{D})$ is dense in $K^\gamma_{p,\theta,\Theta}(\mathcal{D})$.*

(iii) *For any $\mu \in \mathbb{R}$,*

$$\|\psi^\mu f\|_{K^\gamma_{p,\theta,\Theta}(\mathcal{D})} \sim \|f\|_{K^\gamma_{p,\theta+\mu p,\Theta+\mu p}(\mathcal{D})}, \tag{2.16}$$

where ψ satisfies (2.8). The same statement holds for ℓ_2-valued functions.

(iv) *(Pointwise multiplier) Let $\gamma\in\mathbb{R}$, $n \in \mathbb{N}_0$ with $|\gamma| \le n$. If $|a|^{(0)}_n := \sup_{\mathcal{D}} \sum_{|\alpha|\le|n|} \rho^{|\alpha|}|D^\alpha a| < \infty$, then*

$$\|af\|_{K^\gamma_{p,\theta,\Theta}(\mathcal{D})} \le C(n,p,d)|a|^{(0)}_n \|f\|_{K^\gamma_{p,\theta,\Theta}(\mathcal{D})}. \tag{2.17}$$

(v) *The operator* $D_i : K^{\gamma}_{p,\theta,\Theta}(\mathcal{D}) \to K^{\gamma-1}_{p,\theta+p,\Theta+p}(\mathcal{D})$ *is bounded for any* $i = 1, \dots, d$. *In general, for any multi-index* α *we have*

$$\|D^{\alpha} f\|_{K^{\gamma-|\alpha|}_{p,\theta+|\alpha|p,\Theta+|\alpha|p}(\mathcal{D})} \le C\|f\|_{K^{\gamma}_{p,\theta,\Theta}(\mathcal{D})}. \tag{2.18}$$

The same statement holds for ℓ_2*-valued functions.*

(vi) *(Sobolev-Hölder embedding) Let* $\gamma - \frac{d}{p} \ge n + \delta$, *where* $n \in \mathbb{N}_0$ *and* $\delta \in (0, 1)$. *Then for any* $f \in K^{\gamma}_{p,\theta-p,\Theta-p}(\mathcal{D})$,

$$\sum_{k\le n} |\rho^{k-1+\frac{\Theta}{p}} \rho_{\circ}^{(\theta-\Theta)/p} D^k f|_{\mathcal{C}(\mathcal{D})} + [\rho^{n-1+\delta+\frac{\Theta}{p}} \rho_{\circ}^{(\theta-\Theta)/p} D^n f]_{\mathcal{C}^{\delta}(\mathcal{D})} \le C\|f\|_{K^{\gamma}_{p,\theta-p,\Theta-p}(\mathcal{D})}, \tag{2.19}$$

where $C = C(d, \gamma, p, \theta, \Theta, \mathcal{M})$.

Proof. All the results follow from Definition 2.4 and properties of the weighted Sobolev space $H^{\gamma}_{p,\Theta}(\mathcal{O})$ (cf. [23,20,16,13]). See e.g. [23, Proposition 2.2] for (i)-(iii) and see [23, Theorem 3.1] for (iv).

To prove (v), we put $\xi = \rho_{\circ}^{(\theta-\Theta)/p}$. Then, using $\xi Df = D(\xi f) - \xi(\xi^{-1} D\xi) f$ and (2.14), we get

$$\|Df\|_{K^{\gamma-1}_{p,\theta+p,\Theta+p}(\mathcal{D})} \le \|D(\xi f)\|_{H^{\gamma-1}_{p,\Theta+p}(\mathcal{D})} + \|(\xi^{-1} D\xi) f\|_{K^{\gamma-1}_{p,\theta+p,\Theta+p}(\mathcal{D})}.$$

By [23, Theorem 3.1],

$$\|D(\xi f)\|_{H^{\gamma-1}_{p,\Theta+p}(\mathcal{D})} \le C\|\xi f\|_{H^{\gamma}_{p,\Theta}(\mathcal{D})} = C\|f\|_{K^{\gamma}_{p,\theta,\Theta}(\mathcal{D})}.$$

Using (2.6), one can check $|\psi \xi^{-1} D\xi|^{(0)}_m < \infty$ for any $m \in \mathbb{N}$. Thus, by (2.16) and (2.17),

$$\|(\xi^{-1} D\xi) f\|_{K^{\gamma-1}_{p,\theta+p,\Theta+p}(\mathcal{D})} \le C\|(\psi \xi^{-1} D\xi) f\|_{K^{\gamma-1}_{p,\theta,\Theta}(\mathcal{D})} \le C\|f\|_{K^{\gamma-1}_{p,\theta,\Theta}(\mathcal{D})}.$$

Thus (v) is proved.

Finally we prove (vi). Put $g = \xi f$. Then by [23, Theorem 4.3],

$$\sum_{k\le n} |\rho^{k-1+\frac{\Theta}{p}} D^k g|_{\mathcal{C}(\mathcal{D})} + [\rho^{n-1+\delta+\frac{\Theta}{p}} D^n g]_{\mathcal{C}^{\delta}(\mathcal{D})} \le C\|g\|_{H^{\gamma}_{p,\Theta-p}(\mathcal{D})}. \tag{2.20}$$

Hence, to prove (vi), it is enough to note that the left hand side of (2.19) is bounded by a constant times of the left hand side of (2.20). The lemma is proved. □

Using the aforementioned spaces, we now introduce the function spaces for the solutions u to equation (2.2) as well as the function spaces for the inputs f^0, $\mathbf{f}$, and g. To make equation (2.2) well-defined after all, we restrict $p \in [2, \infty)$; see Remark 2.9 (i) below. With such p and a fixed time $T \in (0, \infty)$ we first define

$$\mathbb{H}_p^\gamma(T) := L_p(\Omega \times (0,T], \mathcal{P}; H_p^\gamma),$$
$$\mathbb{H}_p^\gamma(T,\ell_2) := L_p(\Omega \times (0,T], \mathcal{P}; H_p^\gamma(\ell_2)).$$

Next, for $\theta, \Theta, \gamma \in \mathbb{R}$ we define the function spaces

$$\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T) := L_p(\Omega \times (0,T], \mathcal{P}; K^\gamma_{p,\theta,\Theta}(\mathcal{D})),$$
$$\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T,d) := L_p(\Omega \times (0,T], \mathcal{P}; K^\gamma_{p,\theta,\Theta}(\mathcal{D};\mathbb{R}^d)),$$
$$\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T,\ell_2) := L_p(\Omega \times (0,T], \mathcal{P}; K^\gamma_{p,\theta,\Theta}(\mathcal{D};\ell_2)),$$

and denote

$$\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T) := \mathbb{K}^0_{p,\theta,\Theta}(\mathcal{D},T), \quad \mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,d) := \mathbb{K}^0_{p,\theta,\Theta}(\mathcal{D},T,d),$$
$$\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2) := \mathbb{K}^0_{p,\theta,\Theta}(\mathcal{D},T,\ell_2).$$

Also, by $\mathbb{K}_c^\infty(\mathcal{D},T)$ we denote the space of all functions f of the form

$$f(\omega,t,x) = \sum_{i=1}^m \mathbf{1}_{(\tau_{i-1}(\omega),\tau_i(\omega)]}(t) f_i(x),$$

where $\tau_0 \le \cdots \le \tau_m$ is a finite sequence of bounded stopping times with respect to the filtration $(\mathscr{F}_t)_{t\ge 0}$, and $f_i \in \mathcal{C}_c^\infty(\mathcal{D})$, $i = 1,\dots,m$. Similarly, we define $\mathbb{K}_c^\infty(\mathcal{D},T,\ell_2)$ as the space of ℓ_2-valued functions $g = (g^1, g^2, \dots)$ such that the first finite number of g^k are in $\mathbb{K}_c^\infty(\mathcal{D},T)$ and the rest are all identically zero. We also define $\mathbb{K}_c^\infty(\mathcal{D},T,d)$ for $\mathbb{R}^d$-valued functions $\mathbf{f} = (f^1,\dots,f^d)$ in the same manner. Moreover, by $\mathbb{K}_c^\infty(\mathcal{D})$ we denote the space of all functions f of the form

$$f(\omega,x) = \sum_{i=1}^m \mathbf{1}_{A_i}(\omega) f_i(x),$$

where $A_i \in \mathscr{F}_0$ and $f_i \in \mathcal{C}_c^\infty(\mathcal{D})$, $i = 1,\dots,m$.

Remark 2.6. For any $\theta, \Theta, \gamma \in \mathbb{R}$, $\mathbb{K}_c^\infty(\mathcal{D},T)$ is dense in $\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T)$ and so is $\mathbb{K}_c^\infty(\mathcal{D},T,\ell_2)$ in $\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)$. Indeed, by the definition of $\mathcal{P}$, any function $f \in \mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T)$ can be approximated by functions of the type

$$\sum_{i=1}^m 1_{(\tau_i(\omega),\tau_{i+1}(\omega)]}(t) h_i(x),$$

where τ_m are bounded stopping times and $h_i \in K^\gamma_{p,\theta,\Theta}(\mathcal{D})$, $i = 1,\dots,m$. Thus the claim follows from Lemma 2.5 (ii). Similarly, $\mathbb{K}_c^\infty(\mathcal{D})$ is dense in $L_p(\Omega; K^\gamma_{p,\theta,\Theta}(\mathcal{D})) := L_p(\Omega, \mathscr{F}_0, \mathbb{P}; K^\gamma_{p,\theta,\Theta}(\mathcal{D}))$.

From now on we will also use the notation

$$U^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}) := K^{\gamma+2-2/p}_{p,\theta+2-p,\Theta+2-p}(\mathcal{D}).$$

The following definition frames the spaces for the solutions of our SPDE.

Definition 2.7. Let $p\in[2,\infty)$ and $\theta, \Theta, \gamma\in\mathbb{R}$. We write $u\in\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ if $u\in\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$, $u(0,\cdot)\in\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}):=L_p(\Omega,\mathscr{F}_0,\mathbb{P};U^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}))$, and there exists $(\tilde{f},\tilde{g})\in\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)\times\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)$ such that

$$du = \tilde{f}\,dt + \sum_k \tilde{g}^k dw^k_t, \quad t\in(0,T]$$

in the sense of distributions on $\mathcal{D}$, that is, for any $\varphi\in\mathcal{C}^\infty_c(\mathcal{D})$ the equality

$$(u(t,\cdot),\varphi) = (u(0,\cdot),\varphi) + \int_0^t (\tilde{f}(s,\cdot),\varphi)ds + \sum_{k=1}^\infty \int_0^t (\tilde{g}^k(s,\cdot),\varphi)dw^k_s \tag{2.21}$$

holds for all $t\in(0,T]$ (a.s.). In this case we write

$$\mathbb{D}u := \tilde{f} \quad \text{and} \quad \mathbb{S}u := \tilde{g}.$$

The norm in $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ is given by

$$\begin{aligned}\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)} &= \|u\|_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \|\mathbb{D}u\|_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbb{S}u\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)}\\ &\quad + \|u(0,\cdot)\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}.\end{aligned}$$

Remark 2.8. Let us go back to our main equation (2.2). Let $f^0\in\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)$, $\mathbf{f}=(f^1,\cdots,f^d)\in\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,d)$, $g\in\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)$, $u(0,\cdot)\in\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})$, and u belong to $\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$ and be a solution to equation (2.2), that is, u satisfies

$$du = \left(\mathcal{L}u + f^0 + \sum_{i=1}^d f^i_{x^i}\right)dt + \sum_{k=1}^\infty g^k dw^k_t, \quad t\in(0,T]$$

in the sense of distributions on $\mathcal{D}$. Then by (2.18) in Lemma 2.5 (v), we have

$$\mathcal{L}u + f^0 + \sum_{i=1}^d f^i_{x^i} \in \mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)$$

and consequently u belongs to $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ with the accompanied inequality

$$
\begin{aligned}
&\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)}\\
&\leq C\Big(\|u\|_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}+\|f^0\|_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)}+\sum_{i=1}^{d}\|f^i\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T)}\\
&\qquad+\|g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)}+\|u(0,\cdot)\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}\Big).
\end{aligned}
\tag{2.22}
$$

Remark 2.9.

(i) Note that for any $m,n\in\mathbb{N}$ with $m>n$, the quadratic variation of the continuous martingale $\sum_{k=n}^{m}\int_0^t(\tilde g^k,\varphi)dw^k_s$ is $\sum_{k=n}^{m}\int_0^t(\tilde g^k(s),\varphi)^2ds$. Following the lines in [17, Remark 3.2] and using the condition $p\geq 2$, one can easily check

$$
\mathbb{E}\sum_{k=1}^{\infty}\int_0^T(\tilde g^k(t),\varphi)^2dt\leq N(\varphi,p,T)\|\tilde g\|^p_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)},
$$

which implies the infinite series $\sum_{k=1}^{\infty}\int_0^t(\tilde g^k(s),\varphi)dw^k_s$ converges in $L_2\big(\Omega;\mathcal{C}([0,T])\big)$ and in probability uniformly in $t\in[0,T]$. As a consequence, $(u(t,\cdot),\varphi)$ in (2.21) is a continuous semi-martingale on $[0,T]$.

(ii) In Definition 2.7, $\mathbb{D}u$ and $\mathbb{S}u$ are uniquely determined. This can be seen by using the same arguments in [17, Remark 3.3].

Theorem 2.10. *For any $p\in[2,\infty)$ and $\theta,\Theta,\gamma\in\mathbb{R}$, $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ is a Banach space.*

Proof. We only need to prove the completeness. This can be proved by repeating argument in Remark 3.8 of [16], which treats the case $\theta=\Theta$ and $\mathcal{D}=\mathbb{R}^d_+$. The argument in this proof is quite universal and, without any changes, works on any conic domain $\mathcal{D}$ with any $\theta,\Theta\in\mathbb{R}$. □

The following theorem addresses important temporal properties of the functions in $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$. See Section 1 for the notations $[\cdot]_{\mathcal{C}^\alpha}$ and $|\cdot|_{\mathcal{C}^\alpha}$.

Theorem 2.11. *Let $p\in[2,\infty)$ and $\theta,\Theta,\gamma\in\mathbb{R}$.*

(i) *If $2/p<\alpha<\beta\leq 1$, then for any $u\in\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$,*

$$
\mathbb{E}[\psi^{\beta-1}u]^p_{\mathcal{C}^{\alpha/2-1/p}\left([0,T];K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D})\right)}\leq C\,T^{(\beta-\alpha)p/2}\|u\|^p_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)},
\tag{2.23}
$$

and in addition, if $\psi^{\beta-1}u(0,\cdot)\in L_p(\Omega;K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D}))$,

$$
\begin{aligned}
\mathbb{E}|\psi^{\beta-1}u|^p_{\mathcal{C}\left([0,T];K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D})\right)}&\leq C\mathbb{E}\|\psi^{\beta-1}u(0,\cdot)\|^p_{K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D})}\\
&\quad+CT^{p\beta/2-1}\|u\|^p_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)},
\end{aligned}
\tag{2.24}
$$

where ψ satisfies (2.8) and constants C are independent of T and u.

(ii) *For any $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ with $u(0,\cdot)=0$, u belongs to $L_p(\Omega;\mathcal{C}([0,T];K^{\gamma}_{p,\theta,\Theta}(\mathcal{D}))$ and*

$$\mathbb{E}\sup_{t\le T}\|u(t)\|^p_{K^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D})}\le C\|u\|^p_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)},$$

where $C=C(d,p,n,\theta,\Theta,\mathcal{D},T)$. In particular, for any $t\le T$,

$$\|u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},t)}\le\int_0^t\mathbb{E}\sup_{r\le s}\|u(r)\|^p_{K^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},r)}ds\le C\int_0^t\|u\|^p_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},s)}ds. \tag{2.25}$$

Proof. We follow the argument in [16, Section 6] (or the proof of [9, Theorem 2.8]), using [16, Corollary 4.12].

(i). As usual, we suppress the argument ω. Put $\xi(x)=|x|^{(\theta-\Theta)/p}$ and set $v=\xi u$, $\bar f=\xi\mathbb{D}u$, $\bar g=\xi\mathbb{S}u$. Then we have

$$dv=\bar f dt+\sum_{k=1}^{\infty}\bar g^k dw^k_t,\quad t\in(0,T]$$

in the sense of distributions on $\mathcal{D}$ with the initial condition $v(0,\cdot)=\xi u(0,\cdot)$. By (2.16) and Definition 2.4, we have

$$\begin{aligned}I_1&:=\mathbb{E}\left[\psi^{\beta-1}u\right]^p_{\mathcal{C}^{\alpha/2-1/p}([0,T],K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D}))}\\&\sim\mathbb{E}\left[v\right]^p_{\mathcal{C}^{\alpha/2-1/p}([0,T],H^{\gamma+2-\beta}_{p,\Theta+p(\beta-1)}(\mathcal{D}))}\\&\le C\sum_n e^{n(\Theta+p(\beta-1))}\mathbb{E}\left[v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\right]^p_{\mathcal{C}^{\alpha/2-1/p}([0,T];H^{\gamma+2-\beta}_p)}.\end{aligned} \tag{2.26}$$

Now, by assumption, the function $v_n(t,x):=v(t,e^nx)\zeta(e^{-n}\psi(e^nx))$ belongs to $\mathbb{H}^{\gamma+2}_p(T)$ and satisfies

$$dv_n=\bar f(t,e^nx)\zeta(e^{-n}\psi(e^nx))dt+\sum_{k=1}^{\infty}\bar g^k(t,e^nx)\zeta(e^{-n}\psi(e^nx))dw^k_t,\quad t>0 \tag{2.27}$$

on the entire space $\mathbb{R}^d$. Then, by [16, Corollary 4.12] and (2.27), there exists a constant $N>0$, independent of T and u, so that for any constant $a>0$,

$$\begin{aligned}&\mathbb{E}\left[v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\right]^p_{\mathcal{C}^{\alpha/2-1/p}([0,T];H^{\gamma+2-\beta}_p)}\\&\le C\,T^{(\beta-\alpha)p/2}a^{\beta-1}\Big(a\|v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma+2}_p(T)}\\&\qquad+a^{-1}\|\bar f(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma}_p(T)}+\|\bar g^k(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)}\Big)\end{aligned}$$

holds. Taking $a = e^{-np}$, we note that (2.26) yields

$$
\begin{aligned}
I_1 &\le C\,T^{(\beta-\alpha)p/2}\Big(\sum_n e^{n(\Theta-p)}\|v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma+2}_p(T)} \\
&\qquad + \sum_n e^{n(\Theta+p)}\|\bar f(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma}_p(T)} \\
&\qquad + \sum_n e^{n\Theta}\|\bar g^k(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)}\Big) \\
&= C\,T^{(\beta-\alpha)p/2}\Big(\|u\|^p_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \|\mathbb{D}u\|^p_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbb{S}u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)}\Big) \\
&\le C\,T^{(\beta-\alpha)p/2}\|u\|^p_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)}.
\end{aligned}
$$

Thus (2.23) is proved.

If $\psi^{\beta-1}u(0,\cdot) \in L_p(\Omega; K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D}))$, then we note that $\psi^{\beta-1}u$ belongs to $\mathcal{C}\Big([0,T]; K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D})\Big)$ now. For estimate (2.24), we have

$$
\begin{aligned}
I_2 &:= \mathbb{E}|\psi^{\beta-1}u|^p_{\mathcal{C}([0,T];K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D}))} \\
&\le C\sum_n e^{n(\Theta+p(\beta-1))}\mathbb{E}|v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))|^p_{\mathcal{C}([0,T];H^{\gamma+2-\beta}_p)}
\end{aligned}
\tag{2.28}
$$

and by [16, Corollary 4.12] again, for any constant $a > 0$,

$$
\begin{aligned}
&\mathbb{E}|v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))|^p_{\mathcal{C}([0,T];H^{\gamma+2-\beta}_p)} \\
&\le C\,\mathbb{E}\|v(0,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{H^{\gamma+2-\beta}_p} \\
&\quad + C\,T^{p\beta/2-1}a^{\beta-1}\Big(a\|v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma+2}_p(T)} \\
&\qquad + a^{-1}\|\bar f(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma}_p(T)} + \|\bar g^k(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}^{\gamma+1}_p(T,\ell_2)}\Big).
\end{aligned}
$$

This, (2.28), and the same argument above, especially the adjustment $a = e^{-np}$ for each n, lead us to (2.24).

(ii). We use the notations used in (i). Obviously,

$$
\mathbb{E}\sup_{t\le T}\|u(t)\|^p_{K^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D})} \le C\sum_n e^{n\Theta}\mathbb{E}\sup_{t\le T}\|v(t,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{H^{\gamma+1}_p}.
$$

By Remark 4.14 in [16] with $\beta = 1$ there, $v_n \in L_p(\Omega; \mathcal{C}([0,T]; H^{\gamma+1}_p))$ and for any $a > 0$,

$$\mathbb{E}\sup_{t\leq T}\|v(t,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{H_p^{\gamma+1}} \leq C\Big(a\|v(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}_p^{\gamma+2}(T)}$$
$$+a^{-1}\|\bar{f}(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}_p^{\gamma}(T)} + \|\bar{g}^k(\cdot,e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))\|^p_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)}\Big).$$

Again, taking $a = e^{-np}$ and following the above arguments, we get

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}\|u(t)\|^p_{K^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D})}\\ &\leq C\left(\|u\|^p_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \|\mathbb{D}u\|^p_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbb{S}u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)}\right)\\ &= C\,\|u\|^p_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)}.\end{aligned}$$

The theorem is proved. □

Remark 2.12. The additional condition $\psi^{\beta-1}u(0,\cdot) \in L_p(\Omega; K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D}))$ for (2.24) does not follow from the assumption $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$. This condition is unnecessary when we prove the corresponding result on polygonal domains. See Remark 5.3 for detail.

Remark 2.13. Theorems 2.10 and 2.11 hold for any $\theta, \Theta \in \mathbb{R}$, but certain restrictions will be given later for our main results, Theorems 2.19 and 2.21. Actually the admissible range of θ for our Sobolev-regularity theory of equation (2.2) is affected by *the shape of* $\mathcal{D} = \mathcal{D}(\mathcal{M})$, the uniform parabolicity of the leading coefficients, the space dimension d, and the summability parameter p. On the hand, the admissible range of Θ depends only on d and p, that is,

$$d-1 < \Theta < d-1+p.$$

To explain the admissible range of θ for equation (2.2) we need the following definitions. For some of the notations in them one can refer to Section 1.

Definition 2.14 *(cf. Section 2 of [15]).* Let $L = \sum_{i,j=1}^d \alpha^{ij}(t)D_{ij}$ be a uniformly parabolic "deterministic" operator with bounded coefficients α^{ij}s.

(i) By $\lambda^+_{c,L} = \lambda^+_{c,L,\mathcal{D}}$ we denote the supremum of all $\lambda \geq 0$ such that for some constant $K_0 = K_0(\lambda, L, \mathcal{M})$ it holds that

$$|v(t,x)| \leq K_0\left(\frac{|x|}{R}\right)^{\lambda} \sup_{Q^{\mathcal{D}}_{\frac{3R}{4}}(t_0,0)} |v|, \quad \forall\,(t,x) \in Q^{\mathcal{D}}_{R/2}(t_0,0) \tag{2.29}$$

for any $R > 0$, t_0, and the deterministic function $v = v(t,x)$ belonging to $\mathcal{V}_{loc}(Q^{\mathcal{D}}_R(t_0,0))$ and satisfying

$$v_t = Lv \quad \text{in } Q^{\mathcal{D}}_R(t_0,0) \quad ; \quad v(t,x) = 0 \quad \text{for } x \in \partial\mathcal{D}. \tag{2.30}$$

(ii) By $\lambda^-_{c,L}$ we denote the supremum of $\lambda \geq 0$ with above property for the operator

$$\hat{L} := \sum_{i,j} \alpha^{ij}(-t) D_{ij}.$$

Note that K_0 in (2.29) may depend on the operator L. Such dependency on L is one of major obstacles when one handles SPDE having random coefficients, since it naturally involves infinitely many operators at the same time. To treat such case, which is in fact our case in this article, we design the following definition.

Definition 2.15.

(i) By $\mathcal{T}_{\nu_1,\nu_2}$ we denote the collection of all "deterministic" operators in the form $L = \sum_{i,j=1}^d \alpha^{ij}(t) D_{ij}$, where $\alpha^{ij}(t)$ are measurable in t and satisfy Assumption 2.2 with the fixed constants ν_1, ν_2 in the uniform parabolicity condition (2.3).

(ii) For a fixed $\mathcal{D} = \mathcal{D}(\mathcal{M})$, by $\lambda_c(\nu_1, \nu_2) = \lambda_c(\nu_1, \nu_2, \mathcal{D})$ we denote the supremum of all $\lambda \geq 0$ such that for some constant $K_0 = K_0(\lambda, \nu_1, \nu_2, \mathcal{M})$ it holds that for any operator $L \in \mathcal{T}_{\nu_1,\nu_2}$, $R > 0$ and t_0,

$$|v(t,x)| \leq K_0 \left(\frac{|x|}{R}\right)^{\lambda} \sup_{Q^{\mathcal{D}}_{\frac{3R}{4}}(t_0,0)} |v|, \quad \forall\, (t,x) \in Q^{\mathcal{D}}_{R/2}(t_0,0), \tag{2.31}$$

provided that v is a deterministic function in $\mathcal{V}_{loc}(Q^{\mathcal{D}}_R(t_0,0))$ satisfying

$$v_t = Lv \quad \text{in } Q^{\mathcal{D}}_R(t_0,0) \quad ; \quad v(t,x) = 0 \quad \text{for } x \in \partial\mathcal{D}.$$

Remark 2.16.

(i) Note that the dependency of K_0 in Definition 2.15 is more explicit compared to that of Definition 2.14. By definitions, if L is an operator in $\mathcal{T}_{\nu_1,\nu_2}$, then

$$\lambda^{\pm}_{c,L} \geq \lambda_c(\nu_1, \nu_2).$$

(ii) The values of $\lambda^{\pm}_{c,L}$ and $\lambda_c(\nu_1, \nu_2)$ do not change if one replaces $\frac{3}{4}$ in (2.29) and (2.31) by any number in $(1/2, 1)$ (see [15, Lemma 2.2]). Following the proof of [15, Lemma 2.2], one can also show that for any constant $\beta > 0$

$$\lambda^{\pm}_{c,\beta L} = \lambda^{\pm}_{c,L}, \qquad \lambda_c(\beta\nu_1, \beta\nu_2) = \lambda_c(\nu_1, \nu_2).$$

Below are some sharp estimates for $\lambda^{\pm}_{c,L}$ and $\lambda_c(\nu_1, \nu_2)$. See [15] for more information.

Proposition 2.17.

(i) *If* $L = \Delta_x$*, then*

$$\lambda^{\pm}_{c,L} = -\frac{d-2}{2} + \sqrt{\Lambda + \frac{(d-2)^2}{4}} > 0,$$

where $\Lambda = \Lambda_{\mathcal{D}}$ *is the first eigenvalue of Laplace-Beltrami operator with the Dirichlet condition on* $\mathcal{M}$*. In particular, if* $d = 2$ *and* $\mathcal{D} = \mathcal{D}^{(\kappa)}$ *(see (2.1)), then*

$$\lambda^{\pm}_{c,L} = \frac{\pi}{\kappa}.$$

(ii) *Let* $0 < \nu_1 \le \nu_2 < \infty$*. Then we have* $\lambda_c(\nu_1, \nu_2) > 0$ *and*

$$\lambda_c(\nu_1, \nu_2) \ge -\frac{d}{2} + \sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda + \frac{(d-2)^2}{4}}. \tag{2.32}$$

Proof. (i) follows from [15, Theorem 2.4.3]. (ii) also follows from the proofs of [15, Theorem 2.4.1, Theorem 2.4.7], which only consider the case $\nu_2 = 1/\nu_1$. Inspecting the proofs of [15, Theorem 2.4.1, Theorem 2.4.7] one can easily check

$$\lambda^{\pm}_{c,L} \ge -\frac{d}{2} + \sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda + \frac{(d-2)^2}{4}}, \text{ and } \lambda^{\pm}_{c,L} > c > 0 \quad \text{if} \quad L \in \mathcal{T}_{\nu_1,\nu_2},$$

where the constant c is the Hölder exponent of solutions to equation (2.30), and it can be chosen so that it depends only on ν_1, ν_2 and $\mathcal{M}$. Moreover, for $\lambda > 0$ satisfying

$$\lambda < c \vee \Big(-\frac{d}{2} + \sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda + \frac{(d-2)^2}{4}} \Big)$$

the constant K_0 in (2.31) can be chosen so that it depends only on ν_1, ν_2 and $\mathcal{M}$. This proves (2.32). □

Example 2.18 *($d = 2$).* For $\kappa \in (0, 2\pi)$ and $\alpha \in [0, 2\pi)$, we consider

$$\mathcal{D} = \mathcal{D}_{\kappa,\alpha} := \Big\{ x = (r\cos\theta,\ r\sin\theta) \in \mathbb{R}^2 \,|\, r \in (0,\ \infty),\ -\frac{\kappa}{2} + \alpha < \theta < \frac{\kappa}{2} + \alpha \Big\}$$

and the constant operator

$$L = aD_{x_1x_1} + b(D_{x_1x_2} + D_{x_2x_1}) + cD_{x_2x_2},$$

where a, b, c are constants such that $a + c > 0$ and $ac - b^2 > 0$. Then, by [7, Proposition 4.1], we have

$$\lambda^{\pm}_{c,L} = \lambda^{\pm}_{c,L,\mathcal{D}_{\kappa,\alpha}} = \frac{\pi}{\widetilde{\kappa}},$$

where

$$\widetilde{\kappa} = \pi - \arctan\Big(\frac{\bar{c}\,\cot(\kappa/2) + \bar{b}}{\sqrt{\det(A)}}\Big) - \arctan\Big(\frac{\bar{c}\,\cot(\kappa/2) - \bar{b}}{\sqrt{\det(A)}}\Big)$$

with constants $\bar{a}, \bar{b}, \bar{c}$ from the relation

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

In particular, we have $\tilde{\kappa} = \pi$ if $\kappa = \pi$.

Now, let $\kappa \neq \pi$, $\alpha = 0$ for $\mathcal{D}$. Also, let $b = 0$ in L. In this case we can take $\nu_1 = a \wedge c$ and $\nu_2 = a \vee c$ in (2.3). We note that $\tilde{\kappa}$ is determined by the simple relation

$$\tan\Big(\frac{\widetilde{\kappa}}{2}\Big) = \sqrt{\frac{a}{c}}\tan\Big(\frac{\kappa}{2}\Big).$$

We are ready to pose our Sobolev regularity results on conic domains. We formulate them into two theorems to handle random and non-random coefficients separately. The proofs of them are located in Section 4. Note that the admissible range of θ for non-random coefficients is relatively wider than that of random coefficients.

Theorem 2.19 *(SPDE on conic domains with non-random coefficients). Let $\mathcal{L} = \sum_{ij} a^{ij}(t)D_{ij}$ be non-random, $p \in [2,\infty)$, and $\gamma \geq -1$. Also assume that Assumptions 2.1 and 2.2 hold, and $\theta, \Theta \in \mathbb{R}$ satisfy*

$$p(1 - \lambda^{+}_{c,\mathcal{L}}) < \theta < p(d - 1 + \lambda^{-}_{c,\mathcal{L}}), \qquad d - 1 < \Theta < d - 1 + p. \tag{2.33}$$

Then for any $f^0 \in \mathbb{K}^{\gamma\vee 0}_{p,\theta+p,\Theta+p}(\mathcal{D},T)$, $\mathbf{f}=(f^1,\cdots,f^d) \in \mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,d)$, $g \in \mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,l_2)$, and $u_0 \in \mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})$, equation (2.2) has a unique solution u in the class $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ and moreover we have

$$\begin{aligned}\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)} \leq C\big(&\|f^0\|_{\mathbb{K}^{\gamma\vee 0}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbf{f}\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,d)} + \|g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,l_2)} \\ &+\|u_0\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}\big),\end{aligned} \tag{2.34}$$

where the constant C depends only on $\mathcal{M}, d, p, \theta, \Theta, \mathcal{L}, \gamma$. In particular, it is independent of T.

Remark 2.20.

(i) A particular result of the above theorem is introduced in [2] (cf. [3]). More precisely, the combination of Theorem 2.8 and Corollary 2.11 in [2] covers the case

$$\mathcal{L} = \Delta, \quad \Theta = d = 2, \quad \mathcal{D} = \mathcal{D}^{(\kappa)} \text{ of (2.1)}.$$

(ii) If $\gamma \geq 0$, the separation of two terms f^0 and $\mathbf{f} = (f^1, \cdots, f^d)$ in our equation is redundant and we simply pose $f \in \mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D}, T)$ instead. This is because, by (2.18), we have $h^0 + \sum_{i=1}^d h^i_{x^i} \in K^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D})$ for $h^0 \in K^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D})$, $h^i \in K^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D})$, $i = 1, \ldots, d$. The corresponding change in the estimate (2.34) is clear.

Theorem 2.21 *(SPDE on conic domains with random coefficients). Let $\mathcal{L} = \sum_{ij} a^{ij}(\omega, t) D_{ij}$ be random, $p \in [2, \infty)$, and $\gamma \geq -1$. Also assume that Assumptions 2.1 and 2.2 hold, $d - 1 < \Theta < d - 1 + p$, and*

$$p\big(1 - \lambda_c(\nu_1, \nu_2)\big) < \theta < p\big(d - 1 + \lambda_c(\nu_1, \nu_2)\big). \tag{2.35}$$

Then all the claims of Theorem 2.19 hold with a constant $N = N(\mathcal{M}, d, p, \gamma, \theta, \Theta, \nu_1, \nu_2)$.

Remark 2.22. By Proposition 2.17, (2.35) is fulfilled if

$$p\left(\frac{d+2}{2} - \sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda_{\mathcal{D}} + \frac{(d-2)^2}{4}}\right) < \theta < p\left(\frac{d-2}{2} + \sqrt{\frac{\nu_1}{\nu_2}}\sqrt{\Lambda_{\mathcal{D}} + \frac{(d-2)^2}{4}}\right). \tag{2.36}$$

In the case of $L = \Delta$, by Proposition 2.17, (2.33) is fulfilled if

$$p\left(\frac{d}{2} - \sqrt{\Lambda_{\mathcal{D}} + \frac{(d-2)^2}{4}}\right) < \theta < p\left(\frac{d}{2} + \sqrt{\Lambda_{\mathcal{D}} + \frac{(d-2)^2}{4}}\right).$$

Remark 2.23. By (2.4), inequality (2.34) yields (1.7). In particular, if $\gamma = -1$ and $u(0, \cdot) \equiv 0$, then we have

$$\begin{aligned} &\mathbb{E}\int_0^T \int_{\mathcal{D}} \left(|\rho^{-1} u|^p + |u_x|^p\right) \rho_\circ^{\theta-\Theta} \rho^{\Theta-d}\, dx\, dt \\ &\leq C\, \mathbb{E}\int_0^T \int_{\mathcal{D}} \left(|\rho f^0|^p + \sum_{i=1}^d |f^i|^p + |g|^p_{\ell_2}\right) \rho_\circ^{\theta-\Theta} \rho^{\Theta-d}\, dx\, dt. \end{aligned}$$

Remark 2.24. The solutions u in Theorems 2.19 and 2.21 satisfy zero Dirichlet boundary condition. Indeed, under the assumption $d - 1 < \Theta < d - 1 + p$, [6, Theorem 2.8] implies that the trace operator is well defined for functions in $\mathbb{K}^1_{p,\theta-p,\Theta-p}(\mathcal{D}, T)$, and hence by Lemma 2.5 (iv) we have $u|_{\partial\mathcal{D}} = 0$.

Here comes our Hölder regularity properties of solutions on conic domains.

Theorem 2.25 *(Hölder estimates on conic domains). Let $p \in [2, \infty)$, $\theta, \Theta \in \mathbb{R}$, and $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ be the solution taken from Theorem 2.19 (or from Theorem 2.21).*

(i) *If* $\gamma+2-\frac{d}{p}\geq n+\delta$, *where* $n\in\mathbb{N}_0$ *and* $\delta\in(0,1]$, *then for any* $0\leq k\leq n$,

$$|\rho^{k-1+\frac{\Theta}{p}}\rho_\circ^{(\theta-\Theta)/p}D^k u(\omega,t,\cdot)|_{\mathcal{C}(\mathcal{D})}+[\rho^{n-1+\delta+\frac{\Theta}{p}}\rho_\circ^{(\theta-\Theta)/p}D^n(\omega,t,\cdot)]_{\mathcal{C}^\delta(\mathcal{D})}<\infty$$

holds for a.e. (ω,t), *in particular,*

$$|u(\omega,t,x)|\leq C(\omega,t)\rho^{1-\frac{\Theta}{p}}(x)\rho_\circ^{(-\theta+\Theta)/p}(x)\quad \textit{for all } x\in\mathcal{D}. \tag{2.37}$$

(ii) *Let*

$$2/p<\alpha<\beta\leq 1,\quad \gamma+2-\beta-d/p\geq m+\varepsilon,$$

where $m\in\mathbb{N}_0$ *and* $\varepsilon\in(0,1]$. *Put* $\eta=\beta-1+\Theta/p$. *Then for any* $0\leq k\leq m$,

$$\mathbb{E}\sup_{t\neq s\leq T}\frac{\left|\rho^{\eta+k}\rho_\circ^{(\theta-\Theta)/p}\left(D^k u(t,\cdot)-D^k u(s,\cdot)\right)\right|^p_{\mathcal{C}(\mathcal{D})}}{|t-s|^{p\alpha/2-1}}<\infty, \tag{2.38}$$

$$\mathbb{E}\sup_{t\neq s\leq T}\frac{\left[\rho^{\eta+m+\varepsilon}\rho_\circ^{(\theta-\Theta)/p}\left(D^m u(t,\cdot)-D^m u(s,\cdot)\right)\right]^p_{\mathcal{C}^\varepsilon(\mathcal{D})}}{|t-s|^{p\alpha/2-1}}<\infty. \tag{2.39}$$

Proof. (i) By definition, for almost all (ω,t), we have $u(\omega,t,\cdot)\in K^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D})$. Thus (i) is a consequence of (2.19). Similarly, the claims of (ii) follow from (2.19) (2.23), and the observation

$$\mathbb{E}\sup_{t\neq s\leq T}\frac{\|\psi^{\beta-1}(u(t)-u(s))\|^p_{K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D})}}{|t-s|^{(\alpha/2-1/p)p}}$$
$$\sim\mathbb{E}\sup_{t\neq s\leq T}\frac{\|u(t)-u(s)\|^p_{K^{\gamma+2-\beta}_{p,\theta+\beta p-p,\Theta+\beta p-p}(\mathcal{D})}}{|t-s|^{(\alpha/2-1/p)p}}. \qquad\square$$

Remark 2.26.

(i) (2.37) tells how fast the solution from Theorem 2.19 (or Theorem 2.21) vanishes near the boundary. Near boundary points away from the vertex, u is controlled by $\rho^{1-\Theta/p}$ and, if $p>\Theta$, the decay near the vertex is not slower than $\rho_\circ^{1-\theta/p}$.

(ii) In (2.38) and (2.39), $\alpha/2-1/p$ is the Hölder exponent in time and η is related to the decay rate near the boundary. As $\alpha/2-1/p\to 1/2-1/p$, η must increase accordingly.

(iii) Suppose $\theta=d$ satisfies (2.36), and let $u\in\mathcal{K}^1_{p,d,d}(\mathcal{D},T)$ be the solution from Theorem 2.21. Assume

$$\kappa_0:=1-\frac{(d+2)}{p}>0.$$

Then for any $\kappa \in (0, \kappa_0)$, we have

$$\mathbb{E} \sup_{t\le T} \sup_{x,y\in\mathcal{D}} \left| \frac{|u(t,x)-u(t,y)|}{|x-y|^{\kappa}} \right|^p + \mathbb{E} \sup_{t\ne s\le T} \sup_{x\in\mathcal{D}} \left| \frac{|u(t,x)-u(s,x)|}{|t-s|^{\kappa/2}} \right|^p < \infty. \tag{2.40}$$

Indeed, (2.40) can be obtained from (2.38) and (2.39) with appropriate choices of α, β. For the first part, to apply (2.39) we take $\beta = \kappa_0 - \kappa + 2/p$ such that $2/p < \beta < 1$, and take $\varepsilon = 1 - \beta - d/p = \kappa = -\eta$. For the second part, we use (2.38) with $\alpha = \kappa + 2/p$, $\beta = 1 - d/p$ so that $1 - \alpha p/2 = -p\kappa/2$.

3. Key estimates on conic domains

In this section we consider the solutions to SPDEs having a non-random operator. We fix a deterministic operator

$$L_0 := \sum_{i,j} \alpha^{ij}(t) D_{ij} \in \mathcal{T}_{\nu_1,\nu_2}. \tag{3.1}$$

See Definition 2.15. We will estimate the zeroth order derivative of the solution of the equation

$$du = \left(L_0 u + f^0 + \sum_{i=1}^{d} f^i_{x^i} \right) dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t > 0,\ x \in \mathcal{D}(\mathcal{M}). \tag{3.2}$$

Let $G(t,s,x,y)$ denote the Green's function for the operator $\partial_t - L_0$ on $\mathcal{D} = \mathcal{D}(\mathcal{M})$. By definition (cf. [15, Lemma 3.7]), G is a nonnegative function such that for any fixed $s \in \mathbb{R}$ and $y \in \mathcal{D}$, the function $v(t,x) = G(t,s,x,y)$ satisfies

$$\begin{aligned} &\big(\partial_t - L_0\big) v(t,x) = \delta(x-y)\delta(t-s) \quad \text{in} \quad \mathbb{R}\times\mathcal{D}, \\ &v(t,x) = 0 \quad \text{on} \quad \mathbb{R}\times\partial\mathcal{D}\,; \quad v(t,x) = 0 \quad \text{for} \quad t < s. \end{aligned}$$

Now, for any given

$$\begin{aligned} &f^0 \in \mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{D},T), \quad \mathbf{f} = (f^1,\cdots,f^d) \in \mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,d), \\ &g \in \mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2), \quad u_0 \in L_p(\Omega; K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D})), \end{aligned}$$

we define the function $\mathcal{R}(u_0, f^0, \mathbf{f}, g)$ by

$$\begin{aligned} &\mathcal{R}(u_0, f^0, \mathbf{f}, g)(t,x) \\ &:= \int_{\mathcal{D}} G(t,0,x,y) u_0(y) dy \\ &+ \int_0^t \int_{\mathcal{D}} G(t,s,x,y) f(s,y) dy ds - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{D}} G_{y^i}(t,s,x,y) f^i(s,y) dy ds \end{aligned}$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_{\mathcal{D}} G(t,s,x,y)g^k(s,y)dy\,dw_s^k. \tag{3.3}$$

One immediately notices that the function $\mathcal{R}(u_0, f^0, \mathbf{f}, g)$ is a representation of a solution of (3.2) with zero boundary condition and initial condition $u(0,\cdot) = u_0(\cdot)$; see Lemma 4.3 in the next section. Our main result of this section is about this representation and it is given in the following lemma.

Lemma 3.1. *Let $T < \infty$, $p \in [2,\infty)$ and let $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$ satisfy*

$$p(1-\lambda^+_{c,L_0}) < \theta < p(d-1+\lambda^-_{c,L_0}) \quad \textit{and} \quad d-1 < \Theta < d-1+p.$$

If $f^0 \in \mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{D},T)$, $\mathbf{f} \in \mathbb{L}^d_{p,\theta,\Theta}(\mathcal{D},T,d)$, $g \in \mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)$, and $u_0 \in L_p(\Omega; K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D})) := L_p(\Omega,\mathscr{F}_0; K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D}))$, then $u := \mathcal{R}(u_0, f^0, \mathbf{f}, g)$ belongs to $\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$ and the estimate

$$\begin{aligned}\|u\|_{\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq C\Big(&\|f^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbf{f}\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,d)} \\ &+\|g\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)} + \|u_0\|_{L_p(\Omega;K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D}))}\Big)\end{aligned}$$

holds, where $C = C(\mathcal{M}, d, p, \theta, \Theta, L_0)$. Moreover, if

$$p\,(1-\lambda_c(\nu_1,\nu_2)) < \theta < p\,(d-1+\lambda_c(\nu_1,\nu_2)),$$

then the constant C depends only on $\mathcal{M}, d, p, \theta, \Theta, \nu_1$ and ν_2.

To prove Lemma 3.1, we use the following two results. Lemma 3.2 gathers rather technical but important inequalities we keep using in this section.

Lemma 3.2.

(i) *Let $\alpha+\beta > 0$, $\beta > 0$, and $\gamma > 0$. Then for any $a \geq b > 0$*

$$\int_0^\infty \frac{1}{\left(a+\sqrt{t}\right)^{\alpha}\left(b+\sqrt{t}\right)^{\beta+\gamma} t^{1-\frac{\gamma}{2}}}dt \leq \frac{C}{a^\alpha b^\beta},$$

where $C = C(\alpha,\beta,\gamma)$.

(ii) *Let $\sigma > 0$, $\alpha+\gamma > -d$, $\gamma > -1$ and β, $\nu \in \mathbb{R}$. Then for any $x \in \mathcal{D}$,*

$$\int_{\mathcal{D}} \frac{|y|^\alpha}{(|y|+1)^\beta}\frac{\rho(y)^\gamma}{(\rho(y)+1)^\nu}\, e^{-\sigma|x-y|^2}dy \leq C\,(|x|+1)^{\alpha-\beta}\,(\rho(x)+1)^{\gamma-\nu},$$

where $C = C(\mathcal{M}, d, \alpha, \beta, \gamma, \nu, \sigma)$.

Proof. See Lemma 3.2 and Lemma 3.7 in [8]. □

For the operator L_0, we take the constants $K_0, \lambda^+_{c,L_0}, \lambda^-_{c,L_0}$ and the operator $\hat{L}_0$ from Definition 2.14.

Lemma 3.3. *Let $\lambda^+ \in \big(0, \lambda^+_{c,L_0}\big)$ and $\lambda^- \in \big(0, \lambda^-_{c,L_0}\big)$. Denote*

$$K_0^+ = K_0(L_0, \mathcal{M}, \lambda^+), \quad K_0^- = K_0(\hat{L}_0, \mathcal{M}, \lambda^-).$$

Then, there exist positive constants $C = C(\mathcal{M}, \nu_1, \nu_2, \lambda^\pm, K_0^\pm)$ and $\sigma = \sigma(\nu_1, \nu_2)$ such that for any $t > s$ and $x, y \in \mathcal{D}(\mathcal{M})$, the estimates

$$(i) \quad G(t,s,x,y) \le \frac{C}{(t-s)^{d/2}} J_{t-s,x}\, J_{t-s,y}\, R^{\lambda^+-1}_{t-s,x}\, R^{\lambda^--1}_{t-s,y}\, e^{-\sigma\frac{|x-y|^2}{t-s}}$$

$$(ii) \quad \big|\nabla_y G(t,s,x,y)\big| \le \frac{C}{(t-s)^{(d+1)/2}} J_{t-s,x}\, R^{\lambda^+-1}_{t-s,x}\, R^{\lambda^--1}_{t-s,y}\, e^{-\sigma\frac{|x-y|^2}{t-s}}$$

hold, where

$$R_{t,x} := \frac{\rho_\circ(x)}{\rho_\circ(x)+\sqrt{t}}, \quad J_{t,x} := \frac{\rho(x)}{\rho(x)+\sqrt{t}}.$$

In particular, if $\lambda^\pm \in (0, \lambda_c(\nu_1, \nu_2))$, then C depends only on $\mathcal{M}, \nu_1, \nu_2, \lambda^\pm$.

Proof. (i) See inequality (2.8) in [7].

(ii) Denote $\hat{G}(t,s,x,y) = G(-s,-t,y,x)$. Then $\hat{G}$ is the Green's function of the operator $\partial_t - \hat{L}_0$, where $\hat{L}_0 := \sum_{i,j} \alpha^{ij}(-t)D_{ij}$. Then by inequality (2.14) of [7] applied to $\hat{G}$, for any $\lambda^+ \in (0, \lambda_c^+)$ and $\lambda^- \in (0, \lambda_c^-)$, there exist constant $C, \sigma > 0$, with the dependencies prescribed in the lemma, such that

$$|\nabla_x \hat{G}(t,s,x,y)| \le \frac{C}{(t-s)^{(d+1)/2}} J_{t-s,y} R^{\lambda^--1}_{t-s,x} R^{\lambda^+-1}_{t-s,y} e^{-\sigma\frac{|x-y|^2}{t-s}}$$

for any $t > s$ and $x, y \in \mathcal{D}$. This and the fact $\nabla_y G(t,s,x,y) = \nabla_x \hat{G}(-s,-t,y,x)$ prove (ii). □

Since $\mathcal{R}(u_0, f^0, \mathbf{f}, g) = \mathcal{R}(u_0, 0,0,0) + \mathcal{R}(0, f^0, \mathbf{f}, 0) + \mathcal{R}(0,0,0,g)$ with 0 as zero functions in their corresponding function spaces, we will treat these three parts separately in following three lemmas and then combine them to obtain the claim of Lemma 3.1. Especially, the stochastic part $\mathcal{R}(0,0,0,g)$ is important in this article and elaborated thoroughly in Lemma 3.7.

Lemma 3.4. *Let $p \in (1, \infty)$, and let $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$ satisfy*

$$p(1-\lambda^+_{c,L_0}) < \theta < p(d-1+\lambda^-_{c,L_0}) \quad and \quad d-1 < \Theta < d-1+p.$$

If $u_0 \in L_p(\Omega; K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D}))$, then $u = \mathcal{R}(u_0, 0,0,0)$ belongs to $\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D}, T)$ and

$$\|u\|_{\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq C\|u_0\|_{L_p(\Omega;K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D}))}$$

holds, where $C = C(\mathcal{M}, d, p, \theta, \Theta, L_0)$. Moreover, if

$$p\big(1-\lambda_c(\nu_1,\nu_2)\big) < \theta < p\big(d-1+\lambda_c(\nu_1,\nu_2)\big), \tag{3.4}$$

then the constant C depends only on $\mathcal{M}, d, p, \theta, \Theta, \nu_1$ and ν_2.

Proof. Green's function itself is not random. Hence, recalling the definitions of $\mathcal{R}(u_0, 0, 0, 0)$ and $\mathbb{L} = \mathbb{K}^0$, for simplicity we may assume that u_0 and hence u are non-random and we just prove

$$\int_0^T\int_{\mathcal{D}} |\rho^{-1}u|^p \rho_\circ^{\theta-\Theta}\rho^{\Theta-d}dxdt \leq N\int_{\mathcal{D}} |\rho^{-1+\frac{2}{p}}u_0|^p\rho_\circ^{\theta-\Theta}\rho^{\Theta-d}dx. \tag{3.5}$$

1. Let us denote $\mu := -1 + (\theta - d + 2)/p$, $\alpha := -1 + (\Theta - d + 2)/p$, and

$$h(x) := \rho_\circ(x)^{\mu-\alpha}\rho(x)^\alpha u_0(x).$$

Then the claimed estimate (3.5) turns into a simpler form of

$$\left\|\rho_\circ^{\mu-\alpha}\rho^{\alpha-\frac{2}{p}}u\right\|_{L_p([0,T]\times\mathcal{D})} \leq N\|h\|_{L_p(\mathcal{D})}. \tag{3.6}$$

On the other hand, by the range of θ given in the condition, we can always find $\lambda^+ \in \big(0, \lambda^+_{c,L_0}\big)$ and $\lambda^- \in \big(0, \lambda^-_{c,L_0}\big)$ satisfying

$$-\frac{d-2}{p} - \lambda^+ < \mu < \frac{d-2}{p} + \lambda^-. \tag{3.7}$$

Also, by the given range of Θ we have

$$-1 + \frac{1}{p} < \alpha < \frac{1}{p}. \tag{3.8}$$

Hence, we can choose and fix the constants γ, β satisfying

$$0 < \gamma < \lambda^- + \frac{d-2}{p} - \mu\,, \quad 0 < \beta < \frac{1}{p} - \alpha.$$

Noting $\frac{d-2}{p} < d - \frac{d}{p}$, $\frac{1}{p} < 2 - \frac{1}{p}$ which is due to condition $p \in (1, \infty)$, we then have

$$0 < \gamma < \lambda^- + d - \frac{d}{p} - \mu\,, \quad 0 < \beta < 2 - \frac{1}{p} - \alpha. \tag{3.9}$$

Moreover, as $\lambda^+ \in (0, \lambda^+_{c,L_0})$ and $\lambda^- \in (0, \lambda^-_{c,L_0})$, by Lemma 3.3 there exist constants $C = C(\mathcal{M}, L_0, \nu_1, \nu_2, \lambda^{\pm})$, $\sigma = \sigma(\nu_1, \nu_2) > 0$ such that

$$\begin{aligned} G(t,0,x,y) &\leq C\, t^{-\frac{d}{2}}\, R_{t,x}^{\lambda^+-1} R_{t,y}^{\lambda^--1}\, J_{t,x} J_{t,y}\, e^{-\sigma\frac{|x-y|^2}{t}} \\ &= C\, t^{-\frac{d}{2}}\, R_{t,x}^{\lambda^+-1} J_{t,x}\, R_{t,y}^{\gamma} \left(\frac{J_{t,y}}{R_{t,y}}\right)^{\beta} R_{t,y}^{\lambda^--\gamma} \left(\frac{J_{t,y}}{R_{t,y}}\right)^{1-\beta} e^{-\sigma\frac{|x-y|^2}{t}} \end{aligned} \tag{3.10}$$

holds for all $t > s$ and $x, y \in \mathcal{D}$. Let us prove estimate (3.6).

2. Using Hölder inequality and (3.10), we have

$$\begin{aligned} |u(t,x)| &= \left|\int_{\mathcal{D}} G(t,0,x,y)u_0(y)dy\right| \\ &\leq \int_{\mathcal{D}} G(t,0,x,y)|y|^{-\mu+\alpha}\rho(y)^{-\alpha}|h(y)|dy \\ &\leq C\cdot I_1(t,x)\cdot I_2(t,x), \end{aligned}$$

where $q = p/(p-1)$; $\frac{1}{p}+\frac{1}{q}=1$,

$$I_1(t,x) = \left(\int_{\mathcal{D}} t^{-\frac{d}{2}}\, e^{-\sigma\frac{|x-y|^2}{t}}\cdot R_{t,x}^{(\lambda^+-1)p}\, J_{t,x}^{p}\cdot K_1(t,y)\cdot|h(y)|^p dy\right)^{1/p},$$

and

$$I_2(t,x) = \left(\int_{\mathcal{D}} t^{-\frac{d}{2}}\, e^{-\sigma\frac{|x-y|^2}{t}}\cdot K_2(t,y)\cdot|y|^{(-\mu+\alpha)q}\rho^{-\alpha q}(y)dy\right)^{1/q}$$

with

$$K_1(t,y) = R_{t,y}^{\gamma p}\left(\frac{J_{t,y}}{R_{t,y}}\right)^{\beta p}, \quad K_2(t,y) = R_{t,y}^{(\lambda^--\gamma)q}\left(\frac{J_{t,y}}{R_{t,y}}\right)^{(1-\beta)q}.$$

3. We show that there exists a constant C depending only on $\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2$ and λ^- such that

$$I_2(t,x) \leq C\left(|x|+\sqrt{t}\right)^{-\mu+\alpha}\left(\rho(x)+\sqrt{t}\right)^{-\alpha}.$$

This is done by Lemma 3.2 (ii). Indeed, by change of variables $y/\sqrt{t} \to y$ and the fact $\rho(y)/\sqrt{t} = \rho(y/\sqrt{t})$, we have

$$
\begin{aligned}
I_2^q(t,x) &= t^{-\frac{d}{2}}\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t}} K_2(t,y)|y|^{(-\mu+\alpha)q}|\rho(y)|^{-\alpha q}dy \\
&= t^{-\mu q/2}\int_{\mathcal{D}} e^{-\sigma|\frac{x}{\sqrt{t}}-y|^2}\frac{|y|^{(\lambda^- -\mu-\gamma-1+\alpha+\beta)q}}{(|y|+1)^{(\lambda^- -\gamma-1+\beta)q}}\cdot\frac{\rho(y)^{(1-\alpha-\beta)q}}{(\rho(y)+1)^{(1-\beta)q}}dy,
\end{aligned}
$$

for which we can apply Lemma 3.2 since (3.9) implies $(\lambda^- -\mu-\gamma)q > -d$ and $(1-\alpha-\beta)q > -1$. Thus we get constant $C = C(\mathcal{M}, d, p, \theta, \Theta, \lambda^-, \sigma)$ such that

$$
I_2^q(t,x) \le C\left(|x|+\sqrt{t}\right)^{(-\mu+\alpha)q}\left(\rho(x)+\sqrt{t}\right)^{-\alpha q}
$$

holds for all t, x.

4. To prove estimate (3.6), by Step 3 we first note

$$
\begin{aligned}
|x|^{\mu-\alpha}\rho(x)^{\alpha-\frac{2}{p}}\cdot|u(t,x)| &\le C\,|x|^{\mu-\alpha}\rho(x)^{\alpha-\frac{2}{p}}\cdot I_1(t,x)\cdot I_2(t,x) \\
&\le C\,\rho(x)^{-2/p}R_{t,x}^{\mu-\alpha}J_{t,x}^{\alpha}\cdot I_1(t,x)
\end{aligned}
$$

for any t, x. Using this and Fubini's Theorem, we have

$$
\begin{aligned}
\|\rho_\circ^{\mu-\alpha}\rho^{\alpha-\frac{2}{p}}u\|_{L_p([0,T]\times\mathcal{D})}^p &\le C\int_0^T\int_{\mathcal{D}}|\rho(x)|^{-2}\left(R_{t,x}^{\mu-\alpha}J_{t,x}^{\alpha}\,I_1(t,x)\right)^p dxdt \\
&= C\int_{\mathcal{D}} I_3(y)\cdot|h(y)|^p dy,
\end{aligned}
$$

where

$$
I_3(y) = \int_0^T t^{-\frac{d}{2}}K_1(t,y)\left(\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t}}R_{t,x}^{(\lambda^+ +\mu-\alpha-1)p}J_{t,x}^{(\alpha+1)p}\rho(x)^{-2}dx\right)dt.
$$

Since (3.7) and (3.8) imply $(\lambda^+ +\mu)p-2 > -d$ and $(\alpha+1)p-2 > -1$, by change of variables $x/\sqrt{t}\to x$, the fact $\rho(x)/\sqrt{t} = \rho(x/\sqrt{t})$, and Lemma 3.2 (ii), we have

$$
\begin{aligned}
I_3(y) &= \int_0^T \frac{1}{t}K_1(t,y)\int_{\mathcal{D}} e^{-\sigma|x-\frac{y}{\sqrt{t}}|^2}\frac{|x|^{(\lambda^+ +\mu-\alpha-1)p}}{(|x|+1)^{(\lambda^+ +\mu-\alpha-1)p}}\frac{\rho(x)^{(\alpha+1)p-2}}{(\rho(x)+1)^{(\alpha+1)p}}dx\,dt \\
&\le C\int_0^\infty K_1(t,y)\left(\rho(y)+\sqrt{t}\right)^{-2}dt \\
&= C\int_0^\infty \frac{|y|^{(\gamma-\beta)p}}{\left(|y|+\sqrt{t}\right)^{(\gamma-\beta)p}}\cdot\frac{\rho(y)^{\beta p}}{\left(\rho(y)+\sqrt{t}\right)^{\beta p+2}}dt.
\end{aligned}
$$

Lastly, owing to $\gamma p > 0$, $\beta p > 0$, and the fact $|y| \geq \rho(y)$ in $\mathcal{D}$, we can apply Lemma 3.2 (i) and we obtain

$$I_3(y) \leq C(\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2, \lambda^{\pm}).$$

Hence, there exists a constant C having the dependency described in the lemma such that

$$\left\| \rho_\circ^{\mu-\alpha} \rho^{\alpha-\frac{2}{p}} u \right\|^p_{L_p([0,T]\times\mathcal{D})} \leq C \|h\|^p_{L_p(\mathcal{D})}.$$

Estimate (3.6) and the lemma are proved.

5. When θ obeys (3.4), we choose $\lambda^{\pm}$ in the interval $(0, \lambda_c(\nu_1, \nu_2))$. Then the constant C of Green's function estimates in Lemma 3.3 depends only on $\mathcal{M}, \nu_1, \nu_2, \lambda^{\pm}$. Therefore, in particular, constant C in (3.10) does not depend on L_0. Tracking the constants down through Steps 1, 2, 3, 4, we note that the constant in (3.6) does not depend on the particular operator L_0. Rather, it depends on ν_1, ν_2 and hence $C = C(\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2)$. □

Remark 3.5. For $\gamma \geq 0$, $\|u\|_{L_p(\mathbb{R}^d)} \leq \|u\|_{H_p^\gamma(\mathbb{R}^d)}$ is a basic property of the space of Bessel potentials. This with Lemma 2.5 and Definition 2.4, in the context of Lemma 3.4, yields

$$\|u_0\|_{L_p(\Omega; K^0_{\theta+2-p,\Theta+2-p}(\mathcal{D}))} \leq \|u_0\|_{L_p(\Omega; K^{1-2/p}_{\theta+2-p,\Theta+2-p}(\mathcal{D}))} = \|u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{D})}$$

if $p \geq 2$.

Lemma 3.6. *Let $p \in (1, \infty)$ and let $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$ satisfy*

$$p\big(1 - \lambda^+_{c,L_0}\big) < \theta < p\big(d - 1 + \lambda^-_{c,L_0}\big) \quad \text{and} \quad d - 1 < \Theta < d - 1 + p.$$

If $f^0 \in \mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{D}, T)$, $\mathbf{f} \in \mathbb{L}^d_{p,\theta,\Theta}(\mathcal{D}, T, d)$, then $u := \mathcal{R}(0, f^0, \mathbf{f}, 0)$ belongs to $\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D}, T)$ and the estimate

$$\|u\|_{\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq C\big(\|f^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbf{f}\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,d)}\big)$$

holds, where $C = C(\mathcal{M}, d, p, \theta, \Theta, L_0)$. Moreover, if

$$p\,(1 - \lambda_c(\nu_1, \nu_2)) < \theta < p\,(d - 1 + \lambda_c(\nu_1, \nu_2)),$$

then the constant C depends only on $\mathcal{M}, d, p, \theta, \Theta, \nu_1$ and ν_2.

Proof. By the same reason explained in the beginning of the proof of Lemma 3.4, we can assume $f^0, \mathbf{f}$, and hence u are non-random and we just prove

$$\int_0^T \int_{\mathcal{D}} |\rho^{-1} u|^p \rho_0^{\theta-\Theta} \rho^{\theta-d} dx dt \leq C \int_0^T \int_{\mathcal{D}} \left(|\rho f|^p + |\mathbf{f}|^p\right) \rho_0^{\theta-\Theta} \rho^{\theta-d} dx dt. \tag{3.11}$$

Furthermore, when $\mathbf{f} = 0$ estimate (3.11) is already proved in [8, Lemma 3.1], the deterministic counterpart of this article. Hence, we may assume $f^0 = 0$. Finally, for simplicity we further assume $f^2 = \cdots = f^d = 0$.

1. We denote $\mu := (\theta - d)/p$ and $\alpha := (\Theta - d)/p$ and set

$$h(t,x) = \rho_\circ^{\mu-\alpha}(x)\rho^\alpha(x) f^1(t,x).$$

Then (3.11) turns into

$$\left\| \rho_0^{\mu-\alpha} \rho^{\alpha-1} u \right\|_{L_p([0,T]\times\mathcal{D})} \le C \|h\|_{L_p([0,T]\times\mathcal{D})}. \tag{3.12}$$

We prepare a few things as we did in Step 1 of the proof of Lemma 3.4. By the range of θ given in the statement, we can find $\lambda^+ \in (0, \lambda^+_{c,L_0})$ and $\lambda^- \in (0, \lambda^-_{c,L_0})$ satisfying

$$1 - \frac{d}{p} - \lambda^+ < \mu < d - 1 - \frac{d}{p} + \lambda^-.$$

Also, by the range of Θ given we have

$$-\frac{1}{p} < \alpha < 1 - \frac{1}{p}.$$

Then we can choose and fix the constants γ_1, γ_2, β_1 and β_2 satisfying

$$\begin{aligned} -\frac{d-1}{p} &< \gamma_1 < \lambda^+ - 1 + \mu + \frac{1}{p}, & 0 &< \gamma_2 < \lambda^- + d - 1 - \frac{d}{p} - \mu \\ 0 &< \beta_1 < \alpha + \frac{1}{p}, & 0 &< \beta_2 < 1 - \frac{1}{p} - \alpha. \end{aligned} \tag{3.13}$$

Moreover, since $\lambda^+ \in (0, \lambda_c^+)$ and $\lambda^- \in (0, \lambda_c^-)$, by Lemma 3.3 there exist constants $C = C(\mathcal{M}, L_0, \nu_1, \nu_2, \lambda^\pm)$, $\sigma = \sigma(\nu_1, \nu_2) > 0$ such that

$$\begin{aligned} |\nabla_y G(t,s,x,y)| &\le \frac{C}{(t-s)^{(d+1)/2}} e^{-\sigma \frac{|x-y|^2}{t-s}} J_{t-s,x} R_{t-s,x}^{\lambda^+-1} R_{t-s,y}^{\lambda^--1} \\ &= \frac{C}{(t-s)^{(d+1)/2}} e^{-\sigma \frac{|x-y|^2}{t-s}} R_{t-s,x}^{\gamma_1} \left(\frac{J_{t-s,x}}{R_{t-s,x}}\right)^{\beta_1} R_{t-s,y}^{\gamma_2} \left(\frac{J_{t-s,y}}{R_{t-s,y}}\right)^{\beta_2} \\ &\quad \times R_{t-s,x}^{\lambda^+-\gamma_1} \left(\frac{J_{t-s,x}}{R_{t-s,x}}\right)^{1-\beta_1} R_{t-s,y}^{\lambda^--1-\gamma_2} \left(\frac{J_{t-s,y}}{R_{t-s,y}}\right)^{-\beta_2} \end{aligned} \tag{3.14}$$

holds for all $t > s$ and $x, y \in \mathcal{D}$. Now, we start proving (3.12).

2. By Hölder inequality and (3.14), we have

$$\begin{aligned}|u(t,x)| &= \Big|\int_0^t\int_{\mathcal{D}} G_{y^1}(t,s,x,y) f^1(s,y)dyds\Big| \\ &\le \int_0^t\int_{\mathcal{D}} |\nabla_y G(t,s,x,y)|\cdot|y|^{-\mu+\alpha}\rho(y)^{-\alpha}|h(s,y)|dyds \\ &\le C\, I_1(t,x)\cdot I_2(t,x),\end{aligned} \tag{3.15}$$

where $q = p/(p-1)$,

$$I_1(t,x) = \left(\int_0^t\int_{\mathcal{D}} \frac{1}{(t-s)^{(d+1)/2}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{1,1}(t-s,x)K_{1,2}(t-s,y)|h(s,y)|^p dyds\right)^{1/p}$$

and

$$I_2(t,x) = \left(\int_0^t\int_{\mathcal{D}} \frac{1}{(t-s)^{(d+1)/2}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{2,1}(t-s,x)K_{2,2}(t-s,y)|y|^{(-\mu+\alpha)q}\rho^{\alpha q}(y)dyds\right)^{1/q}$$

with

$$K_{1,1}(t,x) = R_{t,x}^{\gamma_1 p}\left(\frac{J_{t,x}}{R_{t,x}}\right)^{\beta_1 p}, \quad K_{1,2}(t,y) = R_{t,y}^{\gamma_2 p}\left(\frac{J_{t,y}}{R_{t,y}}\right)^{\beta_2 p},$$

$$K_{2,1}(t,x) = R_{t,x}^{(\lambda^+-\gamma_1)q}\left(\frac{J_{t,x}}{R_{t,x}}\right)^{(1-\beta_1)q}, \quad K_{2,2}(t,y) = R_{t,y}^{(\lambda^- -1-\gamma_2)q}\left(\frac{J_{t,y}}{R_{t,y}}\right)^{-\beta_2 q}.$$

3. We show that there exists a constant $C = C(\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2) > 0$ such that

$$I_2(t,x) \le C|x|^{-\mu+\alpha}\rho(x)^{-\alpha+\frac{1}{q}} \tag{3.16}$$

holds for all t, x; we note that the right hand side is independent of t.

First, by change of variables $y/\sqrt{t-s} \to y$ and Lemma 3.2 (ii), which we can apply since (3.13) gives $(\lambda^- - 1 - \mu - \gamma_2)q > -d$ and $(-\alpha - \beta_2)q > -1$, we have

$$\frac{1}{(t-s)^{(d+1)/2}}\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{2,2}(t-s,y)|y|^{(-\mu+\alpha)q}|\rho(y)|^{-\alpha q}dy$$

$$= (t-s)^{-(\mu q+1)/2} \int_{\mathcal{D}} e^{-\sigma|\frac{x}{\sqrt{t-s}}-y|^2} \frac{|y|^{(\lambda^- -\mu-1-\gamma_2+\alpha+\beta_2)q}}{(|y|+1)^{(\lambda^- -1-\gamma_2+\beta_2)q}} \cdot \frac{\rho(y)^{(-\alpha-\beta_2)q}}{(\rho(y)+1)^{-\beta_2 q}} dy$$

$$\leq C(t-s)^{-1/2} \left(|x|+\sqrt{t-s}\right)^{(-\mu+\alpha)q} \left(\rho(x)+\sqrt{t-s}\right)^{-\alpha q},$$

where $C = C(\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2)$. Using this, we have

$$\begin{aligned} &I_2^q(t,x) \\ &\leq C \int_0^t K_{2,1}(t-s,x) \cdot (t-s)^{-1/2} \left(|x|+\sqrt{t-s}\right)^{(-\mu+\alpha)q} \left(\rho(x)+\sqrt{t-s}\right)^{-\alpha q} ds \\ &\leq C \int_{-\infty}^t \frac{|x|^{(\lambda^+-1-\gamma_1+\beta_1)q}}{(|x|+\sqrt{t-s})^{(\lambda^+-1+\mu-\alpha-\gamma_1+\beta_1)q}} \cdot \frac{\rho(x)^{(1-\beta_1)q}}{(\rho(x)+\sqrt{t-s})^{(1+\alpha-\beta_1)q}} \cdot \frac{1}{(t-s)^{1/2}} ds. \end{aligned}$$

Then, the change of variable $t-s \to s$ and Lemma 3.2 (i), which we can apply since we have $(\lambda^+ + \mu - \gamma_1)q > 1$ and $(1+\alpha-\beta_1)q > 1$ from (3.13), we further obtain

$$I_2^q(t,x) \leq C|x|^{(-\mu+\alpha)q} \rho(x)^{-\alpha q+1},$$

which is equivalent to (3.16).

4. Now, by (3.16) and (3.15) we have

$$|u(t,x)| \leq C\, I_1(t,x) \cdot I_2(t,x) \leq C\, |x|^{-\mu+\alpha} \rho(x)^{-\alpha+\frac{1}{q}}\, I_1(t,x)$$

and hence

$$\begin{aligned} \|\rho_0^{\mu-\alpha} \rho^{\alpha-1} u\|_{L_p([0,T]\times\mathcal{D})}^p &\leq C \int_0^T \int_{\mathcal{D}} |\rho(x)|^{-1} I_1^p(t,x)\, dx dt \\ &= C \int_0^T \int_{\mathcal{D}} I_3(s,y) \cdot |h(s,y)|^p dy ds, \end{aligned}$$

where

$$I_3(s,y) = \int_s^T \int_{\mathcal{D}} \frac{1}{(t-s)^{(d+1)/2}} e^{-\sigma \frac{|x-y|^2}{t-s}} K_{1,1}(t-s,x) K_{1,2}(t-s,y) \rho(x)^{-1}\, dx dt.$$

By change of variables $t-s \to t$ followed by $x/\sqrt{t} \to x$ and Lemma 3.2 (ii) with $\gamma_1 p - 1 > -d$ and $\beta_1 p - 1 > -1$ from (3.13), we have

$$
\begin{aligned}
I_3(s,y) &= \int_s^T \frac{1}{(t-s)^{(d+1)/2}} K_{1,2}(t-s,y)\left(\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{1,1}(t-s,x)\rho(x)^{-1}\,dx\right)dt \\
&\leq \int_0^\infty \frac{1}{t} K_{1,2}(t,y)\left(\int_{\mathcal{D}} \frac{|x|^{(\gamma_1-\beta_1)p}}{(|x|+1)^{(\gamma_1-\beta_1)p}} \frac{\rho(x)^{\beta_1 p-1}}{(\rho(x)+1)^{\beta_1 p}} e^{-\sigma|x-\frac{y}{\sqrt{t}}|^2}\,dx\right)dt \\
&\leq C\int_0^\infty K_{1,2}(t,y)\left(\rho(y)+\sqrt{t}\right)^{-1} t^{-1/2}dt \\
&= C\int_0^\infty \frac{|y|^{(\gamma_2-\beta_2)p}}{\left(|y|+\sqrt{t}\right)^{(\gamma_2-\beta_2)p}} \cdot \frac{\rho(y)^{\beta_2 p}}{\left(\rho(y)+\sqrt{t}\right)^{\beta_2 p+1}} \cdot \frac{1}{t^{1/2}}dt.
\end{aligned}
$$

Lastly, due to $\gamma_2 p > 0$ and $\nu_2 p > 0$, Lemma 3.2 (i) yields

$$I_3(s,y) \leq C(\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2).$$

Hence, there exists a constant C having the dependency described in the lemma such that

$$\left\| \rho_\circ^{\mu-\alpha}\rho^{\alpha-1}u \right\|^p_{L_p([0,T]\times\mathcal{D})} \leq C\|h\|^p_{L_p([0,T]\times\mathcal{D})}.$$

(3.12) and the lemma are proved.

5. The last part of the claim related to the range of θ holds by the same reason explained in Step 5 of the proof of Lemma 3.4. □

Now, we move on to the stochastic part, the most important and involved one.

Lemma 3.7. *Let $p \in [2,\infty)$ and let $\theta \in \mathbb{R}$, $\Theta \in \mathbb{R}$ satisfy*

$$p(1-\lambda^+_{c,L_0}) < \theta < p(d-1+\lambda^-_{c,L_0}) \quad \textit{and} \quad d-1 < \Theta < d-1+p.$$

If $g \in \mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)$, then $u := \mathcal{R}(0,0,0,g)$ belongs to $\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$ and the estimate

$$\|u\|_{\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq C\|g\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)} \tag{3.17}$$

holds, where $C = C(\mathcal{M}, d, p, \theta, \Theta, L_0)$. Moreover, if

$$p\big(1-\lambda_c(\nu_1,\nu_2)\big) < \theta < p\big(d-1+\lambda_c(\nu_1,\nu_2)\big),$$

then the constant C depends only on $\mathcal{M}, d, p, \theta, \Theta, \nu_1$, and ν_2.

Proof. 1. Again, we denote $\mu := (\theta-d)/p$ and $\alpha := (\Theta-d)/p$. We put $h(\omega,t,x) = \rho_\circ^{\mu-\alpha}(x)\rho(x)^\alpha g(\omega,t,x)$ and recall

$$\Omega_T = \Omega \times (0,T], \quad L_p(\Omega_T \times \mathcal{D}) := L_p(\Omega_T \times \mathcal{D}, d\mathbb{P}dtdx).$$

Then (3.17) is the same as

$$\left\| \rho_\circ^{\mu-\alpha} \rho^{\alpha-1} u \right\|^p_{L_p(\Omega_T \times \mathcal{D})} \le C \left\| |h|_{\ell_2} \right\|^p_{L_p(\Omega_T \times \mathcal{D})}. \tag{3.18}$$

As we did in the proof of Lemma 3.6, we prepare few things. By the range of θ given, we can find constants $\lambda^+ \in (0, \lambda^+_{c,L_0})$ and $\lambda^- \in (0, \lambda^-_{c,L_0})$ satisfying

$$1 - \frac{d}{p} - \lambda^+ < \mu < d - \frac{d}{p} + \lambda^-.$$

Also, by the range of Θ we have

$$-\frac{1}{p} < \alpha < 1 - \frac{1}{p}.$$

Then we can choose and fix the constants γ_1, γ_2, β_1, and β_2 satisfying

$$\begin{aligned} -\frac{d-2}{p} < \gamma_1 < \lambda^+ - 1 + \mu + \frac{2}{p}, \qquad & 0 < \gamma_2 < \lambda^- + d - \frac{d}{p} - \mu \\ \frac{1}{p} < \beta_1 < \alpha + \frac{2}{p}, \qquad & 0 < \beta_2 < 2 - \frac{1}{p} - \alpha. \end{aligned} \tag{3.19}$$

Further, by Lemma 3.3 there exist constants $C = C(\mathcal{M}, L_0, \nu_1, \nu_2, \lambda^\pm)$, $\sigma = \sigma(\nu_1, \nu_2) > 0 >$ 0 such that for any $t > s$ and $x, y \in \mathcal{D}$,

$$\begin{aligned} G(t,s,x,y) \le & \frac{C}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{t-s}} J_{t-s,x} J_{t-s,y} R^{\lambda^+-1}_{t-s,x} R^{\lambda^- -1}_{t-s,y} \\ = & C\,(t-s)^{-d/2} e^{-\sigma \frac{|x-y|^2}{t-s}} R^{\gamma_1}_{t-s,x} \left(\frac{J_{t-s,x}}{R_{t-s,x}} \right)^{\beta_1} R^{\gamma_2}_{t-s,y} \left(\frac{J_{t-s,y}}{R_{t-s,y}} \right)^{\beta_2} \\ & \times R^{\lambda^+-\gamma_1}_{t-s,x} \left(\frac{J_{t-s,x}}{R_{t-s,x}} \right)^{1-\beta_1} R^{\lambda^- -\gamma_2}_{t-s,y} \left(\frac{J_{t-s,y}}{R_{t-s,y}} \right)^{1-\beta_2} \end{aligned} \tag{3.20}$$

holds.

2. We first estimate the p-th moment $\mathbb{E}|u(t,x)|^p$ for any given t and x. Using Burkholder-Davis-Gundy inequality and Minkowski's integral inequality, we have

$$\begin{aligned} \mathbb{E}|u(t,x)|^p = & \mathbb{E}\Big| \sum_{k\in\mathbb{N}} \int_0^t \int_{\mathcal{D}} G(t,s,x,y) g^k(s,y) dy dw^k_s \Big|^p \\ \le & C\mathbb{E} \left(\int_0^t \sum_{k\in\mathbb{N}} \left(\int_{\mathcal{D}} G(t,s,x,y) g^k(s,y) dy \right)^2 ds \right)^{p/2} \end{aligned}$$

$$\leq C\mathbb{E}\left(\int_0^t\left(\int_{\mathcal{D}} G(t,s,x,y)|g(s,y)|_{\ell_2}dy\right)^2 ds\right)^{p/2}$$

$$= C\mathbb{E}\left(\int_0^t\left(\int_{\mathcal{D}} G(t,s,x,y)|y|^{-\mu+\alpha}\rho(y)^{-\alpha}|h(s,y)|_{\ell_2}dy\right)^2 ds\right)^{p/2}.$$

We denote

$$I(\omega,t,x) := \left(\int_0^t\left(\int_{\mathcal{D}} G(t,s,x,y)|y|^{-\mu+\alpha}\rho(y)^{-\alpha}|h(\omega,s,y)|_{\ell_2}dy\right)^2 ds\right)^{1/2}.$$

Then, using (3.20) and applying Hölder inequality twice for x and then t, we get

$$\begin{aligned} I(\omega,t,x) &\leq C\left(\int_0^t\Big(\int_{\mathcal{D}} I_1\cdot I_2\,dy\Big)^2 ds\right)^{1/2} \\ &\leq C\|I_1(\omega,t,\cdot,x,\cdot)\|_{L_p((0,t)\times\mathcal{D},ds\,dy)}\Big\|\|I_2(t,\cdot,x,\cdot)\|_{L_q(\mathcal{D},dy)}\Big\|_{L_r((0,t),ds)} \end{aligned} \tag{3.21}$$

where $q=\frac{p}{p-1}$, $r=\frac{2p}{p-2}$ $(=\infty$ if $p=2)$,

$$\begin{aligned} &I_1^p(\omega,t,s,x,y) \\ &= (t-s)^{-d/2}e^{-\sigma\frac{|x-y|^2}{t-s}}\left(R_{t-s,x}^{\gamma_1}\left(\frac{J_{t-s,x}}{R_{t-s,x}}\right)^{\beta_1}R_{t-s,y}^{\gamma_2}\left(\frac{J_{t-s,y}}{R_{t-s,y}}\right)^{\beta_2}\right)^p|h(\omega,s,y)|_{\ell_2}^p \\ &= (t-s)^{-d/2}e^{-\sigma\frac{|x-y|^2}{t-s}}\;K_{1,1}(t-s,x)\,K_{1,2}(t-s,y)\;|h(\omega,s,y)|_{\ell_2}^p, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} &I_2^q(t,s,x,y) \\ &= (t-s)^{-d/2}e^{-\sigma\frac{|x-y|^2}{t-s}} \\ &\quad\times\left(R_{t-s,x}^{\lambda_1-\gamma_1}\left(\frac{J_{t-s,x}}{R_{t-s,x}}\right)^{1-\beta_1}R_{t-s,y}^{\lambda_2-\gamma_2}\left(\frac{J_{t-s,y}}{R_{t-s,y}}\right)^{1-\beta_2}\right)^q|y|^{(-\mu+\alpha)q}\rho(y)^{-\alpha q} \\ &= (t-s)^{-d/2}e^{-\sigma\frac{|x-y|^2}{t-s}}\;K_{2,1}(t-s,x)\,K_{2,2}(t-s,y)\;|y|^{(-\mu+\alpha)q}\rho(y)^{-\alpha q}, \end{aligned}$$

with

$$K_{1,1}(t,x)=R_{t,x}^{\gamma_1 p}\left(\frac{J_{t,x}}{R_{t,x}}\right)^{\beta_1 p}, \quad K_{1,2}(t,y)=R_{t,y}^{\gamma_2 p}\left(\frac{J_{t,y}}{R_{t,y}}\right)^{\beta_2 p},$$

$$K_{2,1}(t,x)=R_{t,x}^{(\lambda^+-\gamma_1)q}\left(\frac{J_{t,x}}{R_{t,x}}\right)^{(1-\beta_1)q}, \quad K_{2,2}(t,y)=R_{t,y}^{(\lambda^--\gamma_2)q}\left(\frac{J_{t,y}}{R_{t,y}}\right)^{(1-\beta_2)q}.$$

Note, by (3.21) we have

$$\begin{aligned}\mathbb{E}|u(t,x)|^p &\le C\mathbb{E}I^p(t,x) \\ &\le C\Big\| \|I_2(t,\cdot,x,\cdot)\|_{L_q(\mathcal{D},dy)}\Big\|^p_{L_r((0,t),ds)}\mathbb{E}\|I_1(\omega,t,\cdot,x,\cdot)\|^p_{L_p((0,t)\times\mathcal{D},ds\,dy)}.\end{aligned} \tag{3.23}$$

3. In this step we will show that there exists a constant $C=C(\mathcal{M},d,p,\theta,\Theta,\nu_1,\nu_2)>0$ such that

$$\Big\|\|I_2(t,\cdot,x,\cdot)\|_{L_q(\mathcal{D},dy)}\Big\|_{L_r((0,t),ds)} \le C|x|^{-\mu+\alpha}\rho(x)^{-\alpha+1-2/p}. \tag{3.24}$$

In particular, the right hand side is independent of t.

Case 1. Assume $p=2$ (hence, $q=2$ and $r=\infty$). First, we consider

$$\begin{aligned}&\int_{\mathcal{D}} I_2^2(t,s,x,y)\,dy \\ &= K_{2,1}(t-s,x)\cdot\frac{1}{(t-s)^{d/2}}\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t-s}}K_{2,2}(t-s,y)|y|^{2(-\mu+\alpha)}|\rho(y)|^{-2\alpha}\,dy.\end{aligned}$$

Since $2(\lambda^- -\mu-\gamma_2)>-d$ and $2(1-\alpha-\beta_2)>-1$ from (3.19), by change of variables $y/\sqrt{t-s}\to y$ and Lemma 3.2 (ii), we have

$$\begin{aligned}&\frac{1}{(t-s)^{d/2}}\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t-s}}K_{2,2}(t-s,y)|y|^{2(-\mu+\alpha)}|\rho(y)|^{-2\alpha}\,dy \\ &=(t-s)^{-\mu}\int_{\mathcal{D}} e^{-\sigma|\frac{x}{\sqrt{t-s}}-y|^2}\frac{|y|^{2(\lambda^--\mu-\gamma_2-1+\alpha+\beta_2)}}{(|y|+1)^{2(\lambda^--\gamma_2-1+\beta_2)}}\cdot\frac{\rho(y)^{2(1-\alpha-\beta_2)}}{(\rho(y)+1)^{2(1-\beta_2)}}dy \\ &\le C\left(|x|+\sqrt{t-s}\right)^{2(-\mu+\alpha)}\left(\rho(x)+\sqrt{t-s}\right)^{-2\alpha}.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\sup_{s\in[0,t]}\left(\int_{\mathcal{D}} I_2^2\,dy\right)^{1/2} \\ &\le C\sup_{s\in[0,t]}\left(K_{2,1}(t-s,x)\cdot\left(|x|+\sqrt{t-s}\right)^{2(-\mu+\alpha)}\left(\rho(x)+\sqrt{t-s}\right)^{-2\alpha}\right)^{1/2}\end{aligned}$$

$$
\begin{aligned}
&= C \sup_{s\in[0,t]} \left(\frac{|x|^{\lambda^+-1-\gamma_1+\beta_1}}{(|x|+\sqrt{t-s})^{\lambda^+-1+\mu-\gamma_1-\alpha+\beta_1}} \cdot \frac{\rho(x)^{1-\beta_1}}{(\rho(x)+\sqrt{t-s})^{\alpha+1-\beta_1}} \right) \\
&= C\,|x|^{-\mu+\alpha}\rho(x)^{-\alpha} \sup_{s\in[0,t]} \left(R_{t-s,x}^{\lambda^+-1+\mu-\gamma_1} \left(\frac{J_{t-s,x}}{R_{t-s,x}} \right)^{\alpha+1-\beta_1} \right) \\
&\le C\,|x|^{-\mu+\alpha}\rho(x)^{-\alpha}
\end{aligned}
$$

due to $\lambda^+ - 1 + \mu - \gamma_1 > 0$, $\alpha + 1 - \beta_1 > 0$ and $0 \le J_{t-s,x} \le R_{t-s,x} \le 1$. Thus (3.24) holds.

Case 2. Let $p > 2$. Again, since $(\lambda^- - \mu - \gamma_2)q > -d$ and $(1 - \alpha - \beta_2)q > -1$, by change of variables and Lemma 3.2 (ii), we observe

$$
\begin{aligned}
&\frac{1}{(t-s)^{d/2}} \int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{2,2}(t-s,y)|y|^{(-\mu+\alpha)q}\rho(y)^{-\alpha q}dy \\
&= (t-s)^{-\mu q/2} \int_{\mathcal{D}} e^{-\sigma|\frac{x}{\sqrt{t-s}}-y|^2} \frac{|y|^{(\lambda^- -\mu-\gamma_2-1+\alpha+\beta_2)q}}{(|y|+1)^{(\lambda^- -\gamma_2-1+\beta_2)q}} \cdot \frac{\rho(y)^{(1-\alpha-\beta_2)q}}{(\rho(y)+1)^{(1-\beta_2)q}} dy \\
&\le C\left(|x|+\sqrt{t-s}\right)^{(-\mu+\alpha)q}\left(\rho(x)+\sqrt{t-s}\right)^{-\alpha q}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\int_0^t \|I_2(t,s,x,\cdot)\|^r_{L_q(\mathcal{D},dy)} ds \\
&\le C \int_0^t \left\{ K_{2,1}(t-s,x)\cdot\left(|x|+\sqrt{t-s}\right)^{(-\mu+\alpha)q}\left(\rho(x)+\sqrt{t-s}\right)^{-\alpha q} \right\}^{r/q} ds \\
&= C \int_0^t \frac{|x|^{(\lambda^+-1-\gamma_1+\beta_1)r}}{(|x|+\sqrt{t-s})^{(\lambda^+-1+\mu-\gamma_1-\alpha+\beta_1)r}} \cdot \frac{\rho(x)^{(1-\beta_1)r}}{(\rho(x)+\sqrt{t-s})^{(\alpha+1-\beta_1)r}} ds.
\end{aligned}
$$

Moreover, since (3.19) also gives $\left(\lambda^+ + \mu - \gamma_1\right) r > 2$ and $(\alpha + 1 - \beta_1)\, r > 2$, using Lemma 3.2 we again obtain

$$
\begin{aligned}
\big\| \|I_2(t,\cdot,x,\cdot)\|_{L_q(dy;\mathcal{D})} \big\|_{L_r((0,t),ds)} &= \left(\int_0^t \|I_2\|^r_{L_q(\mathcal{D},dy)} ds \right)^{1/r} \\
&\le C|x|^{-\mu+\alpha}\rho(x)^{-\alpha+1-2/p}.
\end{aligned}
$$

4. Now, by (3.23) and (3.24) we have

$$
\mathbb{E}\left||x|^{\mu-\alpha}\rho(x)^{\alpha-1}u(t,x)\right|^p
$$

$$\le C\big(|x|^{\mu-\alpha}\rho(x)^{\alpha-1}\big)^p \cdot \mathbb{E}\int_0^t\int_{\mathcal{D}} I_1^p(t,s,x,y)dy\,ds \cdot \big(|x|^{-\mu+\alpha}\rho(x)^{-\alpha+1-2/p}\big)^p$$

$$= C\,\rho(x)^{-2}\,\mathbb{E}\int_0^t\int_{\mathcal{D}} I_1^p(t,s,x,y)dy\,ds.$$

Therefore, taking integrations with respect to x and t, using Fubini theorem and recalling (3.22), we have

$$\begin{aligned}\mathbb{E}\,\|\rho_0^{\mu-\alpha}\rho^{\alpha-1}u\|^p_{L_p(\Omega_T\times\mathcal{D})} &\le C\,\mathbb{E}\int_0^T\int_{\mathcal{D}}\int_0^t\int_{\mathcal{D}} |\rho(x)|^{-2} I_1^p\,dyds\,dxdt \\ &= C\,\mathbb{E}\int_0^T\int_{\mathcal{D}} I_3(s,y)\cdot|h(s,y)|^p_{\ell_2}dyds, \end{aligned} \tag{3.25}$$

where

$$I_3(s,y) := \int_s^T\int_{\mathcal{D}} \frac{1}{(t-s)^{d/2}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{1,1}(t,s,x,y)K_{1,2}(t,s,x,y)\rho(x)^{-2}\,dxdt.$$

Since (3.19) also implies $\gamma_1 p - 2 > -d$ and $\beta_1 p - 2 > -1$, by change of variables $T - t \to t$ followed by $x/\sqrt{t} \to t$ and Lemma 3.2 (ii), we have

$$\begin{aligned} I_3(s,y) &= \int_s^T \frac{1}{(t-s)^{d/2}} K_{1,2}(t-s,y)\left(\int_{\mathcal{D}} e^{-\sigma\frac{|x-y|^2}{t-s}} K_{1,1}(t-s,x)\rho(x)^{-2}\,dx\right)dt \\ &\le \int_0^\infty \frac{1}{t} K_{1,2}(t,y)\left(\int_{\mathcal{D}} \frac{|x|^{(\gamma_1-\beta_1)p}}{(|x|+1)^{(\gamma_1-\beta_1)p}}\frac{\rho(x)^{\beta_1 p-2}}{(\rho(x)+1)^{\beta_1 p}} e^{-\sigma'|x-\frac{y}{\sqrt{t}}|^2}\,dx\right)dt \\ &\le C\int_0^\infty K_{1,2}(t,y)\left(\rho(y)+\sqrt{t}\right)^{-2}dt \\ &= C\int_0^\infty \frac{|y|^{(\gamma_2-\beta_2)p}}{\left(|y|+\sqrt{t}\right)^{(\gamma_2-\beta_2)p}}\cdot\frac{\rho(y)^{\beta_2 p}}{\left(\rho(y)+\sqrt{t}\right)^{\beta_2 p+2}}\,dt. \end{aligned}$$

Hence, by Lemma 3.2 (i) with the conditions $\gamma_2 p > 0$ and $\beta_2 p > 0$, we finally get

$$I_3(s,y) \le C(\mathcal{M}, d, p, \theta, \Theta, \nu_1, \nu_2).$$

This and (3.25) lead to (3.18) and the lemma is proved.

5. Again, the last part of the claim related to the range of θ holds by the same reason explained in Step 5 of the proof of Lemma 3.4. □

4. Proof of Theorems 2.19 and 2.21

In this section we prove Theorems 2.19 and 2.21, following the strategy below:

1. *A priori estimate and the uniqueness*:

- In Lemma 4.2 below, we first prove that for any solution $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$ to equation (2.2) equipped with the general operator $\mathcal{L} = \sum_{i,j=1}^{d} a^{ij}(\omega,t)D_{ij}$, we have

$$\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)} \leq C\big(\|u\|_{\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \text{norms of the free terms}\big). \tag{4.1}$$

- If $\mathcal{L}$ is non-random, we estimate $\|u\|_{\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}$ based on Lemma 3.1.
- To treat the SPDE with random coefficients, we introduce a SPDE having non-random coefficients and the same free terms f^0, $\mathbf{f}$, g, u_0. Then we prove a priori estimate for the original SPDE based on the fact that the difference between the new SPDE and the original SPDE becomes a PDE (with random coefficients).
- The uniqueness of solution to the original SPDE follows from the uniqueness result of PDEs.

2. *The existence*:

- If the coefficients of $\mathcal{L}$ are non-random, we use the representation formula.
- For general case, we use the method of continuity with the help of the a priori estimate.

Now we start our proofs. The following lemma is what we meant in (4.1). We emphasize that the lemma holds for any $\theta, \Theta \in \mathbb{R}$ and the condition $\partial\mathcal{M} \in C^2$ is not needed in the proof.

Lemma 4.1. *Let $p \in [2,\infty)$, $\gamma,\mu,\theta,\Theta \in \mathbb{R}$, $\mu < \gamma$, and the diffusion coefficients $a^{ij} = a^{ij}(\omega,t)$ satisfy Assumption 2.2. Assume that $f^0 \in \mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)$, $\mathbf{f} = (f^1,\cdots,f^d) \in \mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,d)$, $g \in \mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)$, $u(0,\cdot) \in \mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})$, and $u \in \mathbb{K}^{\mu+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$ satisfies*

$$du = (\mathcal{L}u + f^0 + \sum_{i=1}^{d} f^i_{x^i})\,dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t \in (0,T] \tag{4.2}$$

in the sense of distributions on $\mathcal{D}$. Then $u \in \mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$, hence $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)$, and the estimate

$$\|u\|_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq C\Big(\|u\|_{\mathbb{K}^{\mu+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + \|f^0\|_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbf{f}\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,d)} + \|g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)} + \|u(0,\cdot)\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}\Big)$$

holds with $C = C(\mathcal{M},p,n,\theta,\Theta,\nu_1,\nu_2)$.

The proof of Lemma 4.1 is based on the following result on $\mathbb{R}^d$.

Lemma 4.2. *Let $p \in [2, \infty)$, $\gamma \in \mathbb{R}$, and Assumption 2.2 hold. Assume $f \in \mathbb{H}_p^{\gamma}(T)$, $g \in \mathbb{H}_p^{\gamma+1}(T, \ell_2)$, $u(0, \cdot) \in L_p(\Omega; H_p^{\gamma+2-2/p})$, and $u \in \mathbb{H}_p^{\gamma+1}(T)$ satisfies*

$$du = (\mathcal{L}u + f)\,dt + \sum_{k=1}^{\infty} g^k dw_t^k, \quad t \in (0, T]$$

in the sense of distributions on the whole space $\mathbb{R}^d$. Then $u \in \mathbb{H}_p^{\gamma+2}(T)$ and

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(T)} \leq C\Big(\|u\|_{\mathbb{H}_p^{\gamma+1}(T)} + \|f\|_{\mathbb{H}_p^{\gamma}(T)} + \|g\|_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)} + \|u(0,\cdot)\|_{L_p(\Omega; H_p^{\gamma+2-2/p})}\Big), \tag{4.3}$$

where $C = C(d, p, \nu_1, \nu_2)$ is independent of T.

Proof. **1**. First, we consider the case $u(0, \cdot) \equiv 0$. Then, by e.g. [17, Theorem 4.10], $u \in \mathbb{H}_p^{\gamma+2}(T)$ and

$$\|u_{xx}\|_{\mathbb{H}_p^{\gamma}(T)} \leq C(d, p, \nu_1, \nu_2)(\|f\|_{\mathbb{H}_p^{\gamma}(T)} + \|g\|_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)}).$$

This and the inequality

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(T)} \leq (\|u_{xx}\|_{\mathbb{H}_p^{\gamma}(T)} + \|u\|_{\mathbb{H}_p^{\gamma}(T)})$$

together with the inequality $\|u\|_{\mathbb{H}_p^{\gamma}(T)} \leq \|u\|_{\mathbb{H}_p^{\gamma+1}(T)}$, which due to a basic property of the space of Bessel potentials, yield the claim of the lemma.

2. For the case of general $u(0, \cdot) \not\equiv 0$, we use the solution $v = v(\omega, t, x)$ to the equation

$$dv = \mathcal{L}v\,dt, \quad t \in (0, T]$$

with $v(\omega, 0, \cdot) = u(\omega, 0, \cdot)$ for all $\omega \in \Omega$ (see [17, Theorem 5.2]). From a classical theory of PDE, which we apply for each ω, we have

$$\|v\|_{\mathbb{H}_p^{\gamma+2}(T)} \leq C\|u_0\|_{L_p(\Omega; H_p^{\gamma+2-2/p})}.$$

Then for the function $u - v$, which has zero initial condition, we can apply Step 1 and we obtain estimate (4.3) for u simply by triangle inequality. □

Proof of Lemma 4.1. We first note that we only need to consider the case $\mu = \gamma - 1$. Indeed, suppose that the lemma holds true if $\mu = \gamma - 1$. Now let $\mu = \gamma - n$, $n \in \mathbb{N}$. Then applying the result for $\mu' = \gamma - k$ and $\gamma' = \mu' + 1$ with $k = n, n-1, \cdots, 1$ in order, we get the claim when $\mu = \gamma - n$. Now suppose that the difference between γ and μ is not an integer, i.e. $\gamma - \mu = n + \delta$, $n = 0, 1, 2, \cdots$ and $\delta \in (0, 1)$. Then, since $\mu > \gamma - (n+1) =: \mu'$ and $\|\cdot\|_{\mathbb{K}^{\mu'+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} \leq \|\cdot$

$\|_{\mathbb{K}^{\mu+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}$, we conclude that our assumption holds for μ', that is, $u \in \mathbb{K}^{\mu'+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$. Therefore, the case $\gamma-\mu \notin \mathbb{N}$ is also covered by what we just discussed.

Now we prove the lemma when $\mu=\gamma-1$, i.e. $u \in \mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$. As usual, we omit the argument ω for the simplicity of presentation.

1. For $u \in \mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)$, put

$$\xi(x)=|x|^{(\theta-\Theta)/p}, \quad v:=\xi u, \quad f:=f^0+\sum_{i=1}^d f^i_{x^i}, \quad v_0:=\xi u_0.$$

Using Definition 2.4, Definition 2.3, and the change of variables $t\to e^{2n}t$, we have

$$\begin{aligned}
&\|u\|^p_{\mathbb{K}^{\gamma+2}_{p,\Theta-p,\Theta-p}(\mathcal{D},T)} = \|v\|^p_{\mathbb{H}^{\gamma+2}_{p,\Theta-p}(\mathcal{D},T)}\\
&=\sum_{n\in\mathbb{Z}} e^{n(\Theta-p)}\|\zeta(e^{-n}\psi(e^n\cdot))v(\cdot,e^n\cdot)\|^p_{\mathbb{H}^{\gamma+2}_p(T)}\\
&=\sum_{n\in\mathbb{Z}} e^{n(\Theta-p+2)}\|\zeta(e^{-n}\psi(e^n\cdot))v(e^{2n}\cdot,e^n\cdot)\|^p_{\mathbb{H}^{\gamma+2}_p(e^{-2n}T)}.
\end{aligned} \tag{4.4}$$

For each $n\in\mathbb{Z}$, we denote

$$v_n(t,x):=\zeta(e^{-n}\psi(e^nx))v(e^{2n}t,e^nx), \quad v_{0,n}(x)=\zeta(e^{-n}\psi(e^nx))v_0(e^nx).$$

Then using equation (4.2) and the product rule of differentiation, one can easily check that v_n satisfies

$$dv_n=(\mathcal{L}_nv_n+f_n)dt+\sum_{k=1}^\infty g^k_n dw^{n,k}_t \quad t\in(0,e^{-2n}T]$$

in the sense of distributions on $\mathbb{R}^d$ with the initial condition $v_n(0,\cdot)=v_{0,n}(\cdot)$, where

$$\mathcal{L}_n:=\sum_{i,j}a^{ij}_n(t)D_{ij}, \quad a^{ij}_n(t):=a^{ij}(e^{2n}t),$$

$$g^k_n(t,x):=e^n\zeta(e^{-n}\psi(e^nx))\xi(e^nx)g^k(e^{2n}t,e^nx), \qquad w^{n,k}_t:=e^{-n}w^k_{e^{2n}t},$$

and, with Einstein's summation convention with respect to i,j,

$$\begin{aligned}
f_n(t,x):=\ & e^{2n}\zeta(e^{-n}\psi(e^nx))\xi(e^nx)f(e^{2n}t,e^nx)\\
&+e^na^{ij}_n(t)D_iu(e^{2n}t,e^nx)\zeta'(e^{-n}\psi(e^nx))D_j\psi(e^nx)\xi(e^nx)\\
&+e^{2n}a^{ij}_n(t)D_iu(e^{2n}t,e^nx)\zeta(e^{-n}\psi(e^nx))D_j\xi(e^nx)\\
&+e^na^{ij}_n(t)u(e^{2n}t,e^nx)\zeta'(e^{-n}\psi(e^nx))D_i\psi(e^nx)D_j\xi(e^nx)\\
&+e^{2n}a^{ij}_n(t)u(e^{2n}t,e^nx)\zeta(e^{-n}\psi(e^nx))D_{ij}\xi(e^nx)
\end{aligned}$$

$$
\begin{aligned}
&+a_n^{ij}(t)u(e^{2n}t,e^nx)\zeta''(e^{-n}\psi(e^nx))D_i\psi(e^nx)D_j\psi(e^nx)\xi(e^nx)\\
&+e^na_n^{ij}(t)u(e^{2n}t,e^nx)\zeta'(e^{-n}\psi(e^nx))D_{ij}\psi(e^nx)\\
=:&\sum_{l=1}^{7}f_n^l(t,x).
\end{aligned}
$$

Here, ζ' and ζ'' denote the first and second derivative of ζ, respectively. We note that for each $n\in\mathbb{Z}$, the operator $\mathcal{L}_n$ still satisfies the uniform parabolicity condition (2.3) and $\{w_t^{n,k}:k\in\mathbb{N}\}$ is a sequence of independent Brownian motions. Hence, we can apply Lemma 4.2 and from (4.4) we get

$$
\begin{aligned}
\|u\|^p_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}\leq C\sum_{n\in\mathbb{Z}}&e^{n(\Theta-p+2)}\|v_n\|^p_{\mathbb{H}_p^{\gamma+1}(e^{-2n}T)}\\
&+C\sum_{l=1}^{7}\sum_{n\in\mathbb{Z}}e^{n(\Theta-p+2)}\|f_n^l\|^p_{\mathbb{H}_p^{\gamma}(e^{-2n}T)}\\
&+C\sum_{n\in\mathbb{Z}}e^{n(\Theta-p+2)}\|g_n\|^p_{\mathbb{H}_p^{\gamma+1}(e^{-2n}T,\ell_2)}\\
&+C\sum_{n\in\mathbb{Z}}e^{n(\Theta-p+2)}\|v_{0,n}\|^p_{L_p(\Omega;H_p^{\gamma+2-2/p})}
\end{aligned}
\tag{4.5}
$$

provided that

$$
v_n\in\mathbb{H}_p^{\gamma+1}(e^{-2n}T),\quad f_n^l\in\mathbb{H}_p^{\gamma}(e^{-2n}T),\quad g_n\in\mathbb{H}_p^{\gamma+1}(e^{-2n}T,\ell_2),\quad (l=1,\dots,7).\tag{4.6}
$$

It turns out that the claims in (4.6) hold true. Indeed, the change of variable $e^{2n}t\to t$ and Definition 2.4 yield

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}e^{n(\Theta-p+2)}\|v_n\|^p_{\mathbb{H}_p^{\gamma+1}(e^{-2n}T)}\\
&=\sum_{n\in\mathbb{Z}}e^{n(\Theta-p)}\|\zeta(e^n\psi(e^n\cdot))v(\cdot,e^n\cdot)\|^p_{\mathbb{H}_p^{\gamma+1}(T)}=\|u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}
\end{aligned}
\tag{4.7}
$$

and

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}}e^{n(\Theta-p+2)}\|g_n\|^p_{\mathbb{H}_p^{\gamma+1}(e^{-2n}T,\ell_2)}\\
&=\sum_{n\in\mathbb{Z}}e^{n\Theta}\|\zeta(e^n\psi(e^n\cdot))\xi(e^n\cdot)g(\cdot,e^n\cdot)\|^p_{\mathbb{H}_p^{\gamma+1}(T,\ell_2)}=\|g\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)}.
\end{aligned}
\tag{4.8}
$$

In particular,

$$
v_n\in\mathbb{H}_p^{\gamma+1}(e^{-2n}T),\quad g_n\in\mathbb{H}_p^{\gamma+1}(e^{-2n}T,\ell_2),\quad \forall n\in\mathbb{Z}.
$$

Next, we show that f_n^l belong to $\mathbb{H}_p^\gamma(e^{-2n}T)$ in the following manner. For $l=1$, by Definition 2.4 and the change of variables $e^{2n}t \to t$, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} e^{n(\Theta-p+2)} \|f_n^1\|^p_{\mathbb{H}_p^\gamma(e^{-2n}T)} \\
&= \sum_{n\in\mathbb{Z}} e^{n(\Theta+p)} \|\zeta(e^{-n}\psi(e^n\cdot))\xi(e^n\cdot)f(e^{2n}\cdot, e^n\cdot)\|^p_{\mathbb{H}_p^\gamma(T)} = \|f\|^p_{\mathbb{K}^\gamma_{p,\theta+p,\Theta+p}(\mathcal{D},T)}.
\end{aligned}$$

For $l=2$, by Definition 2.4 and (2.15), we get

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} e^{n(\Theta-p+2)} \|f_n^2\|^p_{\mathbb{H}_p^\gamma(e^{-2n}T)} \\
&\le C \sum_{n\in\mathbb{Z}} \sum_{i,j} e^{n\Theta} \|D_i u(\cdot, e^n\cdot)\zeta'(e^{-n}\psi(e^n\cdot))\xi(e^n\cdot)D_j\psi(e^n\cdot)\|^p_{\mathbb{H}_p^\gamma(T)} \\
&\le C\|\psi_x \xi u_x\|^p_{\mathbb{H}^\gamma_{p,\Theta}(\mathcal{D},T)} = N\|\psi_x u_x\|^p_{\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T)} \\
&\le C\|u_x\|^p_{\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T)} \le C\|u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)},
\end{aligned}$$

where the last two inequalities are due to (2.8), (2.17), and (2.18). For $l=3$, by definitions of norms, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} e^{n(\Theta-p+2)} \|f_n^2\|^p_{\mathbb{H}_p^\gamma(e^{-2n}T)} \\
&\le C \sum_{n\in\mathbb{Z}} \sum_{i,j} e^{n(\Theta+p)} \|D_i u(\cdot, e^n\cdot)\zeta(e^{-n}\psi(e^n\cdot))D_j\xi(e^n\cdot)\|^p_{\mathbb{H}_p^\gamma(T)} \\
&= C\|u_x\xi_x\|^p_{\mathbb{H}^\gamma_{p,\Theta+p}(\mathcal{D},T)} = C\|\xi\xi^{-1}\xi_x u_x\|^p_{\mathbb{H}^\gamma_{p,\Theta+p}(\mathcal{D},T)} \\
&= C\|\xi^{-1}\xi_x u_x\|^p_{\mathbb{K}^\gamma_{p,\theta+p,\Theta+p}(\mathcal{D},T)} \le C\|\psi\xi^{-1}\xi_x u_x\|^p_{\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T)},
\end{aligned} \tag{4.9}$$

where the last inequality is due to (2.16). Now we note that for any $n \in \mathbb{N}$,

$$|\psi\xi^{-1}\xi_x|_n^{(0)} + |\psi^2\xi^{-1}\xi_{xx}|_n^{(0)} \le C(n,\xi) < \infty.$$

Thus, by (2.17) the last term in (4.9) is bounded by

$$C\|u_x\|^p_{\mathbb{K}^\gamma_{p,\theta,\Theta}(\mathcal{D},T)} \le C\|u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}.$$

For other ls one can argue similarly and we gather the results:

$$\begin{aligned}
&\sum_{l=1}^{7}\sum_{n\in\mathbb{Z}} e^{n(\Theta-p+2)} \|f_n^l\|^p_{\mathbb{H}_p^\gamma(e^{-2n}T)} \\
&\le C\|u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)} + C\|f\|^p_{\mathbb{K}^\gamma_{p,\theta+p,\Theta+p}(\mathcal{D},T)}.
\end{aligned} \tag{4.10}$$

Consequently, coming back to (4.5) and using (4.7), (4.8), and (4.10), we get

$$\begin{aligned}&\|u\|^p_{\mathbb{H}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}\\ &\leq C\Big(\|u\|^p_{\mathbb{K}^{\gamma+1}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}+\|f\|^p_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D},T)}+\|g\|^p_{\mathbb{H}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,\ell_2)}+\|u_0\|^p_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}\Big).\end{aligned}$$

This yields what we want to have since $\|f^i_{x^i}\|_{K^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{D})} \leq C\|f^i\|_{K^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D})}$. The lemma is proved. □

Now, we take the deterministic operator L_0 introduced in (3.1) and the Green function G related to L_0. Also, recall the representation $\mathcal{R}(u_0, f^0, \mathbf{f}, g)$ defined in (3.3) in connection with L_0.

Lemma 4.3. *If $f^0 \in \mathbb{K}^{\infty}_c(\mathcal{D}, T)$, $\mathbf{f} \in \mathbb{K}^{\infty}_c(\mathcal{D}, T, d)$, $g \in \mathbb{K}^{\infty}_c(\mathcal{D}, T, \ell_2)$, and $u_0 \in \mathbb{K}^{\infty}_c(\mathcal{D})$, then $u = \mathcal{R}(u_0, f^0, \mathbf{f}, g)$ belongs to $\mathcal{K}^0_{p,\theta,\Theta}(\mathcal{D}, T)$ and satisfies*

$$du = \left(L_0u + f^0 + \sum_{i=1}^{d} f^i_{x^i}\right)dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t \in (0, T] \tag{4.11}$$

in the sense of distributions on $\mathcal{D}$ with $u(0,\cdot) = u_0$.

Proof. First, we note that

$$\begin{aligned}\mathcal{R}(u_0, f^0, \mathbf{f}, g) &= \mathcal{R}(u_0, 0, 0, 0) + \mathcal{R}(0, f^0, \mathbf{f}, 0) + \mathcal{R}(0, 0, 0, g)\\ &=: v_1 + v_2 + v_3.\end{aligned}$$

By considering v_1 for each ω and by the definition of Green's function with the condition $u_0 \in \mathbb{K}^{\infty}_c(\mathcal{D})$, we note that v_1 satisfies

$$dv_1 = L_0v_1dt, \quad t > 0; \quad v_1(0,\cdot) = u_0(\cdot)$$

in the sense of distributions on $\mathcal{D}$. Then Lemma 3.4 and the facts that $\mathbb{K}^{\infty}_c(\mathcal{D})$ is dense in $L_p(\Omega; K^0_{p,\theta+2-p,\Theta+2-p}(\mathcal{D}))$ and $\|u_0\|_{\mathbb{U}^0_{p,\theta,\Theta}(\mathcal{D})} \leq \|u_0\|_{L_p(\Omega;K^0_{p,\theta+2-p,\Theta+2-p}(\mathcal{D}))}$ confirm $v_1 \in \mathcal{K}^0_{p,\theta,\Theta}(\mathcal{D}, T)$. Similarly, v_2 satisfies

$$dv_2 = (L_0v_2 + f^0 + \sum_{i=1}^{d} f^i_{x^i})dt, \quad t > 0$$

in the sense of distributions on $\mathcal{D}$ with zero initial condition and Lemma 3.6 leads us to have $v_2 \in \mathcal{K}^0_{p,\theta,\Theta}(\mathcal{D}, T)$. The fact that v_3 satisfies

$$dv_3 = L_0v_3dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t > 0$$

in the sense of distributions on $\mathcal{D}$ with zero initial condition can be proved by the same way in the proof of [3, Lemma 3.11], which deals with the case $d = 2$. Then Lemma 3.7 gives $v_2 \in \mathcal{K}^0_{p,\theta,\Theta}(\mathcal{D}, T)$. Hence, $u = v_1 + v_2 + v_3$ satisfies the assertions and the lemma is proved. $\square$

Proof of Theorem 2.19. Note that, since $\mathcal{L}$ is non-random, we can take $L_0 = \mathcal{L}$ (see (3.1)).

1. *Existence and estimate* (2.34):

First, we assume that $f^0 \in \mathbb{K}^\infty_c(\mathcal{D}, T)$, $\mathbf{f} \in \mathbb{K}^\infty_c(\mathcal{D}, T, d)$, $g \in \mathbb{K}^\infty_c(\mathcal{D}, T, \ell_2)$, and $u_0 \in \mathbb{K}^\infty_c(\mathcal{D})$. Then by Lemma 4.3, $u = \mathcal{R}(u_0, f^0, \mathbf{f}, g) \in \mathcal{K}^0_{p,\theta,\Theta}(\mathcal{D}, T)$ satisfies equation (4.11) in the sense of distributions on $\mathcal{D}$ with initial condition u_0. Then, we use Lemma 4.1 with $\mu = -2$. As $\gamma + 2 \geq 1$, Lemma 3.1 and Remark 3.5 imply $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ and (2.34).

The general case can be easily handled by standard approximation argument. Indeed, take $f^0_n \in \mathbb{K}^\infty_c(\mathcal{D}, T)$, $\mathbf{f}_n \in \mathbb{K}^\infty_c(\mathcal{D}, T, d)$, $g_n \in \mathbb{K}^\infty_c(\mathcal{D}, T, \ell_2)$, and $u_{0,n} \in \mathbb{K}^\infty_c(\mathcal{D})$ such that $f^0_n \to f^0$, $\mathbf{f}_n \to \mathbf{f}$, $g_n \to g$, and $u_{0,n} \to u_0$, as $n \to \infty$, in the corresponding spaces. Now let $u_n := \mathcal{R}(u_0, f^0_n, \mathbf{f}_n, g_n)$. Then, estimate (2.34) applied for $u_n - u_m$ shows that $\{u_n\}$ is a Cauchy sequence in $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$. Taking u as the limit of u_n in $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$, we find that u is a solution to equation (4.11). Estimate (2.34) for u also follows from those of u_n.

2. *Uniqueness*:

Let $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ be a solution to equation (4.11) with $f^0 \equiv 0$, $\mathbf{f} \equiv 0$, $g \equiv 0$, and $u_0 \equiv 0$. Due to $\gamma + 2 \geq 1$, u at least belongs to $\mathbb{L}_{p,\theta-p,\Theta-p}(\mathcal{D}, T)$, and therefore by Lemma 4.1 we have $u \in \mathcal{K}^2_{p,\theta,\Theta}(\mathcal{D}, T)$ as all the inputs are zeros. Hence, for almost all $\omega \in \Omega$, $u^\omega := u(\omega, \cdot, \cdot) \in L_p((0, T]; K^2_{p,\theta-p,\Theta-p}(\mathcal{D}))$, and satisfies

$$u^\omega_t = \mathcal{L}u^\omega, \quad t \in (0, T] \quad ; \quad u^\omega(0, \cdot) = 0.$$

Hence, from the uniqueness result for the deterministic parabolic equation (see [8, Theorem 2.12]), we conclude $u^\omega = 0$ for almost all ω. This handles the uniqueness. $\square$

Remark 4.4. The approximation argument and uniqueness result in the above proof show that if $\mathcal{L}$ is non-random, then the solution in Theorem 2.19 is given by the formula

$$u = \mathcal{R}(u_0, f^0, \mathbf{f}, g), \quad \text{where} \quad \mathbf{f} = (f^1, \cdots, f^d).$$

Proof of Theorem 2.21. 1. *The a priori estimate*:

Having the method of continuity in mind, we consider the following operators. Denote $L_0 = \nu_1 \Delta$, and for $\lambda \in [0, 1]$ denote

$$\mathcal{L}_\lambda = (1 - \lambda)L_0 + \lambda\mathcal{L}$$

Obviously,

$$\mathcal{L}_\lambda(\omega, \cdot) \in \mathcal{T}_{\nu_1,\nu_2}, \quad \forall \lambda \in [0, 1], \, \omega \in \Omega.$$

Now we prove that the a priori estimate

$$\|v\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)} \leq C\Big(\|f^0\|_{\mathbb{K}^{\gamma\vee 0}_{p,\theta+p,\Theta+p}(\mathcal{D},T)} + \|\mathbf{f}\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,d)} + \|g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T,l_2)} + \|u_0\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}\Big) \tag{4.12}$$

holds with $C = C(\mathcal{M}, d, p, \gamma, \theta, \Theta, \nu_1, \nu_2)$, provided that $v \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ is a solution to the equation

$$dv = \left(\mathcal{L}_\lambda v + f^0 + \sum_{i=1}^{d} f^i_{x^i}\right) dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t \in (0, T] \ ; \quad v(0,\cdot) = u_0(\cdot). \tag{4.13}$$

To prove (4.12), we take $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ from Theorem 2.19, which is the solution to equation (4.11) with the operator $L_0 = \nu_1\Delta$ and the initial condition $u(0,\cdot) = u_0$. Then $\bar{v} := v - u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ satisfies

$$\bar{v}_t = \mathcal{L}_\lambda \bar{v} + \bar{f} = \mathcal{L}_\lambda \bar{v} + \sum_{i=1}^{d} \bar{f}^i_{x^i}, \quad t \in (0, T] \quad ; \quad \bar{v}(0,\cdot) = 0$$

where

$$\bar{f} := (L_0 - \mathcal{L}_\lambda)u = \sum_{i=1}^{d} \left(\sum_{j=1}^{d} [\nu_1 \delta^{ij} - a^{ij}(\omega,t)] u_{x^j}\right)_{x^i} =: \sum_{i=1}^{d} \bar{f}^i_{x^i}.$$

Note that for each fixed ω, $\bar{v}(\omega,\cdot)$ satisfies a deterministic PDE with non-random operator $\mathcal{L}_\lambda(\omega,\cdot)$ and non-random free terms $\bar{f}^i(\omega,\cdot)$. Hence, using the deterministic counterpart of Theorem 2.19 for each ω, and then taking the expectation, we get

$$\|v - u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)} = \|\bar{v}\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D},T)} \leq C \sum_{i=1}^{d} \|\bar{f}^i\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{D},T)} \leq C\|u\|_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{D},T)}.$$

For the last inequality above we used (2.18). This with estimate (2.34) obtained for u finally gives (4.12).

2. *Existence, uniqueness and the estimate*:

Estimate (2.34) and uniqueness result of solution are direct consequences of a priori estimate (4.12), for which the constant C is independent of $\mathcal{L}$ and λ. Thus we only need to prove the existence result.

Let J denote the set of $\lambda \in [0, 1]$ such that for any given $f^0, \mathbf{f}, g, u_0$ in their corresponding spaces, equation (4.13) with given λ has a solution v in $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$. Then by Theorem 2.19, $0 \in J$. Hence, the method of continuity (see e.g. proof of [17, Theorem 5.1]) and a priori estimate (4.12) together yield $J = [0, 1]$, and in particular $1 \in J$. This proves the existence result. The theorem is proved. □

In the next section, we use the result of Theorem 2.21 to study the regularity of SPDEs on polygonal domains in $\mathbb{R}^2$. We also use the following result which helps us prove the existence of a solution on polygonal domains.

Lemma 4.5. *For $j = 1, 2$, let $p_j \geq 2$ and $\theta_j, \Theta_j \in \mathbb{R}$, and $d-1 < \Theta_j < d-1+p_j$. Also let θ_j ($j = 1, 2$) satisfy*

$$p_j(1-\lambda_c^+) < \theta_j < p_j(d-1+\lambda_c^-) \quad \textit{if } \mathcal{L} \textit{ is non-random},$$

and

$$p_j(1-\lambda_c(\nu_1,\nu_2)) < \theta_j < p_j(d-1+\lambda_c(\nu_1,\nu_2)) \quad \textit{if } \mathcal{L} \textit{ is random}.$$

Then, if $u \in \mathcal{K}^1_{p_1,\theta_1,\Theta_1}(\mathcal{D},T)$ is a solution to equation (2.2) with the initial condition $u(0,\cdot) = u_0(\cdot)$ and $f^0, \mathbf{f} = (f^1, \cdots, f^d)$, g, u_0 satisfying

$$f^0 \in \mathbb{L}_{p_j,\theta_j+p_j,\Theta_j+p_j}(\mathcal{D},T), \quad \mathbf{f} \in \mathbb{L}_{p_j,\theta_j,\Theta_j}(\mathcal{D},T,d),$$

$$g \in \mathbb{L}_{p_j,\theta_j,\Theta_j}(\mathcal{D},T,\ell_2), \quad u_0 \in \mathbb{U}^1_{p,\theta,\Theta}(\mathcal{D})$$

for both $j = 1$ and $j = 2$, then $u \in \mathcal{K}^1_{p_2,\theta_2,\Theta_2}(\mathcal{D},T)$.

Proof. If $\mathcal{L}$ is non-random, the lemma follows from Remark 4.4. In general, as before we fix a deterministic operator $L_0(t) = \sum_{i,j} \alpha^{ij}(t) \in \mathcal{T}_{\nu_1,\nu_2}$ and $v = \mathcal{R}(u_0, f^0, \mathbf{f}, g)$. Then, since L_0 is non-random, by Remark 4.4

$$v \in \mathcal{K}^1_{p_1,\theta_1,\Theta_1}(\mathcal{D},T) \cap \mathcal{K}^1_{p_2,\theta_2,\Theta_2}(\mathcal{D},T). \tag{4.14}$$

Put $\bar{u}_1 := u - v$. Then $\bar{u} = \bar{u}_1$ satisfies

$$d\bar{u} = \left[\mathcal{L}\bar{u} + \sum_{i=1}^d \Big(\sum_{j=1}^d [\alpha^{ij}(t) - a^{ij}(\omega,t)] v_{x^j} \Big)_{x^i} \right] dt, \quad t \in (0,T]. \tag{4.15}$$

Also, due to (4.14), equation (4.15) has a solution $\bar{u}_2 \in \mathcal{K}^1_{p_2,\theta_2,\Theta_2}(\mathcal{D},T)$. Now note that for each fixed ω, both $\bar{u}_1(\omega,\cdot,\cdot)$ and $\bar{u}_2(\omega,\cdot,\cdot)$ satisfy equation (4.15), which we can consider as a deterministic equation with non-random operator. By the above result for non-random operator we conclude

$$\bar{u}_1(\omega,\cdot,\cdot) = \bar{u}_2(\omega,\cdot,\cdot)$$

for almost all ω. From this we conclude that both v and $u - v$ are in $\mathcal{K}^1_{p_2,\theta_2,\Theta_2}(\mathcal{D},T)$, and therefore the lemma is proved. □

5. SPDE on polygonal domains

In this section, based on Theorem 2.21, we develop a regularity theory of the stochastic parabolic equations on polygonal domains in $\mathbb{R}^2$. This development is an enhanced version of the corresponding result in [2] in which $\mathcal{L} = \Delta_x$ and $\Theta = d$. Our generalization is as follows:

- $\Delta \quad \to \quad \mathcal{L} = \sum_{i,j} a^{ij}(\omega, t) D_{ij}$; operator with (random) predictable coefficients
- $\Theta = 2 \quad \to \quad 1 < \Theta < 1 + p$
- The restriction on θ is weakened
- Sobolev regularity with $\gamma \in \{-1, 0, \cdots\}$
  $\to$ Sobolev and Hölder regularities with real number $\gamma \geq -1$

Let $\mathcal{O} \subset \mathbb{R}^2$ be a bounded polygonal domain with a finite number of vertices $\{p_1, \dots, p_M\} \subset \partial\mathcal{O}$. For any $x \in \mathcal{O}$, we denote

$$\rho(x) := \rho_{\mathcal{O}}(x) := d(x, \partial\mathcal{O}).$$

In the polygonal domain, the function of x defined by

$$\min_{1 \leq m \leq M} |x - p_m|$$

will play the role of $\rho_{\circ,\mathcal{D}}$, which is the distance to the vertex in an angular domain $\mathcal{D}$. We first construct a smooth version of the function $\min_{1 \leq m \leq M} |x - p_m| g$ as follows. Consider the domain $V := \mathbb{R}^2 \setminus \{p_1, \cdots, p_M\}$ and note that

$$\rho_V(x) := d(x, \partial V) = \min_{1 \leq m \leq M} |x - p_m|.$$

Then, applying (2.9) and (2.10) for ρ_V and the domain V, we define ψ_V and set

$$\rho_\circ = \rho_{\circ,\mathcal{O}} := \psi_V.$$

We can check that for any multi-index α and $\mu \in \mathbb{R}$,

$$\rho_\circ \sim \min_{1 \leq m \leq M} |x - p_m|, \qquad \sup_{\mathcal{O}} \left| \rho_\circ^{|\alpha| - \mu} D^\alpha \rho_\circ^\mu \right| < \infty.$$

On the other hand, we also choose a smooth function $\psi = \psi_{\mathcal{O}}$ such that $\psi \sim \rho_{\mathcal{O}}$ and satisfies (2.8) with $\rho_{\mathcal{O}}$ in place of $\rho_{\mathcal{D}}$.

Then, we recall the norms of the spaces $H^\gamma_{p,\Theta}(\mathcal{O})$ and $H^\gamma_{p,\Theta}(\mathcal{O}; \ell_2)$ introduced in Definition 2.3;

$$\|f\|^p_{H^\gamma_{p,\Theta}(\mathcal{O})} := \sum_{n \in \mathbb{Z}} e^{n\Theta} \|\zeta(e^{-n}\psi(e^n \cdot)) f(e^n \cdot)\|^p_{H^\gamma_p(\mathbb{R}^d)},$$

$$\|g\|^p_{H^\gamma_{p,\Theta}(\mathcal{O}; \ell_2)} := \sum_{n \in \mathbb{Z}} e^{n\Theta} \|\zeta(e^{-n}\psi(e^n \cdot)) g(e^n \cdot)\|^p_{H^\gamma_p(\mathbb{R}^d; \ell_2)},$$

where $\psi=\psi_{\mathcal{O}}$. Using $\rho_{\circ,\mathcal{O}}$ in place of $\rho_{\circ,\mathcal{D}}$, and following Definition 2.4, we define the function spaces

$$K^{\gamma}_{p,\theta,\Theta}(\mathcal{O}),\quad K^{\gamma}_{p,\theta,\Theta}(\mathcal{O};\mathbb{R}^d),\quad K^{\gamma}_{p,\theta,\Theta}(\mathcal{O};\ell_2),$$

as well as the stochastic spaces

$$\mathbb{K}^{\gamma}_{p,\theta,\Theta}(\mathcal{O},T),\quad \mathbb{K}^{\gamma}_{p,\theta,\Theta}(\mathcal{O},T,d),\quad \mathbb{K}^{\gamma}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2),$$
$$\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O},T),\quad \mathbb{K}^{\infty}_{c}(\mathcal{O},T),\quad \mathbb{K}^{\infty}_{c}(\mathcal{O},T,\ell_2),\quad \mathbb{K}^{\infty}_{c}(\mathcal{O}).$$

More specifically, we write $f\in K^{\gamma}_{p,\theta,\Theta}(\mathcal{O})$ if and only if $\rho_\circ^{(\theta-\Theta)/p}f\in H^{\gamma}_{p,\Theta}(\mathcal{O})$, and define

$$\|f\|_{K^{\gamma}_{p,\theta,\Theta}(\mathcal{O})}:=\|\rho_\circ^{(\theta-\Theta)/p}f\|_{H^{\gamma}_{p,\Theta}(\mathcal{O})}.$$

As in Section 2, if $\gamma\in\mathbb{N}_0$, then we have

$$\|f\|^p_{K^{\gamma}_{p,\theta,\Theta}(\mathcal{O})}\sim\sum_{|\alpha|\le\gamma}\int_{\mathcal{O}}|\rho^{|\alpha|}D^\alpha f|^p\rho_\circ^{\theta-\Theta}\rho^{\Theta-d}dx. \tag{5.1}$$

Definition 5.1. We write $u\in\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O},T)$ if $u\in\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{O},T)$ and there exist $(\tilde f,\tilde g)\in\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{O},T)\times\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)$ and $u(0,\cdot)\in\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O})$ satisfying

$$du=\tilde f\,dt+\sum_k\tilde g^k dw^k_t,\quad t\in(0,T]$$

in the sense of distributions on $\mathcal{O}$. The norm is defined by

$$\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O},T)}:=\|u\|_{\mathbb{K}^{\gamma+2}_{p,\theta-p,\Theta-p}(\mathcal{O},T)}+\|\tilde f\|_{\mathbb{K}^{\gamma}_{p,\theta+p,\Theta+p}(\mathcal{O},T)}+\|\tilde g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}$$
$$+\|u(0,\cdot)\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D})}.$$

Theorem 5.2. *With $\mathcal{D}$ replaced by $\mathcal{O}$, all the claims of Lemma 2.5, Remark 2.6, Theorem 2.10, Theorem 2.11, and Lemma 4.1 hold.*

Proof. All of these claims in Section 2 are proved based on (2.12), (2.14), and some properties of weighted Sobolev spaces $H^{\gamma}_{p,\Theta}(\mathcal{D})$ taken e.g. from [23]. Since these properties in [23] hold true on arbitrary domains, the exactly same proofs of Section 2 work with $\mathcal{D}$ replaced by $\mathcal{O}$. □

Remark 5.3. For the analog of Theorem 2.11 in the case of polygonal domain we do not need the additional condition for the initial condition. This is because since ψ is bounded and $\beta>2/p$, by Lemma 2.5 (iv), we have

$$\|\psi^{\beta-1}u(0,\cdot)\|_{L_p(\Omega;K^{\gamma+2-\beta}_{p,\theta,\Theta}(\mathcal{D}))}\le C\|\psi^{2/p-1}u(0,\cdot)\|_{L_p(\Omega;K^{\gamma+2-2/p}_{p,\theta,\Theta}(\mathcal{D}))}\le C\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}}.$$

For $m = 1, \ldots, M$, let κ_m denote the interior angle at the vertex p_m, and denote

$$\kappa_0 := \max_{1 \le m \le M} \kappa_m.$$

Also, for each m, let $\mathcal{D}_m$ denote the conic domain in $\mathbb{R}^2$ such that

$$\mathcal{O} \cap B_\varepsilon(p_m) \cap \{p_m + x : x \in \mathcal{D}_m\} = \mathcal{O} \cap B_\varepsilon(p_m)$$

for all sufficiently small $\varepsilon > 0$. Denote

$$\lambda^{\pm}_{c,\mathcal{L},\mathcal{O}} := \min_m \lambda^{\pm}_{c,\mathcal{L},\mathcal{D}_m} \quad \text{if } \mathcal{L} \text{ is non-random}$$

and

$$\lambda_{c,\mathcal{O}}(\nu_1, \nu_2) := \min_m \lambda_c(\nu_1, \nu_2, \mathcal{D}_m) \quad \text{if } \mathcal{L} \text{ is random.}$$

In Theorem 5.4 below, we pose the condition

$$p(1 - \lambda^{+}_{c,\mathcal{L},\mathcal{O}}) < \theta < p(1 + \lambda^{-}_{c,\mathcal{L},\mathcal{O}}) \tag{5.2}$$

if $\mathcal{L}$ is non-random, and

$$p(1 - \lambda_{c,\mathcal{O}}(\nu_1, \nu_2)) < \theta < p(1 + \lambda_{c,\mathcal{O}}(\nu_1, \nu_2)) \tag{5.3}$$

if $\mathcal{L}$ is random.

Here are our main results on polygonal domains.

Theorem 5.4 *(SPDE on polygonal domains with random or non-random coefficients). Let $p \in [2, \infty)$, $\gamma \ge -1$, and Assumption 2.2 hold. Also assume that*

$$1 < \Theta < p + 1, \tag{5.4}$$

and condition (5.2) holds if $\mathcal{L}$ is non-random, condition (5.3) holds if $\mathcal{L}$ is random. Then for given $f^0 \in \mathbb{K}^{\gamma\vee 0}_{p,\theta+p,\Theta+p}(\mathcal{O}, T)$, $\mathbf{f} = (f^1, \cdots, f^d) \in \mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O}, T, d)$, $g \in \mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O}, T, \ell_2)$, and $u_0 \in \mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O})$, the equation

$$du = \left(\mathcal{L}u + f^0 + \sum_{i=1}^{d} f^i_{x^i}\right) dt + \sum_{k=1}^{\infty} g^k dw^k_t, \quad t \in (0, T] \quad ; \quad u(0, \cdot) = u_0 \tag{5.5}$$

admits a unique solution u in the class $\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O}, T)$. Moreover, the estimate

$$\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O},T)} \le C\big(\|f^0\|_{\mathbb{K}^{\gamma\vee 0}_{p,\theta+p,\Theta+p}(\mathcal{O},T)} + \|\mathbf{f}\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O},T,d)} + \|g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}$$
$$+ \|u_0\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O})}\big)$$

holds with a constant $C = C(\mathcal{O}, p, \gamma, \nu_1, \nu_2, \theta, \Theta, T)$.

Remark 5.5. Since $d = 2$ in this section, the range of Θ in (5.4) coincides with $(d-1, d-1+p)$ which we have kept throughout this article.

Theorem 5.6 *(Hölder estimates on polygonal domains). Let* $p \geq 2$, $\theta, \Theta \in \mathbb{R}$ *and* $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O}, T)$.

(i) *If* $\gamma + 2 - \frac{d}{p} \geq n + \delta$, *where* $n \in \mathbb{N}_0$ *and* $\delta \in (0, 1)$, *then for any* $k \leq n$,

$$|\rho^{k-1+\frac{\Theta}{p}} \rho_\circ^{(\theta-\Theta)/p} D^k u(\omega, t, \cdot)|_{\mathcal{C}(\mathcal{O})} + [\rho^{n-1+\delta+\frac{\Theta}{p}} \rho_\circ^{(\theta-\Theta)/p} D^k u(\omega, t, \cdot)]_{\mathcal{C}^\delta(\mathcal{O})} < \infty$$

holds for almost all (ω, t). *In particular,*

$$|u(\omega, t, x)| \leq C(\omega, t)\rho^{1-\frac{\Theta}{p}}(x)\rho_\circ^{(-\theta+\Theta)/p}(x).$$

(ii) *Let*

$$2/p < \alpha < \beta \leq 1, \quad \gamma + 2 - \beta - d/p \geq m + \varepsilon,$$

where $m \in \mathbb{N}_0$ *and* $\varepsilon \in (0, 1]$. *Put* $\eta = \beta - 1 + \Theta/p$. *Then for any* $k \leq m$,

$$\mathbb{E} \sup_{t,s \leq T} \frac{\left|\rho^{\eta+k} \rho_\circ^{(\theta-\Theta)/p} \left(D^k u(t) - D^k u(s)\right)\right|^p_{\mathcal{C}(\mathcal{O})}}{|t-s|^{p\alpha/2-1}} < \infty,$$

$$\mathbb{E} \sup_{t,s \leq T} \frac{\left[\rho^{\eta+m+\varepsilon} \rho_\circ^{(\theta-\Theta)/p} \left(D^m u(t) - D^m u(s)\right)\right]^p_{\mathcal{C}^\varepsilon(\mathcal{O})}}{|t-s|^{p\alpha/2-1}} < \infty.$$

Proof. The claims follow from the corresponding results of (2.19) and (2.23) mentioned in Theorem 5.2. □

For the proof of Theorem 5.4, we first prove the following estimate.

Lemma 5.7 *(A priori estimate). Let Assumptions in Theorem 5.4 hold. Then there exists a constant* $C = C(d, p, \theta, \Theta, \nu_1, \nu_2, \mathcal{O}, T)$ *such that the a priori estimate*

$$\|u\|_{\mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O},T)} \leq C\Big(\|f^0\|_{\mathbb{K}^{\gamma\vee 0}_{p,\theta+p,\Theta+p}(\mathcal{O},T)} + \|\mathbf{f}\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O},T,d)} + \|g\|_{\mathbb{K}^{\gamma+1}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)} + \|u_0\|_{\mathbb{U}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O})}\Big) \tag{5.6}$$

holds provided that a solution $u \in \mathcal{K}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{D}, T)$ *to equation* (5.5) *exists.*

Proof. First, choose a sufficiently small constant $r > 0$ such that each $B_{3r}(p_m)$ contains only one vertex p_m and intersects with only two edges for each $m = 1, \ldots, M$. Then we choose a function $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ satisfying

$$1_{B_r(0)}(x) \leq \xi(x) \leq 1_{B_{2r}(0)}(x) \quad \text{for all } x \in \mathbb{R}^2.$$

Let $\xi_m(x) := \xi(x - p_m)$ and $\xi_0 := 1 - \sum_{m=1}^{M} \xi_m$. By the choice of r and ξ, $supp(\xi_m)$s are disjoint and hence $0 \leq \xi_0 \leq 1$. Moreover, $\xi_0(x) = 1$ if $\rho_V(x) > 2r$.

For $m = 1, \ldots, M$, let $\mathcal{D}_m$ be the angular (conic) domain centered at p_m with interior angle κ_m such that $\mathcal{D}_m \cap B_{3r}(p_m) = \mathcal{O} \cap B_{3r}(p_m)$.

Now let G be a C^1-domain in $\mathcal{O}$ such that

$$\xi_0(x) = 0 \quad \text{for } x \in \mathcal{O} \setminus G \quad \text{and} \quad \inf_{x \in G} \rho_\circ(x) \geq c > 0 \text{ with a constant } c.$$

Then, due to the choices of ξ_m and $\mathcal{D}_m$ $(m = 1, \ldots, M)$, (2.4) and (5.1) together easily yield

$$\|\xi_m v\|^p_{K^n_{p,\theta,\Theta}(\mathcal{O})} \sim \|\xi_m v\|_{K^n_{p,\theta,\Theta}(\mathcal{D}_m)}, \quad m = 1, \ldots, M,$$

for any $\theta, \Theta \in \mathbb{R}$, $n \in \{0, 1, 2, \ldots\}$, and $v \in K^n_{p,\theta,\Theta}(\mathcal{O})$. Similarly,

$$\|\xi_0 v\|^p_{K^n_{p,\theta,\Theta}(\mathcal{O})} \sim \int_G |\xi_0 v|^p \rho^{\Theta - d} dx \sim \|\xi_0 v\|^p_{H^n_{p,\Theta}(G)},$$

and the same relations hold for ℓ_2-valued functions. Denote

$$\mathbb{H}^\gamma_{p,\Theta}(G, T) := L_p(\Omega \times (0, T], \mathcal{P}; H^\gamma_{p,\Theta}(G)),$$

$$\mathbb{H}^\gamma_{p,\Theta}(G, T, \ell_2) := L_p(\Omega \times (0, T], \mathcal{P}; H^\gamma_{p,\Theta}(G; \ell_2)).$$

Then, the above observations in particular imply

$$\|v\|_{\mathbb{K}^n_{p,\theta,\Theta}(\mathcal{O},T)} \sim \left(\|\xi_0 v\|_{\mathbb{H}^n_{p,\Theta}(G,T)} + \sum_{m=1}^{M} \|\xi_m v\|_{\mathbb{K}^n_{p,\theta,\Theta}(\mathcal{D}_m,T)} \right) \tag{5.7}$$

for any $v \in \mathbb{K}^n_{p,\theta,\Theta}(\mathcal{O}, T)$, where $n \in \{0, 1, 2, \cdots\}$.

Now, for each $m = 1, \ldots, M$ we define $u_m := \xi_m u$. Then, since $\gamma + 2 \geq 1$, u_m belongs to $\mathbb{K}^1_{p,\theta-p,\Theta-p}(\mathcal{D}_m, T)$. Also, $\xi_0 u$ belongs to $\mathbb{H}^1_{p,\Theta-p}(G, T)$. Note that each u_m satisfies

$$d(u_m) = \left(\mathcal{L}u_m + f^0_m + \sum_{i=1}^{d} (f^i_m)_{x^i} \right) dt + \sum_k g^k_m dw^k_t, \quad t \in (0, T] \tag{5.8}$$

in the sense of distributions on $\mathcal{D}_m$ with the initial condition $u_m(0,\cdot)=\xi_m u_0$ and $\xi_0 u$ satisfies

$$d(\xi_0 u)=\Big(\mathcal{L}(\xi_0 u)+f_0^0+\sum_{i=1}^d (f_0^i)_{x^i}\Big)dt+\sum_k g_0^k dw_t^k,\quad t\in(0,T] \tag{5.9}$$

in the sense of distributions on G with the initial condition $w(0,\cdot)=\xi_0 u_0$, where

$$f_m^0=f^0\xi_m-\sum_{i=1}^d f^i(\xi_m)_{x^i}-u\mathcal{L}(\xi_m),\quad f_m^i=2\sum_{j=1}^d a^{ij}u\,(\xi_m)_{x^j},\quad g_m=g\xi_m \tag{5.10}$$

for $m=0,1,2,\dots,M$.

Since $supp(\xi_m)\subset\overline{B_{2r}(p_m)}$ and $(\xi_m)_x=0$ on a neighborhood of p_m for $m=1,\dots,M$, we have

$$\|u(\xi_m)_x\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},t)}+\|u(\xi_m)_{xx}\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{O},t)}\le C\|u\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},t)}$$

for $t\le T$, where C depends only on $\mathcal{O}$, p, θ and Θ.

Hence, for $m=1,\dots,M$, by Theorems 2.19 and 2.21, which our range of θ allows us to use, we have for any $t\le T$,

$$\begin{aligned}
&\|\xi_m u\|_{\mathbb{K}^1_{p,\theta-p,\Theta-p}(\mathcal{D}_m,t)}\\
&\le C\big(\|f_m^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{D}_m,t)}+\sum_{i=1}^d\|f_m^i\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D}_m,t)}+\|g_m\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{D}_m,t,\ell_2)}\\
&\qquad+\|\xi_m u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{D}_m)}\big)\\
&\le C\big(\|u\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T)}+\|f^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{O},T)}+\|\mathbf{f}\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,d)}+\|g\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}\\
&\qquad+\|u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{O})}\big).
\end{aligned}$$

For $m=0$, by [10, Theorem 2.7] (or [11, Theorem 2.9]), we have

$$\begin{aligned}
&\|\xi_0 u\|_{\mathbb{H}^1_{p,\Theta-p}(G,t)}\\
&\le C\Big(\|f_0^0\|_{\mathbb{L}_{p,\Theta+p}(G,t)}+\sum_{i=0}^d\|f_0^i\|_{\mathbb{L}_{p,\Theta}(G,t)}+\|g_0\|_{\mathbb{L}_{p,\Theta}(G,t,\ell_2)}+\|\xi_0 u_0\|_{L_p(\Omega;H^{1-2/p}_{p,\Theta+2-p}(G))}\Big)\\
&\le C\Big(\|u\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T)}+\|f^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{O},T)}+\|\mathbf{f}\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,d)}+\|g\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}\\
&\qquad+\|u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{O})}\Big).
\end{aligned}$$

Summing up over all $m=0,\dots,M$ and using (5.7), for each $t\le T$, we have

$$\begin{aligned}&\|u\|_{\mathbb{K}^1_{p,\theta-p,\Theta-p}(\mathcal{O},t)}\\ &\leq C\Big(\|u\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},t)}+\|f^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{O},T)}+\|\mathbf{f}\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T)}+\|g\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}\\ &\qquad \|u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{O})}\Big).\end{aligned}$$

Using this and the polygonal versions of (2.22) and (2.25), which mentioned in Theorem 5.2, we get, for each $t\leq T$,

$$\begin{aligned}&\|u\|^p_{\mathcal{K}^1_{p,\theta,\Theta}(\mathcal{O},t)}\\ &\leq C\int_0^t \|u\|^p_{\mathcal{K}^1_{p,\theta,\Theta}(\mathcal{O},s)}ds\\ &\quad + C\left(\|f^0\|^p_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{O},T)}+\|\mathbf{f}\|^p_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T)}+\|g\|^p_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}+\|u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{O})}\right).\end{aligned}$$

Applying Gronwall's inequality, we further obtain

$$\begin{aligned}&\|u\|_{\mathcal{K}^1_{p,\theta,\Theta}(\mathcal{O},T)}\\ &\leq C\left(\|f^0\|_{\mathbb{L}_{p,\theta+p,\Theta+p}(\mathcal{O},T)}+\|\mathbf{f}\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T)}+\|g\|_{\mathbb{L}_{p,\theta,\Theta}(\mathcal{O},T,\ell_2)}+\|u_0\|_{\mathbb{U}^1_{p,\theta,\Theta}(\mathcal{O})}\right).\end{aligned}$$

This and the polygonal version of Lemma 4.1, which is mentioned in Theorem 5.2, yield a priori estimate (5.6). The lemma is proved. □

The following is a $\mathcal{C}^1$-domain version of Lemma 4.5. We use it in the proof of Theorem 5.4 below.

Lemma 5.8. *Let G be a bounded $\mathcal{C}^1$ domain in $\mathbb{R}^d$ and let $p_j\in[2,\infty)$, $\Theta_j\in(d-1,d-1+p_j)$ for $j=1,2$. Assume that $u\in\mathbb{H}^1_{p_1,\Theta_1-p_1}(G,T)$ satisfies*

$$du=\Big(\mathcal{L}u+f^0+\sum_{i=1}^d f^i_{x^i}\Big)dt+\sum_k g^k dw^k_t,\quad t\in(0,T]$$

in the sense of distributions on G with the initial condition $u(0,\cdot)=u_0(\cdot)$ and f^0, f^i ($i=1,2,\ldots$), g, u_0 satisfying

$$f^0\in\mathbb{L}_{p_j,\Theta_j+p_j}(G,T)\cap\mathbb{L}_{p_j,d+p_j}(G,T),\quad f^i\in\mathbb{L}_{p_j,\Theta_j}(G,T)\cap\mathbb{L}_{p_j,d}(G,T),\ i=1,\cdots,d,$$
$$g\in\mathbb{L}_{p_j,\Theta_j}(G,T,\ell_2)\cap\mathbb{L}_{p_j,d}(G,T,\ell_2),$$
$$u_0\in L_p(\Omega,\mathscr{F}_0;H^{1-2/p_j}_{p_j,\Theta_j+2-p_j}(G))\cap L_p(\Omega,\mathscr{F}_0;H^{1-2/p_j}_{p_j,d+2-p_j}(G))$$

for both $j=1$ and $j=2$. Then u belongs to $\mathbb{H}^1_{p_2,\Theta_2-p_2}(G,T)$.

Proof. See [2, Lemma 3.8]. We remark that only Δ is considered in [2], however the proof of [2, Lemma 3.8] works for general case without any changes since the proof depends only on [10, Theorem 2.7] (or [11, Theorem 2.9]), which involves operators having coefficients measurable in (ω, t) and continuous in x. $\square$

We recall $d = 2$ in this section.

Proof of Theorem 5.4. Due to Lemma 5.7, we only need to prove the existence result. Furthermore, relying on standard approximation argument, we may assume

$$f^0 \in \mathbb{K}_c^\infty(\mathcal{O}, T), \quad \mathbf{f} \in \mathbb{K}_c^\infty(\mathcal{O}, T, 2), \quad g \in \mathbb{K}_c^\infty(\mathcal{O}, T, \ell_2) \quad u_0 \in \mathbb{K}_c^\infty(\mathcal{O}).$$

Considering $u - u_0$ as usual, we may assume $u_0 \equiv 0$. Also, note that $g^k = 0$ for all large k (say, for all $k > N$), and each g^k is of the type $\sum_{j=1}^{n(k)} 1_{(\tau_{j-1}^k, \tau_j^k]}(t) h^{kj}(x)$, where τ_j^k are bounded stopping times and $h^{jk} \in \mathcal{C}_c^\infty(\mathcal{O})$. Thus the function v defined by

$$v(t,x) := \sum_{k=1}^{\infty} \int_0^t g^k dw_s^k = \sum_{k \le N} \sum_{j \le n(k)} \left(w^k_{\tau_j^k \wedge t} - w^k_{\tau_{j-1}^k \wedge t} \right) h^{kj}(x)$$

is infinitely differentiable in x and vanishes near the boundary of $\mathcal{O}$. Consequently v belongs to $\mathcal{K}_{p,\theta,\Theta}^{\nu+2}(\mathcal{O}, T)$ for any $\nu, \theta, \Theta \in \mathbb{R}$ as we consult with Definition 5.1. Now, u satisfies equation (5.5) if and only if $\bar{u} := u - v$ satisfies

$$d\bar{u} = \Big(\mathcal{L}\bar{u} + \bar{f}^0 + \sum_{i=1}^{2} f^i_{x^i} \Big) dt, \quad t \in (0, T] \quad ; \quad \bar{u}(0, \cdot) = 0,$$

where $\bar{f}^0 = f^0 + \mathcal{L}v$. Hence, considering $\bar{f}^0$ in place of f^0, to prove the existence we may further assume $g = 0$.

Then, by the classical results without weights for $p = 2$, (see, e.g. [25] or [9, Theorem 2.12, Corollary 2.14]), there exists a solution u in $\mathcal{K}_{2,2,2}^1(\mathcal{O}, T)$ to equation (5.5), which now is simplified as

$$u_t = \mathcal{L}u + f^0 + \sum_{i=1}^{2} f^i_{x^i}, \quad t \in (0, T] \quad ; \quad u(0, \cdot) = 0.$$

By Theorem A in [1] (or see estimate (2.11) and proof of Theorem 2.4 in [12] for more detail), for any $r > 4$, we have

$$\mathbb{E} \sup_{t,x} |u(t,x)|^p \le C\mathbb{E} \| \, |f^0| + |\mathbf{f}| \, \|^p_{L_r((0,T]\times\mathcal{O}))} < \infty. \tag{5.11}$$

Now we prove $u \in \mathcal{K}_{p,\theta,\Theta}^1(\mathcal{O}, T)$ using Lemma 4.5 and Lemma 5.8 along with $u \in \mathcal{K}_{2,2,2}^1(\mathcal{O}, T)$. Define $u_m := \xi_m u$ in the same way we did in the proof of Lemma 5.7. Then $\xi_m u$ satisfies (5.8) in the sense of distributions on $\mathcal{D}_m$ for $m = 1, \ldots, M$ and $\xi_0 u$ satisfies (5.9) on

G for $m=0$ with the same f^0_m, f^i_m, $\xi_m u_0$ as in (5.10). Note that since $f^0, \mathbf{f}$ are bounded and $f^0, \mathbf{f}, (\xi_m)_x, (\xi_m)_{xx}$ vanish near vertices, we have for any $\theta \in \mathbb{R}$, $q \geq 2$ and $1 < \Theta < 1+q$,

$$\|f^0_m\|^q_{\mathbb{L}_{q,\theta+q,\Theta+q}(\mathcal{O},T)} + \sum_{i=1}^{2} \|f^i_m\|^q_{\mathbb{L}_{q,\theta,\Theta}(\mathcal{O},T)}$$

$$\leq C\mathbb{E}\int_0^T \int_{\mathcal{O}} (1+|u|^q)\rho^{\Theta-2}dx \leq C\left(\int_{\mathcal{O}} \rho^{\Theta-2}dx\right)\mathbb{E}\sup_{t,x}(1+|u|^q) < \infty.$$

For the last inequality we used (5.11) and the fact $\Theta - 2 > -1$. Hence, f^0_m, f^i_m along with $\xi_m u_0$ satisfy assumptions in Lemma 4.5 and Lemma 5.8. Consequently $\xi_m u \in \mathbb{K}^1_{p,\theta-p,\Theta-p}(\mathcal{D}_m, T)$ as $\xi_m u \in \mathcal{K}^1_{p,\theta,\Theta}(\mathcal{D}_m, T)$ for $m = 1, 2, \cdots, M$ and $\xi_0 u \in \mathbb{H}^1_{p,\Theta-p}(G, T)$. These and (5.7) with $n = 1$ yield $u \in \mathbb{K}^1_{p,\theta-p,\Theta-p}(\mathcal{O}, T)$ and in turn $u \in \mathcal{K}^1_{p,\theta,\Theta}(\mathcal{O}, T)$.

Finally, the analogy of Lemma 4.1 in case of polygonal domains (see Theorem 5.2) proves that the solution u found above actually belongs to the space $u \in \mathcal{H}^{\gamma+2}_{p,\theta,\Theta}(\mathcal{O}, T)$. The theorem is proved. □

Data availability

No data was used for the research described in the article.

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