



A Sobolev space theory for the stochastic partial differential equations with space-time non-local operators

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Abstract. We deal with the Sobolev space theory for the stochastic partial differential equation (SPDE) driven by Wiener processes

$$\partial_t^\alpha u = (\phi(\Delta)u + f(u)) + \partial_t^\beta \sum_{k=1}^{\infty} \int_0^t g^k(u) dw_s^k, \quad t > 0, x \in \mathbb{R}^d$$

as well as the SPDE driven by space-time white noise

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^{\beta-1} h(u) \dot{W}, \quad t > 0, x \in \mathbb{R}^d.$$

Here, $\alpha \in (0, 1)$, $\beta < \alpha + 1/2$, $\{w_t^k : k = 1, 2, \dots\}$ is a family of independent one-dimensional Wiener processes and \dot{W} is a space-time white noise defined on $[0, \infty) \times \mathbb{R}^d$. The time non-local operator ∂_t^α denotes the Caputo fractional derivative of order α , the function ϕ is a Bernstein function, and the spatial non-local operator $\phi(\Delta)$ is the integro-differential operator whose symbol is $-\phi(|\xi|^2)$. In other words, $\phi(\Delta)$ is the infinitesimal generator of the d -dimensional subordinate Brownian motion. We prove the uniqueness and existence results in Sobolev spaces and obtain the maximal regularity results of solutions.

1. Introduction

We study the stochastic partial differential equations with both time and spatial non-local operators. The time and spatial non-local operators we adopt in this article are ∂_t^α and $\phi(\Delta)$, respectively. The Caputo fractional derivative ∂_t^α is used in the time fractional heat equation to describe, e.g., the anomalous diffusion exhibiting subdiffusive behavior caused by particle sticking and trapping effects (e.g., [35, 36]), and the spatial non-local operator $\phi(\Delta)$ is the infinitesimal generator of the subordinate Brownian motion. The operator $\phi(\Delta)$ describes long range jumps of particles, diffusion on fractal structures, and long-time behavior of particles moving in space with quenched and disordered force field (e.g., [3, 15]). For instance, if $\phi(\lambda) = \lambda^{\delta/2}$, then $\phi(\Delta) = \Delta^{\delta/2}$ becomes the fractional Laplacian, which is related to the isotropic δ -stable process. In

Mathematics Subject Classification: 60H15, 35R60, 26A33, 47G20

Keywords: Stochastic partial differential equations, Sobolev space theory, Space-time non-local operators, Maximal L_p -regularity, Space-time white noise.

The authors were supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government (MSIT) (No. NRF-2019R1A5A1028324).

this article we use both ∂_t^α and $\phi(\Delta)$ for the description of the combined phenomena, for instance, jump diffusions with a higher peak and heavier tails (e.g., [6, 9, 17, 34]).

The goal of this article is to present an L_p -theory ($p \geq 2$) for the SPDE driven by Wiener processes

$$\partial_t^\alpha u = (\phi(\Delta)u + f(u)) + \partial_t^\beta \sum_{k=1}^\infty \int_0^t g^k(u) dw_s^k, \quad t > 0, x \in \mathbb{R}^d; \quad u(0, \cdot) = 0 \tag{1.1}$$

as well as for the SPDE driven by multi-dimensional space-time white noise

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^{\beta-1} h(u) \dot{W}, \quad t > 0, x \in \mathbb{R}^d; \quad u(0, \cdot) = 0. \tag{1.2}$$

As mentioned above, $\{w_t^1, w_t^2, \dots\}$ is a family of independent one-dimensional Wiener processes, \dot{W} is a space-time white noise on $[0, \infty) \times \mathbb{R}^d$, and α and β are constants satisfying $\alpha \in (0, 1)$ and $\beta < \alpha + 1/2$, respectively. The nonlinear terms $f(u)$, $g^k(u)$ and $h(u)$ are functions depending on (ω, t, x, u) . Such types of SPDEs can be used, e.g., to describe random effects of particles in medium with thermal memory or particles subject to sticking and trapping (see, e.g., [7]).

We interpret SPDEs (1.1) and (1.2) by their integral forms, and the restriction $\beta < \alpha + 1/2$ is necessary to make sense of the equations. For instance, the integral form of (1.1) is

$$\begin{aligned} u(t, x) - u(0, x) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (\phi(\Delta)u(s, x) + f(s, x, u(s, x))) ds \\ &\quad + \sum_{k=1}^\infty \frac{1}{\Gamma(1+\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta} g^k(s, x, u(s, x)) dw_s^k, \end{aligned}$$

and even if g^k is bounded, say $g^k \equiv 1$, the condition $\alpha - \beta > -1/2$ is needed to make sense of the integral $\int_0^t (t-s)^{\alpha-\beta} dw_s^k$.

In this article, under appropriate continuity of f, g , we prove the unique solvability of equation (1.1) together with the maximal regularity

$$\begin{aligned} &\mathbb{E} \|\phi(\Delta)^{(\gamma+2)/2} u\|_{L_p([0, T]; L_p)}^p \\ &\leq C \mathbb{E} \left(\|\phi(\Delta)^{\gamma/2} f(0)\|_{L_p([0, T]; L_p)}^p + \|\phi(\Delta)^{(\gamma+c_0)/2} g(0)\|_{l_2}^p \right), \end{aligned} \tag{1.3}$$

for any $p \geq 2$ and $\gamma \in \mathbb{R}$. Here $c_0 := (2\beta - 1)^+ / \alpha < 2$. Also, we use estimate (1.3) for $\gamma < 0$ to deal with equation (1.2), that is, the SPDE driven by space-time white noise. This is possible since one can transform equation (1.2) into the one of type (1.1).

Now let us provide a description on the related works and their approaches. The L_p -theory ($p \geq 2$) of the classical stochastic heat equation of the type

$$du = \Delta u dt + g dw_t, \quad t > 0, x \in \mathbb{R}^d; \quad u(0, \cdot) = 0$$

was first introduced by N.V. Krylov [28,30]. Krylov introduced an analytic approach and proved the maximal regularity estimate

$$\mathbb{E}\|\nabla u\|_{L_p((0,T);L_p)}^p \leq C\mathbb{E}\|g\|_{L_p((0,T);L_p)}^p, \quad p \geq 2. \tag{1.4}$$

The essence of Krylov’s approach is to control the sharp function of derivatives of u in terms of the free term, that is

$$(\nabla u)^\sharp(t, x) \leq C \left(\mathbb{M}|g|^2(t, x) \right)^{1/2}, \quad \forall (t, x) \text{ uniformly on } \Omega, \tag{1.5}$$

where \sharp and \mathbb{M} are used to denote the sharp and maximal functions, respectively (see Sect. 3.1). This with the L_p -norm equivalent relation between functions and their sharp and maximal functions leads to (1.4). Since the work of [28,30], the analytic approach has been further used for SPDEs having different spatial operators. The fractional Laplacian $\Delta^{\delta/2}$ is considered in [4,23], fractional Laplacian-like operator is considered in [38], and the operator $\phi(\Delta)$ is considered in [24]. It is also used for SPDE having time non-local operator in [11,26] and [12], in which the spatial operators used are Δ and $\Delta^{\delta/2}$, respectively. As for other approaches on Sobolev regularity theory, the method based on H^∞ -calculus is also available in the literature. This approach was introduced in [39,44,45], in which the maximal regularity of $\sqrt{-A}u$ is obtained for the stochastic convolution

$$u(t) := \int_0^t e^{(t-s)A} g(s) dW_s.$$

Here, W is a Brownian motion and the operator $-A$ is assumed to admit a bounded H^∞ -calculus of angle less than $\pi/2$ on L_p , where $p \geq 2$. The result of [39,44,45] certainly generalizes Krylov’s result [28,30] as one can take $A = \Delta$. The method based on H^∞ -calculus is also used in [13] for the study of the mild solution to the integral equation

$$u(t) + \int_0^t (t-s)^{\alpha-1} Au(s) ds = \int_0^t (t-s)^{\beta-1} g(s) dW_s, \tag{1.6}$$

where the generator A is supposed to satisfy the assumption described above. We also remark that a nonlinear version of equation (1.6) is studied recently in [32] with $A(U)$ in place of AU in the Hilbert space setting, that is, the Gelfand triple setting. Also see [7] for a Hilbert space theory of SPDEs having time non-local operator and the second-order spatial operators with measurable coefficients.

As is expected, our results for nonlinear equations are proved based on those for the corresponding linear equations and certain fixed point argument. To handle linear equations, we use both analytic approach and the one based on H^∞ -calculus. First, speaking of Krylov’s analytic approach, we control the sharp functions of solutions and their fractional derivatives in terms of free terms. In other words, we prove a generalization of (1.5). This approach is elementary; however, the extension to general

equations involves quite non-trivial computations. Moreover, for a technical reason, this approach is carried out under the condition

$$c \left(\frac{R}{r} \right)^{\delta_0} \leq \frac{\phi(R)}{\phi(r)}, \quad \forall 0 < r < R < \infty, \tag{1.7}$$

where $\delta_0 \in (0, 1]$ and $c > 0$. Regarding the second approach based on H^∞ -calculus, we check that $\phi(\Delta)$ admits the bounded H^∞ -calculus on $L_p(\mathbb{R}^d)$ of angle zero. The second approach works without condition (1.7), but it relies on abstract operator theory.

This article is organized as follows. In Sect. 2, we introduce basic facts on time and spatial non-local operators and related function spaces. Then, we present our main results, Theorems 2.12 and 2.19. In Sect. 3, we obtain a priori estimate for the solutions, and finally in Sects. 4 and 5 we prove Theorems 2.12 and 2.19, respectively.

We finish the introduction with notations used in this article. We use “:=” or “=:” to denote a definition. As usual, \mathbb{R}^d stands for the d -dimensional Euclidean space of points $x = (x^1, \dots, x^d)$. We set $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$, and $B_r := B_r(0)$. \mathbb{N} denotes the natural number system and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $i = 1, \dots, d$, multi-indices $\sigma = (\sigma_1, \dots, \sigma_d)$, $\sigma_i \in \{0, 1, 2, \dots\}$, and functions $u(x)$, we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D_x^\sigma u = D_1^{\sigma_1} \cdots D_d^{\sigma_d} u, \quad |\sigma| = \sigma_1 + \cdots + \sigma_d.$$

We also use the notation D_x^m for the set of partial derivatives of order m with respect to x . For a Banach space B , by $C_c^\infty(\mathbb{R}^d; B)$ we denote the collection of B -valued smooth functions having compact support in \mathbb{R}^d . We drop B if $B = \mathbb{R}^d$. $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class on \mathbb{R}^d . By $C_b^2(\mathbb{R}^d)$, we denote the space of twice continuously differentiable functions on \mathbb{R}^d with bounded derivatives. For $p > 1$, we use L_p to denote the set of complex-valued Lebesgue measurable functions u on \mathbb{R}^d satisfying

$$\|u\|_{L_p} := \left(\int_{\mathbb{R}^d} |u(x)|^p dx \right)^{1/p} < \infty.$$

Generally, for a given measure space (X, \mathcal{M}, μ) , $L_p(X, \mathcal{M}, \mu; B)$ denotes the space of all B -valued \mathcal{M}^μ -measurable functions u so that

$$\|u\|_{L_p(X, \mathcal{M}, \mu; B)} := \left(\int_X \|u(x)\|_B^p \mu(dx) \right)^{1/p} < \infty,$$

where \mathcal{M}^μ denotes the completion of \mathcal{M} with respect to the measure μ . If there is no confusion for the given measure and σ -algebra, we usually omit the measure and the σ -algebra. We use the notations

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}(g)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} g(\xi) d\xi$$

to denote the Fourier and the inverse Fourier transforms, respectively. $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $a^+ := a \vee 0$. Also we write $a \sim b$ if there

exists a constant $c > 1$ independent of a, b such that $c^{-1}a \leq b \leq ca$. If we write $C = C(a, b, \dots)$, this means that the constant C depends only on a, b, \dots . Throughout the article, for functions depending on (ω, t, x) , the argument $\omega \in \Omega$ will be usually omitted.

2. Main results

First, we introduce some preliminary facts on the fractional calculus. For $\alpha > 0$ and $\varphi \in L_1((0, T))$, we define the Riemann–Liouville fractional integral of the order α by

$$I_t^\alpha \varphi := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \varphi(s) ds, \quad 0 \leq t \leq T.$$

We also define $I^0 \varphi := \varphi$. Due to Jensen’s inequality, for $p \in [1, \infty]$,

$$\|I_t^\alpha \varphi\|_{L_p(0,T)} \leq C(\alpha, p, T) \|\varphi\|_{L_p(0,T)}. \tag{2.1}$$

One can easily check for any $\alpha, \beta \geq 0$

$$I_t^\alpha I_t^\beta \varphi = I_t^{\alpha+\beta} \varphi. \tag{2.2}$$

Let $\alpha \in [n - 1, n)$ for some $n \in \mathbb{N}$. For a function $\varphi(t)$ which $\left(\frac{d}{dt}\right)^{n-1} I_t^{n-\alpha} \varphi$ is absolutely continuous on $[0, T]$, the Riemann–Liouville fractional derivative D_t^α and the Caputo fractional derivative ∂_t^α of the order α are defined as

$$D_t^\alpha \varphi := \left(\frac{d}{dt}\right)^n (I_t^{n-\alpha} \varphi), \tag{2.3}$$

and

$$\partial_t^\alpha \varphi = D_t^\alpha \left(\varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^{(k)}(0) \right).$$

In particular, if $\alpha \in (0, 1)$, then

$$D_t^\alpha \varphi = \left(I_t^{1-\alpha} \varphi\right)'(t), \quad \partial_t^\alpha \varphi(t) = \left(I_t^{1-\alpha} (\varphi - \varphi(0))\right)'(t).$$

Note that $D_t^\alpha \varphi = \partial_t^\alpha \varphi$ if $\varphi(0) = \varphi^{(1)}(0) = \dots = \varphi^{(n-1)}(0) = 0$. By (2.2) and (2.3), for any $\alpha, \beta \geq 0$, we have

$$D_t^\alpha I_t^\beta \varphi = \begin{cases} D_t^{\alpha-\beta} \varphi & : \alpha > \beta \\ I_t^{\beta-\alpha} \varphi & : \alpha \leq \beta, \end{cases}$$

and $D_t^\alpha D_t^\beta \varphi = D_t^{\alpha+\beta} \varphi$. Also if $\varphi(0) = \varphi^{(1)}(0) = \dots = \varphi^{(n-1)}(0) = 0$ then

$$I_t^\alpha \partial_t^\alpha u = I_t^\alpha D_t^\alpha u = u.$$

Finally we define $I_t^{-\alpha} \varphi := D_t^\alpha \varphi$ for $\alpha > 0$.

Next, we introduce the spatial non-local operator $\phi(\Delta)$, and function spaces related to this operator. Recall that a function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\phi(0+) = 0$ is called a Bernstein function if there exist a constant $b \geq 0$, called a drift, and a Lévy measure μ (i.e., $\int_{(0,\infty)} (1 \wedge t) \mu(dt) < \infty$) such that

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t}) \mu(dt). \tag{2.4}$$

It is known that $\phi(\lambda)$ is a Bernstein function if and only if there is a subordinator (i.e., one-dimensional non-decreasing Lévy process) S_t whose Laplace exponent is $\phi(\lambda)$, that is,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad \forall \lambda > 0. \tag{2.5}$$

From (2.4), we easily have $\phi'(\lambda) > 0$ and

$$(-1)^n \phi^{(n)}(\lambda) \leq 0, \quad \forall \lambda > 0, n \in \mathbb{N}.$$

Actually, for any $n \geq 1$, we also have (see, e.g., [27,42])

$$\lambda^n |\phi^{(n)}(\lambda)| \leq C(n)\phi(\lambda). \tag{2.6}$$

Note also that ϕ^{-1} , the inverse function of ϕ , is well defined since $\phi(0+) = 0$, ϕ is strictly increasing and $\phi(+\infty) = \infty$.

For $f \in \mathcal{S}(\mathbb{R}^d)$, we define $\phi(\Delta)f := -\phi(-\Delta)f$ as

$$\phi(\Delta)f(x) = \mathcal{F}^{-1}(-\phi(|\xi|^2)\mathcal{F}(f)(\xi))(x).$$

It turns out (e.g., [21, Theorem 31.5]) that $\phi(\Delta)$ is an integro-differential operator defined by

$$\phi(\Delta)f(x) = b\Delta f + \int_{\mathbb{R}^d} (f(x+y) - f(x) - \nabla f(x) \cdot y \mathbf{1}_{|y| \leq 1}) J(y) dy \tag{2.7}$$

where $J(x) = j(|x|)$ and $j : (0, \infty) \rightarrow (0, \infty)$ is given by

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt).$$

The non-local operator $\phi(\Delta)$ is related to a certain jump process as follows (see [25,42]). Let W_t be a d -dimensional Brownian motion independent of S_t , and denote $X_t := W_{S_t}$ (d -dimensional subordinate Brownian motion). Then, it holds that $\phi(\Delta)$ is the infinitesimal generator of X_t , that is,

$$\phi(\Delta)f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}f(x + X_t) - f(x)}{t},$$

where \mathbb{E} denotes the expectation. Furthermore, the solution to the equation

$$u_t = \phi(\Delta)u, \quad t > 0; \quad u(0, \cdot) = u_0$$

is given by $u(t, x) = \mathbb{E}u_0(x + X_t)$.

Now we introduce Sobolev spaces related to the operator $\phi(\Delta)$. For $p > 1$ and $\gamma \in \mathbb{R}$, we denote $H_p^{\phi, \gamma}$ by the closure of $\mathcal{S}(\mathbb{R}^d)$ under the norm (cf. [14])

$$\|u\|_{H_p^{\phi, \gamma}} := \|(1 - \phi(\Delta))^{\gamma/2}u\|_{L_p}.$$

Note that if $\phi(\lambda) = \lambda$, then $H_p^{\phi, \gamma}$ is the classical Bessel potential space H_p^γ . For any $u \in H_p^{\phi, \gamma}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, by (u, φ) we denote the value of linear functional u at φ , that is,

$$(u, \varphi) := \left((1 - \phi(\Delta))^{\gamma/2}u, (1 - \phi(\Delta))^{-\gamma/2}\varphi \right)_{L_2(\mathbb{R}^d)}.$$

For any $\gamma, \nu \in \mathbb{R}$ and $u \in H_p^{\phi, \gamma}$, we have $(1 - \phi(\Delta))^{\nu/2}u \in H_p^{\phi, \gamma - \nu}$, and furthermore

$$((1 - \phi(\Delta))^{\nu/2}u, \varphi) = (u, (1 - \phi(\Delta))^{\nu/2}\varphi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.8}$$

Let l_2 denote the set of all sequences $a = (a^1, a^2, \dots)$ such that

$$|a|_{l_2} := \left(\sum_{k=1}^{\infty} |a^k|^2 \right)^{1/2} < \infty.$$

By $H_p^{\phi, \gamma}(l_2) = H_p^{\phi, \gamma}(\mathbb{R}^d; l_2)$ we denote the class of all l_2 -valued tempered distributions $v = (v^1, v^2, \dots)$ on \mathbb{R}^d such that

$$\|v\|_{H_p^{\phi, \gamma}(l_2)} := \| (1 - \phi(\Delta))^{\gamma/2}v \|_{l_2} \|_{L_p} < \infty.$$

The following lemma gives basic properties of $H_p^{\phi, \gamma}$ and $H_p^{\phi, \gamma}(l_2)$.

- Lemma 2.1.** (i) For any $p > 1, \gamma \in \mathbb{R}$, $H_p^{\phi, \gamma}$ is a Banach space.
 (ii) For any $p > 1$ and $\mu, \gamma \in \mathbb{R}$, the map $(1 - \phi(\Delta))^{\mu/2}$ is an isometry from $H_p^{\phi, \gamma}$ to $H_p^{\phi, \gamma - \mu}$.
 (iii) If $p > 1$ and $\gamma_1 \leq \gamma_2$, then $H_p^{\phi, \gamma_2} \subset H_p^{\phi, \gamma_1}$, and there is a constant $C > 0$ independent of u such that

$$\|u\|_{H_p^{\phi, \gamma_1}} \leq C \|u\|_{H_p^{\phi, \gamma_2}}.$$

(iv) If $p > 1$ and $\gamma \geq 0$, then it holds that

$$\|u\|_{H_p^{\phi, \gamma}} \sim \left(\|u\|_{L_p} + \|\phi(\Delta)^{\gamma/2}u\|_{L_p} \right).$$

(v) The assertions in (i)–(iv) also hold true for the l_2 -valued function spaces $H_p^{\phi,\gamma}(l_2)$.

Proof. First, (i) and (ii) easily follow from the definition of $H_p^{\phi,\gamma}$. For (iii) and (iv), see Theorems 2.3.1 and 2.2.7 in [14], respectively. Here, we remark that the proofs are based on Lemma 1.5.6, Theorems 1.5.10 and 2.2.10 in [14], which can be proved for l_2 -valued spaces. \square

Remark 2.2. (i) Following [37, Remark 3], one can show that the embeddings $H_p^{2n} \subset H_p^{\phi,2n}$ and $H_p^{2n}(l_2) \subset H_p^{\phi,2n}(l_2)$ are continuous for any $n \in \mathbb{N}$. Therefore, using this and the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $H_p^{\gamma'}$ for any $\gamma' \in \mathbb{R}$, we deduce that $C_c^\infty(\mathbb{R}^d)$ is dense in $H_p^{\phi,\gamma}$ for all γ .

(ii) Let $\nu \in \mathbb{R}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then for any multi-index σ , $D^\sigma \varphi \in H_p^{\phi,\nu}$ by (i). Therefore, $(1 - \phi(\Delta))^{\nu/2} D^\sigma \varphi \in L_p$, and this implies $(1 - \phi(\Delta))^{\nu/2} \varphi \in H_p^{2n}$ for any $n \in \mathbb{N}$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\{\mathcal{F}_t, t \geq 0\}$ an increasing filtration of σ -fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all $(\mathcal{F}, \mathbb{P})$ -null sets. We assume that a family of independent one-dimensional Wiener processes $\{w_t^k\}_{k \in \mathbb{N}}$ relative to the filtration $\{\mathcal{F}_t, t \geq 0\}$ is given on Ω . By \mathcal{P} , we denote the predictable σ -field generated by \mathcal{F}_t , i.e., \mathcal{P} is the smallest σ -field containing every set $A \times (s, t]$, where $s < t$ and $A \in \mathcal{F}_s$.

Now we define stochastic Banach spaces for $p \geq 2$:

$$\begin{aligned} \mathbb{H}_p^{\phi,\gamma}(T) &:= L_p\left(\Omega \times [0, T], \mathcal{P}; H_p^{\phi,\gamma}\right), \quad \mathbb{L}_p(T) = \mathbb{H}_p^{\phi,0}(T), \\ \mathbb{H}_p^{\phi,\gamma}(T, l_2) &:= L_p\left(\Omega \times [0, T], \mathcal{P}; H_p^{\phi,\gamma}(l_2)\right), \quad \mathbb{L}_p(T, l_2) = \mathbb{H}_p^{\phi,0}(T, l_2). \end{aligned}$$

We write $g = (g^1, g^2, \dots) \in \mathbb{H}_0^\infty(T, l_2)$ if $g^k = 0$ for all sufficiently large k , and each g^k is of the type

$$g^k(t, x) = \sum_{i=1}^{n(k)} 1_{(\tau_{i-1}^k, \tau_i^k]}(t) g^{ik}(x), \quad g^{ik} \in C_c^\infty(\mathbb{R}^d),$$

where $0 \leq \tau_0^k \leq \tau_1^k \leq \dots \leq \tau_{n(k)}^k$ are bounded stopping times. One can check that the space $\mathbb{H}_0^\infty(T, l_2)$ is dense in $\mathbb{H}_p^{\phi,\gamma}(T, l_2)$ (see, e.g., [28, Theorem 3.10]).

Remark 2.3. Let $p \geq 2$, and q denote the conjugate of p . Then, by Minkowski and Hölder inequalities, for any $g \in \mathbb{H}_p^{\phi,\gamma}(T, l_2)$ and $\varphi \in H_q^{\phi,-\gamma}$,

$$\begin{aligned} &\mathbb{E} \left[\sum_{k=1}^\infty \int_0^T (g^k(s, \cdot), \varphi)^2 ds \right] \\ &= \mathbb{E} \left[\sum_{k=1}^\infty \int_0^T \left((1 - \phi(\Delta)^{\gamma/2}) g^k(s, \cdot), (1 - \phi(\Delta)^{-\gamma/2}) \varphi \right)_{L_2}^2 ds \right] \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{E} \int_0^T \|(1 - \phi(\Delta))^{\gamma/2} g\|_{L^2}^2 \|(1 - \phi(\Delta))^{-\gamma/2} \varphi\|_{L^q}^2 dt \\ &\leq C(T) \|\varphi\|_{H_q^{-\gamma, \phi}}^2 \|g\|_{\mathbb{H}_p^{\phi, \gamma}(T, l_2)}^2. \end{aligned} \tag{2.9}$$

Consequently, $(g, \varphi) \in L_2(\Omega \times [0, T], \mathcal{P}; l_2)$, and it also follows that the sequence of stochastic integral $\sum_{i=1}^n \int_0^t (g^k(s, \cdot), \varphi) dw_s^k$ converges in probability uniformly on $[0, T]$, and consequently the infinite series $\sum_{i=1}^\infty \int_0^t (g^k(s, \cdot), \varphi) dw_s^k$ becomes a continuous L_2 -martingale on $[0, T]$.

The following lemma will be used later for certain approximation arguments.

Lemma 2.4. (i) Let $v \geq 0$ and $h \in L_2(\Omega \times [0, T], \mathcal{P}; l_2)$. Then, the equality

$$I_t^v \left(\sum_{k=1}^\infty \int_0^\cdot h^k(s) dw_s^k \right) (t) = \sum_{k=1}^\infty \left(I_t^v \int_0^\cdot h^k(s) dw_s^k \right) (t)$$

holds for all $t \leq T$ (a.s.) and also in $L_2(\Omega \times [0, T])$.

(ii) Suppose $v \geq 0$ and $h_n \rightarrow h$ in $L_2(\Omega \times [0, T], \mathcal{P}; l_2)$ as $n \rightarrow \infty$. Then

$$\sum_{k=1}^\infty \left(I_t^v \int_0^\cdot h_n^k(s) dw_s^k \right) (t) \longrightarrow \sum_{k=1}^\infty \left(I_t^v \int_0^\cdot h^k(s) dw_s^k \right) (t)$$

in probability uniformly on $[0, T]$.

(iii) If $v < 1/2$ and $h \in L_2(\Omega \times [0, T], \mathcal{P}; l_2)$, then

$$\partial_t^v \left(\sum_{k=1}^\infty \int_0^\cdot h^k(s) dw_s^k \right) (t) = \frac{1}{\Gamma(1-v)} \sum_{k=1}^\infty \int_0^t (t-s)^{-v} h^k(s) dw_s^k$$

(a.e.) on $\Omega \times [0, T]$.

(iv) Let $0 < v < 1/2$ and $h_n \rightarrow h$ in $L_2(\Omega \times [0, T], \mathcal{P}; l_2)$ as $n \rightarrow \infty$. Then, there exists a subsequence n_j such that

$$\partial_t^v \left(\sum_{k=1}^\infty \int_0^\cdot h_{n_j}^k(s) dw_s^k \right) (t) \longrightarrow \partial_t^v \left(\sum_{k=1}^\infty \int_0^\cdot h^k(s) dw_s^k \right) (t)$$

(a.e.) on $\Omega \times [0, T]$.

Proof. See [7, Lemma 3.1, Lemma 3.3] for (i)–(iii). We prove (iv). Put $g_n = h_n - h$. Then, for each $t > 0$, by Burkholder–Davis–Gundy inequality,

$$\begin{aligned} \mathbb{E} \left| \sum_{k=1}^\infty \int_0^t (t-s)^{-v} g_n^k(s) dw_s^k \right|^2 &\leq \mathbb{E} \sup_{r \leq t} \left| \sum_{k=1}^\infty \int_0^r (t-s)^{-v} g_n^k(s) dw_s^k \right|^2 \\ &\leq c \mathbb{E} \int_0^t |t-s|^{-2v} |g_n(s)|_{l_2}^2 ds \end{aligned}$$

Therefore, we have the $L_2(\Omega \times [0, T])$ convergence, i.e.,

$$\mathbb{E} \int_0^T \left| \sum_{k=1}^{\infty} \int_0^t (t-s)^{-\nu} g_n^k(s) dw_s^k \right|^2 dt \leq c \mathbb{E} \int_0^T \int_0^t |t-s|^{-2\nu} |g_n(s)|_{l_2}^2 ds dt \rightarrow 0$$

as $n \rightarrow \infty$, and the claim easily follows. □

Now, we explain our sense of solutions.

Definition 2.5. Let $u \in \mathbb{H}_p^{\phi, \gamma_1}(T)$, $f \in \mathbb{H}_p^{\phi, \gamma_2}(T)$, and $g \in \mathbb{H}_p^{\phi, \gamma_3}(T, l_2)$ for some $\gamma_i \in \mathbb{R}$, $i = 1, 2, 3$. Then, we say u satisfies

$$\partial_t^\alpha u(t, x) = f(t, x) + \partial_t^\beta \sum_{k=1}^{\infty} \int_0^t g^k(s, x) dw_s^k, \quad t \in (0, T]; \quad u(0, \cdot) = 0 \quad (2.10)$$

in the sense of distributions if for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ the equality

$$(u(t), \varphi) = I_t^\alpha (f(t), \varphi) + \partial_t^{\beta-\alpha} \sum_{k=1}^{\infty} \int_0^t (g^k(s), \varphi) dw_s^k \quad (2.11)$$

holds ‘‘a.e. on $\Omega \times [0, T]$.’’ Here $\partial_t^{\beta-\alpha} := I_t^{\alpha-\beta}$ if $\beta \leq \alpha$.

Remark 2.6. (i) Due to Remark 2.3 and Lemma 2.4, the infinite series in (2.11) makes sense.

(ii) Let (2.10) hold with u, f, g as in Definition 2.5. Denote $\gamma = \min\{\gamma_1, \gamma_2, \gamma_3\}$. Then, (2.9), Lemma 2.4 and standard approximation argument show that (2.11) holds a.e. on $\Omega \times [0, T]$ for any $\varphi \in H_q^{\phi, -\gamma}$, where $q = p/(p-1)$. In particular, it holds a.e. for any $(1 - \phi(\Delta))^{\nu/2} \bar{\varphi}$, where $\nu \in \mathbb{R}$ and $\bar{\varphi} \in \mathcal{S}(\mathbb{R}^d)$.

Remark 2.7. Let u, f and g be given as in Definition 2.5. Fix $\nu \in \mathbb{R}$, and denote $\bar{\gamma}_i = \gamma_i - \nu$, $i = 1, 2, 3$. Also denote $\bar{u} = (1 - \phi(\Delta))^{\nu/2} u$, and define \bar{f} and \bar{g} in the same way. Then, by Lemma 2.1,

$$\bar{u} \in \mathbb{H}_p^{\phi, \bar{\gamma}_1}(T), \quad \bar{f} \in \mathbb{H}_p^{\phi, \bar{\gamma}_2}(T), \quad \bar{g} \in \mathbb{H}_p^{\phi, \bar{\gamma}_3}(T, l_2).$$

Moreover, due to Remark 2.6 (ii) and (2.8), it follows that (2.10) holds in the sense of distributions with $(\bar{u}, \bar{f}, \bar{g})$, in place of (u, f, g) .

In Definition 2.5, we only require (2.11) holds a.e. on $\Omega \times [0, T]$, not for all $t \leq T$ (a.s.). Below we give an equivalent statement which will clarify our notion of solutions.

Proposition 2.8. Let $u \in \mathbb{H}_p^{\phi, \gamma}(T)$, $f \in \mathbb{H}_p^{\phi, \gamma}(T)$, and $g \in \mathbb{H}_p^{\phi, \gamma}(T, l_2)$ for some $\gamma \in \mathbb{R}$. Then the following are equivalent.

- (i) u satisfies (2.10) in the sense of Definition 2.5 with f , and g .
- (ii) For any constant Λ satisfying

$$\Lambda \geq (\alpha \vee \beta) \text{ and } \Lambda > \frac{1}{p},$$

$I_t^{\Lambda-\alpha} u$ has an $H_p^{\phi,\gamma}$ -valued continuous version in $\mathbb{H}_p^{\phi,\gamma}(T)$, still denoted by $I_t^{\Lambda-\alpha}(u)$, such that for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(I_t^{\Lambda-\alpha} u(t), \varphi) = I_t^\Lambda (f(t, \cdot), \varphi) + \sum_{k=1}^\infty I_t^{\Lambda-\beta} \int_0^t (g^k(s, \cdot), \varphi) dw_s^k \quad (2.12)$$

holds “for all $t \in [0, T]$ (a.s.)”

(iii) The claim of (ii) holds for some Λ satisfying $\Lambda \geq (\alpha \vee \beta)$ and $\Lambda > \frac{1}{p}$.

Proof. Due to Remark 2.7, it suffices to prove only the case $\gamma = 0$. In this case we have $H_p^{\phi,0} = L_p$, and thus the lemma follows from [22, Proposition 2.13]. Actually [22, Proposition 2.13] does not include statement (ii) above. However, the proof “(ii) \rightarrow (i)” only utilizes the result of (iii). \square

Remark 2.9. If $\alpha = \beta = 1$, then we can take $\Lambda = 1$. Then, (2.12) reads as

$$(u(t), \varphi) = \int_0^t (f(s, \cdot), \varphi) ds + \sum_{k=1}^\infty \int_0^t (g^k(s, \cdot), \varphi) dw_s^k, \quad \forall t \leq T \text{ (a.s.)}$$

This might look wrong at first glance because in Definition 2.5 we only require this equality holds a.e. on $\Omega \times [0, T]$. The point of Proposition 2.8 is that u has a continuous version, still denoted by u , such that this equality holds for all $t \leq T$ (a.s.).

Proposition 2.10. *Let assumptions in Proposition 2.8 hold, and let u satisfy (2.10) in the sense of Definition 2.5 with f and g .*

(i) If $\Lambda \geq (\alpha \vee \beta)$ and $\Lambda > \frac{1}{p}$, then

$$\mathbb{E} \sup_{t \leq T} \|I_t^{\Lambda-\alpha}(u(t))\|_{H_p^{\phi,\gamma}}^p \leq C \left(\|f\|_{\mathbb{H}_p^{\phi,\gamma}(T)}^p + \|g\|_{\mathbb{H}_p^{\phi,\gamma}(T,l_2)}^p \right),$$

where $C = C(\Lambda, \alpha, \beta, T, p)$.

(ii) Let $\theta := \min\{1, \alpha, 2(\alpha - \beta) + 1\}$. Then, for any $t \leq T$,

$$\|u\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p \leq C \int_0^t (t-s)^{\theta-1} \left(\|f\|_{\mathbb{H}_p^{\phi,\gamma}(s)}^p + \|g\|_{\mathbb{H}_p^{\phi,\gamma}(s,l_2)}^p \right) ds, \quad (2.13)$$

where the constant C depends only $\alpha, \beta, d, p, \gamma, \phi, T$.

Proof. Again, we only need to consider the case $\gamma = 0$. Therefore, (i) and (ii) follow from [22, Proposition 2.13] and [26, Theorem 2.1 (iv)], respectively. \square

For $\alpha \in (0, 1)$, $\beta < \alpha + \frac{1}{2}$, and $\kappa \in (0, 1)$, define

$$c_0 := c_0(\alpha, \beta, \kappa) = \frac{(2\beta - 1)^+}{\alpha} + \kappa 1_{\beta=1/2}.$$

Below we use notation $f(u)$, and $g(u)$ to denote $f(\omega, t, x, u)$, and $g(\omega, t, x, u)$, respectively.

Assumption 2.11. (i) For any $u \in \mathbb{H}_p^{\phi, \gamma+2}(T)$,

$$f(u) \in \mathbb{H}_p^{\phi, \gamma}(T), \quad g(u) \in \mathbb{H}_p^{\phi, \gamma+c_0}(T, l_2).$$

(ii) For any $\varepsilon > 0$, there exists a constant $N = N(\varepsilon) > 0$ so that

$$\begin{aligned} & \|f(t, u) - f(t, v)\|_{H_p^{\phi, \gamma}} + \|g(t, u) - g(t, v)\|_{H_p^{\phi, \gamma+c_0}(l_2)} \\ & \leq \varepsilon \|u - v\|_{H_p^{\phi, \gamma+2}} + N \|u - v\|_{H_p^{\phi, \gamma}} \end{aligned}$$

for any ω, t , and $u, v \in H_p^{\phi, \gamma+2}$.

Here is our main result for SPDE driven by a family of independent one-dimensional Wiener processes. The proof of Theorem 2.12 is given in Sect. 4.

Theorem 2.12. *Let $\alpha \in (0, 1)$, $\beta < \alpha + 1/2$, $p \geq 2$, $\gamma \in \mathbb{R}$, and $T \in (0, \infty)$. Let Assumption 2.11 hold. Then, the equation*

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^\beta \sum_{k=1}^\infty \int_0^t g^k(u) dw_s^k, \quad t \in (0, T], \tag{2.14}$$

with $u(0, \cdot) = 0$ has a unique solution $u \in \mathbb{H}_p^{\phi, \gamma+2}(T)$ in the sense of distribution, and for this solution we have

$$\|u\|_{\mathbb{H}_p^{\phi, \gamma+2}(T)} \leq C \left(\|f(0)\|_{\mathbb{H}_p^{\phi, \gamma}(T)} + \|g(0)\|_{\mathbb{H}_p^{\phi, \gamma+c_0}(T, l_2)} \right), \tag{2.15}$$

where the constant C depends only on $\alpha, \beta, d, p, \phi, \gamma, \kappa$, and T .

Remark 2.13. Recall $c_0 = \frac{(2\beta-1)^+}{\alpha} + \kappa 1_{\beta=1/2}$. Note that the pair $(\gamma + 2, \gamma + c_0)$ determines the regularity relation between the solution u and forcing term $g(0)$. Here are some comments and details on c_0 :

- if $\beta > 1/2$, then $c_0 = \frac{(2\beta-1)}{\alpha}$. Hence, to have $H_p^{\phi, \gamma+2}$ -valued solution u , we require $g(0)$ to be an $H_p^{\phi, \gamma + \frac{(2\beta-1)}{\alpha}}(l_2)$ -valued process. This relation is optimal and can be easily proved using a scaling argument (see, e.g., [22, Remark 2.20]).
- if $\beta < 1/2$, then $c_0 = 0$. Thus, the stochastic forcing term $g(0)$ is not assumed to be smoother than the deterministic forcing term $f(0)$. Actually, if $\beta < 1/2$, we can transform equation (2.14) into a PDE by absorbing the stochastic term $\partial_t^\beta \sum_{k=1}^\infty \int_0^t g^k(u) dw_s^k$ into $f(u)$.
- if $\beta = 1/2$, we assume $c_0 > 0$. This is a technical assumption: we handle the case $\beta = 1/2$ based on the result for $\beta > 1/2$, and this approach yields extra regularity on $g(0)$.

Remark 2.14. Regarding the equations with nonzero initial values, the similar argument in [16, Sect. 4.4] yields that for the solution to

$$\partial_t^\alpha u = \phi(\Delta)u, \quad t > 0, x \in \mathbb{R}^d; \quad u(0, \cdot) = u_0,$$

we have

$$\|u\|_{\mathbb{H}_p^{\phi, \gamma+2}(T)} \leq C \|u_0\|_{L_p(\Omega, \mathcal{F}_0; B_p^{\phi, \gamma+2-2/\alpha p})},$$

where $B_p^{\phi, \gamma+2-2/\alpha p}$ denotes the Besov spaces related to ϕ (see, e.g., [27, Definition 2.3]). However, for the simplicity, we always assume $u(0) = 0$.

Remark 2.15. By letting $\alpha \rightarrow 1$ and taking $\beta = 1$ and $\phi(\lambda) = \lambda$, we (at least formally) get a classical result by Krylov [28, Theorem 5.2].

Next, we consider the semi-linear SPDE driven by space-time white noise:

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^{\beta-1} h(u) \dot{W}, \quad t > 0; \quad u(0, \cdot) = 0. \quad (2.16)$$

Here \dot{W} is a space-time white noise on $[0, \infty) \times \mathbb{R}^d$, and the functions f and h depend on (ω, t, x, u) .

First, to explain our sense of solutions, let us multiply by a test function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ to the equation, integrate over $[0, t) \times \mathbb{R}^d$, and (at least formally) get

$$(I_t^{1-\alpha} u(t, \cdot), \varphi) = \int_0^t (\phi(\Delta)u + f(u), \varphi) ds + \partial_t^{\beta-1} \int_0^t \int_{\mathbb{R}^d} h(u) \varphi \dot{W}(dx ds),$$

where Walsh’s stochastic integral against the space-time white noise is employed above. Applying $D_t^{1-\alpha}$ we further get

$$(u(t), \varphi) = I_t^\alpha (\phi(\Delta)u + f(u), \varphi) + \partial_t^{\beta-\alpha} \int_0^t \int_{\mathbb{R}^d} h(u) \varphi \dot{W}(dx ds). \quad (2.17)$$

Now, let $\{\eta^k : k = 1, 2, \dots\}$ be an orthogonal basis on $L_2(\mathbb{R}^d)$. Then (see [10, 28]) there exists a sequence of independent one-dimensional Wiener processes $\{w_t^k : k = 1, 2, \dots\}$ such that

$$\int_0^t \int_{\mathbb{R}^d} X(s, x) \dot{W}(dx ds) = \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} X(s, x) \eta^k(x) dx dw_s^k, \quad \forall t \text{ (a.s.)}$$

for any X of the type $X = \zeta(x)1_{(\tau, \sigma]}(t)$, where τ, σ are bounded stopping times and $\zeta \in C_c^\infty(\mathbb{R}^d)$. Thus (2.17) leads to the equation

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^\beta \sum_{k=1}^\infty \int_0^t h(u) \eta^k dw_t^k, \quad t > 0; \quad u(0, \cdot) = 0. \quad (2.18)$$

Definition 2.16. We say u is a solution to equation (2.16) in the sense of distributions if u satisfies equation (2.18) in the sense of Definition 2.5, that is, for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(u(t), \varphi) = I_t^\alpha (\phi(\Delta)u + f(u), \varphi) + \partial_t^{\beta-\alpha} \sum_{k=1}^\infty \int_0^t (h(u) \eta^k, \varphi) dw_s^k$$

holds “a.e. on $\Omega \times [0, T]$.” Here $\partial_t^{\beta-\alpha} := I_t^{\alpha-\beta}$ if $\beta \leq \alpha$.

Here comes our assumption on nonlinear terms $f(u)$ and $h(u)$ together with some restrictions on β and d . The argument ω is omitted as usual.

Assumption 2.17. (i) The functions f and h are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{d+1})$ -measurable.

(ii) For each ω, t, x, u and v ,

$$|f(t, x, u) - f(t, x, v)| \leq K|u - v|, \quad |h(t, x, u) - h(t, x, v)| \leq \xi(t, x)|u - v|,$$

where K is a constant and ξ is a function of (ω, t, x) .

(iii) $\delta_0 \in (1/4, 1], H_p^{\phi,1} \subset H_p^{\delta_0}$,

$$\beta < \left(1 - \frac{1}{4\delta_0}\right)\alpha + \frac{1}{2}, \quad d < 2\delta_0 \left(2 - \frac{(2\beta - 1)^+}{\alpha}\right) =: d_0. \quad (2.19)$$

Remark 2.18. Using the Fourier multiplier theorem, one can easily check that the embedding $H_p^{\phi,1} \subset H_p^{\delta_0}$ in Assumption 2.17 holds under condition (3.3).

Note that $d_0 \in (1, 4]$, and if $\beta < \alpha(1 - \frac{3}{4\delta_0}) + \frac{1}{2}$, then one can take $d = 1, 2, 3$. Recall

$$f(0) = f(t, x, 0), \quad h(0) = h(t, x, 0).$$

Here is our main result for SPDE driven by space-time white noise.

Theorem 2.19. *Suppose Assumption 2.17 holds, and*

$$\|f(0)\|_{\mathbb{H}_p^{\phi,-k_0-c_0}(T)} + \|h(0)\|_{\mathbb{L}_p(T)} + \sup_{\omega,t} \|\xi\|_{L^{2s}} < \infty,$$

where k_0 and s satisfy

$$\frac{d}{2\delta_0} < k_0 < \left(2 - \frac{(2\beta - 1)^+}{\alpha}\right) \wedge \frac{d}{\delta_0}, \quad \frac{d}{2k_0\delta_0 - d} < s. \quad (2.20)$$

Then, equation (2.16) has unique solution $u \in \mathbb{H}_p^{\phi,2-k_0-c_0}(T)$, and for this solution we have

$$\|u\|_{\mathbb{H}_p^{\phi,2-k_0-c_0}(T)} \leq C \left(\|f(0)\|_{\mathbb{H}_p^{\phi,-k_0-c_0}(T)} + \|h(0)\|_{\mathbb{L}_p(T)} \right),$$

where C is a constant independent of u .

Remark 2.20. (i) Due to the second relation in (2.19) one can always choose k_0 satisfying (2.20).

(ii) Note that the space for the solution is $\mathbb{H}_p^{\phi,2-k_0-c_0}(T)$, and the constant $2-k_0-c_0$ represents the regularity (or differentiability) of the solution with respect to the spatial variable. By the definition of c_0 , we have

$$0 < 2 - k_0 - c_0 < \begin{cases} 2 - \frac{d}{2\delta_0} - \frac{2\beta-1}{\alpha} & : \beta > \frac{1}{2} \\ 2 - \frac{d}{2\delta_0} & : \beta \leq \frac{1}{2}. \end{cases}$$

If ξ is bounded, then one can choose $r = 1$. Thus by taking k_0 sufficiently close to $d/(2\delta_0)$, one can make $2 - k_0 - c_0$ as close to the above upper bounds as one wishes.

3. A priori estimate for linear equation

In this section, we obtain a priori estimate for the solution to the linear equation

$$\partial_t^\alpha u = \phi(\Delta)u + \partial_t^\beta \sum_{k=1}^\infty \int_0^t g^k dw_s^k, \quad t > 0; \quad u(0, \cdot) = 0. \tag{3.1}$$

More precisely, we prove if $\frac{1}{2} < \beta < \alpha + \frac{1}{2}$, then

$$\mathbb{E} \|\phi(\Delta)^{1-(2\beta-1)/2\alpha} u\|_{L_p((0,\infty) \times \mathbb{R}^d)}^p \leq C \mathbb{E} \| |g|_{l_2} \|_{L_p((0,\infty) \times \mathbb{R}^d)}^p, \tag{3.2}$$

where C is a constant independent of u .

If $\alpha = \beta = 1$, then a version of (3.2) is obtained in [24]. In this case, the solution to (3.1) is given (at least formally) by the formula

$$u(t) = \sum_{k=1}^\infty \int_0^t S_{t-s} g^k(\cdot, s)(x) dw_s^k,$$

where $S_t : f \rightarrow e^{t\phi(\Delta)} f$, and (3.2) reads as

$$\left\| \sqrt{-\phi(\Delta)} \sum_{k=1}^\infty \int_0^t e^{(t-s)\phi(\Delta)} g^k(s) dw_s^k \right\|_{L_p(\Omega \times (0,\infty) \times \mathbb{R}^d)} \leq c \| |g|_{l_2} \|_{L_p(\Omega \times (0,\infty) \times \mathbb{R}^d)}.$$

We give two independent proofs of (3.2). One is based on Krylov’s analytic approach and the other is based on H^∞ -calculus. The first proof is much elementary, but it requires long calculus and some extra condition on ϕ .

3.1. Analytic approach

In this subsection, we impose the following assumption on ϕ ;

Assumption 3.1. ϕ is a Bernstein function for which there exist constants $\delta_0 \in (0, 1]$ and $\kappa_0 > 0$ such that

$$\kappa_0 \left(\frac{R}{r} \right)^{\delta_0} \leq \frac{\phi(R)}{\phi(r)}, \quad 0 < r < R < \infty. \tag{3.3}$$

By Assumption 3.1 and the concavity of ϕ , we have

$$\kappa_0 \left(\frac{R}{r} \right)^{\delta_0} \leq \frac{\phi(R)}{\phi(r)} \leq \frac{R}{r}, \quad 0 < r < R < \infty. \tag{3.4}$$

Note that we admit the case $\delta_0 = 1$, and we assume (3.3) for all $0 < r < R < \infty$. Here are some examples of Bernstein functions satisfying Assumption 3.1 (see, e.g., [42, Chapter 16] for more examples):

- (1) Stable subordinators : $\phi(\lambda) = \lambda^\beta$, $0 < \beta \leq 1$;

- (2) Sum of stable subordinators : $\phi(\lambda) = \lambda^{\beta_1} + \lambda^{\beta_2}$, $0 < \beta_1, \beta_2 \leq 1$;
- (3) Stable with logarithmic correction : $\phi(\lambda) = \lambda^\beta (\log(1 + \lambda))^\gamma$, $\beta \in (0, 1), \gamma \in (-\beta, 1 - \beta)$;
- (4) Relativistic stable subordinators : $\phi(\lambda) = (\lambda + m^{1/\beta})^\beta - m$, $\beta \in (0, 1), m > 0$;
- (5) Conjugate geometric stable subordinators : $\phi(\lambda) = \frac{\lambda}{\log(1 + \lambda^{\beta/2})}$, $\beta \in (0, 2)$.

Recall that $S = (S_t)_{t \geq 0}$ is a subordinator with Laplace exponent ϕ and $W = (W_t)_{t \geq 0}$ is a d -dimensional Brownian motion, independent of S . It is well known that the subordinate Brownian motion $X_t = W_{S_t}$ is a Lévy process in \mathbb{R}^d with characteristic exponent $\phi(|\xi|^2)$ (see, e.g., [2,21]), that is,

$$\mathbb{E}e^{-iX_t \cdot \xi} = e^{-t\phi(|\xi|^2)}, \quad \forall t > 0, \xi \in \mathbb{R}^d.$$

Here, by $p(t, x) = p_d(t, x)$, we denote the transition density of X_t .

Let Q_t be a subordinator, independent of X_t , having the Laplace transform

$$\mathbb{E} \exp(-\lambda Q_t) = \exp(-t\lambda^\alpha).$$

Such process exists since the function $\lambda \rightarrow \lambda^\alpha$ is a Bernstein function (see (2.5)). Let

$$R_t := \inf\{s > 0 : Q_s > t\}$$

be the inverse process of the subordinator Q_t , and let $\varphi(t, r)$ denote the probability density function of R_t . Then, it is known that (see [27, Lemma 5.1] or [5, Theorem 1.1]), the function

$$q(t, x) := \int_0^\infty p(r, x) d_r \mathbb{P}(R_t \leq r) = \int_0^\infty p(r, x) \varphi(t, r) dr$$

becomes the fundamental solution to equation

$$\partial_t^\alpha u = \phi(\Delta)u, \quad t > 0; \quad u(0, \cdot) = u_0.$$

That is, $q(t, x)$ is the function such that under appropriate smoothness condition on u_0 , the function $u(t, x) := (q(t, \cdot) * u_0(\cdot))(x)$ solves the above equation. Actually, the definition of $q(t, x)$ implies that $q(t, x)$ is the transition density of $Y_t := X_{R_t}$, which is called subordinate Brownian motion delayed by an inverse subordinator.

For $\beta \in \mathbb{R}$, denote

$$\varphi_{\alpha,\beta}(t, r) := D_t^{\beta-\alpha} \varphi(t, r) := (D_t^{\beta-\alpha} \varphi(\cdot, r))(t), \tag{3.5}$$

and for $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}$ define

$$q_{\alpha,\beta}(t, x) := \int_0^\infty p(r, x) \varphi_{\alpha,\beta}(t, r) dr,$$

and

$$q_{\alpha,\beta}^\gamma(t, x) := \int_0^\infty \phi(\Delta)^\gamma p(r, x) \varphi_{\alpha,\beta}(t, r) dr.$$

Below we collect some properties of $q_{\alpha,\beta}$ and $q_{\alpha,\beta}^\gamma$. The proof will be given in Appendix A.

Lemma 3.2. *Let $m \in \mathbb{N}_0$, $\alpha, \gamma \in (0, 1)$, and $\beta \in \mathbb{R}$.*

(i) $D_t^{\beta-\alpha} q(t, x)$ is well-defined for $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}$, and it holds that

$$D_t^{\beta-\alpha} q(t, x) = q_{\alpha, \beta}(t, x).$$

(ii) $D_x^m q_{\alpha, \beta}(t, x)$ is well defined for $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}$, and there exists $C = C(\alpha, \beta, d, \delta_0, \kappa_0, m)$ such that

$$|D_x^m q_{\alpha, \beta}(t, x)| \leq C t^{2\alpha-\beta} \frac{\phi(|x|^{-2})}{|x|^{d+m}}. \tag{3.6}$$

Additionally, if $t^\alpha \phi(|x|^{-2}) \geq 1$, then

$$|D_x^m q_{\alpha, \beta}(t, x)| \leq C t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} dr, \tag{3.7}$$

where ϕ^{-1} denotes the inverse of ϕ .

(iii) $D_x^m q_{\alpha, \beta}^\gamma(t, x)$ is well defined for $(t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}$, and there exists $C = C(\alpha, \beta, d, \delta_0, \kappa_0, \gamma, m)$ such that

$$|D_x^m q_{\alpha, \beta}^\gamma(t, x)| \leq C t^{\alpha-\beta} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}. \tag{3.8}$$

Additionally, if $t^\alpha \phi(|x|^{-2}) \geq 1$, then

$$|D_x^m q_{\alpha, \beta}^\gamma(t, x)| \leq C t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} r^{-\gamma} dr. \tag{3.9}$$

(iv) For any $t > 0$,

$$\int_{\mathbb{R}^d} |q_{\alpha, \beta}(t, x)| dx \leq C t^{\alpha-\beta}, \tag{3.10}$$

and

$$\int_{\mathbb{R}^d} |q_{\alpha, \beta}^\gamma(t, x)| dx \leq C t^{(1-\gamma)\alpha-\beta}, \tag{3.11}$$

where $C = C(\alpha, \beta, d, \delta_0, \kappa_0, \gamma)$.

(v) For any $t > 0$ and $\xi \in \mathbb{R}^d$,

$$\mathcal{F}_d(q_{\alpha, \beta}^\gamma)(t, \xi) = -t^{\alpha-\beta} \phi(|\xi|)^\gamma E_{\alpha, 1-\beta+\alpha}(-t^\alpha \phi(|\xi|^2)), \tag{3.12}$$

$$\mathcal{F}_d(q_{\alpha, \beta})(t, \xi) = t^{\alpha-\beta} E_{\alpha, 1-\beta+\alpha}(-t^\alpha \phi(|\xi|^2)), \tag{3.13}$$

where $E_{\alpha, \beta}$ is the two-parameter Mittag-Leffler function defined as

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \alpha > 0, \beta \in \mathbb{C}.$$

Next, we introduce the representation formula of the solution. For the rest of this section we assume

$$\alpha \in (0, 1), \quad 1/2 < \beta < \alpha + 1/2.$$

Lemma 3.3. *For given $g \in \mathbb{H}_0^\infty(T, l_2)$, define*

$$u(t, x) := \sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^d} q_{\alpha, \beta}(t-s, x-y) g^k(s, y) dy dw_s^k. \tag{3.14}$$

Then, $u \in \mathbb{H}_p^{\phi, 2}(T)$ and satisfies (3.1) in the sense of distributions (see Definition 2.5).

Proof. The proof is almost the same as that of [26, Lemma 4.2] which treats the case $\phi(\lambda) = \lambda$. The only difference is that we need to use (3.12), (3.13) and [27, lemma 4.1] in place of corresponding results when $\phi(\lambda) = \lambda$. □

Denote $c_1 := 2 - (2\beta - 1)/\alpha$ and

$$T_{t-s}^{\alpha, \beta} h(x) := \int_{\mathbb{R}^d} q_{\alpha, \beta}^{c_1/2}(t-s, x-y) h(y) dy, \quad h \in C_c^\infty(\mathbb{R}^d).$$

This is well defined due to Lemma 3.2 (iv). It is also easy to check (cf. Remark 2.2 (ii))

$$\begin{aligned} T_{t-s}^{\alpha, \beta} h(x) &= \int_{\mathbb{R}^d} q_{\alpha, \beta}(t-s, x-y) \phi(\Delta)^{c_1/2} h(y) dy \\ &= \phi(\Delta)^{c_1/2} \int_{\mathbb{R}^d} q_{\alpha, \beta}(t-s, x-y) h(y) dy. \end{aligned}$$

Take the solution u from (3.14). Then, by the Burkholder–Davis–Gundy inequality,

$$\begin{aligned} \|\phi(\Delta)^{c_1/2} u\|_{L_p(T)}^p &\leq C(p) \mathbb{E} \int_{\mathbb{R}^d} \int_0^T \left(\int_0^t \sum_{k=1}^\infty |T_{t-s}^{\alpha, \beta} g^k(s)(x)|^2 ds \right)^{p/2} dt dx \\ &= C(p) \mathbb{E} \left\| \left(\int_0^t |T_{t-s}^{\alpha, \beta} g(s)(x)|_H^2 ds \right)^{1/2} \right\|_{L_p((0, T) \times \mathbb{R}^d)}^p. \end{aligned} \tag{3.15}$$

Now we estimate the right hand side of (3.15) in terms of $\|g\|_{L_p(T, l_2)}$ under a slightly general setting. Let H be a Hilbert space. For functions $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$, we define the operator \mathcal{T} as

$$\mathcal{T}g(t, x) := \left[\int_{-\infty}^t \left| T_{t-s}^{\alpha, \beta} g(s, \cdot)(x) \right|_H^2 ds \right]^{1/2},$$

where $|\cdot|_H$ denotes the given norm in H . Note that \mathcal{T} is sublinear since the Minkowski inequality yields

$$\|f + g\|_{L_2((-\infty, t); H)} \leq \|f\|_{L_2((-\infty, t); H)} + \|g\|_{L_2((-\infty, t); H)}.$$

Note that (3.2) is a consequence of (3.15) and Theorem 3.4.

Theorem 3.4. *Let H be a separable Hilbert space, and $T \in (-\infty, \infty]$. Then, for any $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$,*

$$\int_{\mathbb{R}^d} \int_{-\infty}^T |\mathcal{T}g(t, x)|^p dt dx \leq C \int_{\mathbb{R}^d} \int_{-\infty}^T |g(t, x)|_H^p dt dx, \tag{3.16}$$

where $C = C(\alpha, \beta, d, p, \delta_0, \kappa_0)$. Consequently, the operator \mathcal{T} is continuously extended to $L_p(\mathbb{R}^{d+1}; H)$.

The proof of the theorem is given later after some preparations. The main strategy is as follows.

1. First, we control the sharp function of $\mathcal{T}g$ in terms of maximal function of g (the definitions of the sharp and maximal functions are given below), that is, we prove

$$(\mathcal{T}g)^\sharp(t, x) \leq C(\mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x))^{1/2}, \quad \forall (t, x) \text{ uniformly on } \Omega.$$

2. Then, we apply Fefferman–Stein inequality and Hardy–Littlewood maximal inequality to obtain (3.16).

Recall that $g = (g^1, g^2, \dots) \in \mathbb{H}_0^\infty(T, l_2)$ if $g^k = 0$ for all sufficiently large k , and each g^k is of the type

$$g^k(t, x) = \sum_{i=1}^n 1_{(\tau_{i-1}, \tau_i]}(t) g^{ik}(x),$$

where $0 \leq \tau_0 \leq \tau_1 \leq \dots \leq \tau_n$ are bounded stopping times, and $g^{ik} \in C_c^\infty(\mathbb{R}^d)$.

The following result, Lemma 3.5, is a version of Theorem 3.4 for $p = 2$. For the proof, we use the following fact on the Mittag–Leffler function: if $\alpha \in (0, 1)$ and $b \in \mathbb{R}$, then there exist positive constants $\varepsilon = \varepsilon(\alpha)$ and $C = C(\alpha, b)$ such that

$$|E_{\alpha, b}(z)| \leq C(1 \wedge |z|^{-1}) \quad \text{if } \pi - \varepsilon \leq |\arg(z)| \leq \pi \tag{3.17}$$

(see, e.g., [40, Theorem 1.6]).

Lemma 3.5. *For any $T \in (-\infty, \infty]$ and $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$,*

$$\int_{\mathbb{R}^d} \int_{-\infty}^T |\mathcal{T}g(t, x)|^2 dt dx \leq C \int_{\mathbb{R}^d} \int_{-\infty}^T |g(t, x)|_H^2 dt dx, \tag{3.18}$$

where $C = C(\alpha, \beta, d)$ is independent of T .

Proof. We follow the proof of [26, Lemma 3.5].

Step 1. First, assume $g(t, x) = 0$ for $t \leq 0$. In this case we may further assume $T > 0$ since the left hand side of (3.18) is zero if $T \leq 0$. Since $g(t, x) = \mathcal{T}g(t, x) = 0$ for $t \leq 0$, by Parseval’s identity,

$$\int_{\mathbb{R}^d} \int_{-\infty}^T |\mathcal{T}g(t, x)|^2 dt dx$$

$$\begin{aligned}
 &= \int_0^T \int_0^t \int_{\mathbb{R}^d} \phi(|\xi|^2)^{c_1} |\hat{q}_{\alpha,\beta}(t-s, \cdot)(\xi)|^2 |\hat{g}(s, \xi)|_H^2 d\xi ds dt \\
 &= \int_{\mathbb{R}^d} \int_0^T \int_s^T \phi(|\xi|^2)^{c_1} |\hat{q}_{\alpha,\beta}(t-s, \cdot)(\xi)|^2 |\hat{g}(s, \xi)|_H^2 dt ds d\xi \\
 &= \int_{\mathbb{R}^d} \int_0^T \int_0^{T-s} \phi(|\xi|^2)^{c_1} |\hat{q}_{\alpha,\beta}(t, \cdot)(\xi)|^2 |\hat{g}(s, \xi)|_H^2 dt ds d\xi. \tag{3.19}
 \end{aligned}$$

By Lemma 3.2 (v) and (3.17) (recall $\beta > 1/2$), for $0 < s < T$,

$$\begin{aligned}
 &\int_0^{T-s} \phi(|\xi|^2)^{c_1} |\hat{q}_{\alpha,\beta}(t, \cdot)(\xi)|^2 dt \\
 &\leq \phi(|\xi|^2)^{c_1} \int_0^T \left| t^{\alpha-\beta} E_{\alpha, 1-\beta+\alpha}(-\phi(|\xi|^2)t^\alpha) \right|^2 dt \\
 &\leq C\phi(|\xi|^2)^{c_1} \int_0^{\phi(|\xi|^2)^{-\frac{1}{\alpha}}} t^{2(\alpha-\beta)} dt + C1_A(\xi)\phi(|\xi|^2)^{c_1} \int_{\phi(|\xi|^2)^{-\frac{1}{\alpha}}}^T \left| \frac{t^{\alpha-\beta}}{\phi(|\xi|^2)t^\alpha} \right|^2 dt \\
 &\leq C\phi(|\xi|^2)^{(c_1-2+\frac{2\beta-1}{\alpha})} + C\phi(|\xi|^2)^{c_1-2}\phi(|\xi|^2)^{(\frac{2\beta-1}{\alpha})} \leq C,
 \end{aligned}$$

where $A = \{\xi : \phi(|\xi|^2) \geq T^{-\alpha}\}$. Thus, (3.19) and Parseval’s identity yield

$$\int_{\mathbb{R}^d} \int_{-\infty}^T |\mathcal{T}g(t, x)|^2 dt dx \leq C \int_0^T \int_{\mathbb{R}^d} |g(t, x)|_H^2 dx dt,$$

and (3.18) holds for all $T > 0$ with a constant independent of T . It follows that (3.18) also holds for $T = \infty$.

Step 2. General case. Take $a \in \mathbb{R}$ so that $g(t, x) = 0$ for $t \leq a$. Then obviously, for $\bar{g}(t, x) := g(t + a, x)$ we have $\bar{g}(t) = 0$ for $t \leq 0$. Also note that

$$\begin{aligned}
 (\mathcal{T}g(t+a))^2 &= \left(\int_{-\infty}^{t+a} \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c_1/2}(t+a-s, x-y)g(s, y)dy \right|_H^2 ds \right) \\
 &= \left(\int_{-\infty}^t \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c_1/2}(t-s, x-y)\bar{g}(s, y)dy \right|_H^2 ds \right) \\
 &= (\mathcal{T}\bar{g}(t))^2.
 \end{aligned}$$

Thus, it is enough to apply the result of Step 1 with \bar{g} and $T - a$ in place of g and T , respectively. The lemma is proved. \square

For $b > 0$ and $(t, x) \in \mathbb{R}^{d+1}$, we define

$$\kappa(b) := \left(\phi(b^{-2}) \right)^{-1/\alpha}, \quad B_b(x) = \{y \in \mathbb{R}^d : |y - x| < b\},$$

and

$$I_b(t) = (t - \kappa(b), t), \quad Q_b(t, x) = I_b(t) \times B_b(x).$$

We also denote

$$I_b = I_b(0), \quad B_b = B_b(0), \quad Q_b = Q_b(0, 0).$$

For measurable functions $h(t, x)$ on \mathbb{R}^{d+1} , we define the sharp function

$$h^\#(t, x) = \sup_{(t,x) \in Q} \int_Q |h(r, z) - h_Q| \, dr dz,$$

where

$$h_Q = \int_Q h(s, y) dy ds,$$

and the supremum is taken over all $Q \subset \mathbb{R}^{d+1}$ of the form Q_b containing (t, x) .

For functions h on \mathbb{R}^d , we define the maximal function

$$\mathbb{M}_x h(x) := \sup_{x \in B_r(z)} \frac{1}{|B_r(z)|} \int_{B_r(x)} |h(y)| dy = \sup_{x \in B_r(z)} \int_{B_r(z)} |h(y)| dy.$$

We also use the notation $\mathbb{M}_t h(t)$ when $d = 1$ for functions depending on t . For measurable functions $h(t, x)$ set

$$\mathbb{M}_x h(t, x) = \mathbb{M}_x (h(t, \cdot)) (x), \quad \mathbb{M}_t h(t, x) = \mathbb{M}_t (h(\cdot, x)) (t),$$

and

$$\mathbb{M}_t \mathbb{M}_x h(t, x) = \mathbb{M}_t (\mathbb{M}_x h(\cdot, x)) (t).$$

Below we record some useful computations which are often used in the rest of this section.

Lemma 3.6. (i) *Let f be a nonnegative integrable function on \mathbb{R} . Assume there exists $a > 0$ such that $f(t) = 0$ if $t > -2a$. Then, for any $v > 1$ and $t \in (-a, 0)$,*

$$\begin{aligned} \int_{-\infty}^{-2a} \int_{-a-r}^{-r} f(r) s^{-v} ds dr &= \int_{-\infty}^{-2a} \int_{-a}^0 f(r) (s - r)^{-v} ds dr \\ &\leq C(v) a^{2-v} \mathbb{M}_t f(t). \end{aligned} \tag{3.20}$$

(ii) *For positive real numbers v, θ and r , define*

$$G_{v,\theta}(\rho) := \frac{\phi(\rho^{-2})^v}{\rho^\theta}, \quad \rho > 0 \tag{3.21}$$

and

$$H_{v,\theta}(r, \rho) := \int_{(\phi(\rho^{-2}))^{-1}}^{2r^\alpha} (\phi^{-1}(l^{-1}))^{\theta/2} l^{-v} dl, \quad r^\alpha \phi(\rho^{-2}) \geq 1. \tag{3.22}$$

Then, we have

$$\left| \frac{d}{d\rho} G_{v,\theta}(\rho) \right| \leq C(v, \theta) G_{v,\theta+1}(\rho), \quad \rho > 0, \tag{3.23}$$

$$\left| \frac{d}{d\rho} H_{v,\theta}(r, \rho) \right| \leq C(\alpha, \beta, d, \delta_0, \kappa_0, v, \theta) H_{v,\theta+1}(r, \rho), \quad r^\alpha \phi(\rho^{-2}) \geq 1. \tag{3.24}$$

Proof. (i) Using integration by parts, we get

$$\begin{aligned} & \int_{-\infty}^{-2a} \int_{-a}^0 f(r)(s-r)^{-v} ds dr \\ &= \frac{1}{v-1} \int_{-\infty}^{-2a} f(r) \left((-a-r)^{1-v} - (-r)^{1-v} \right) dr \\ &\leq C(v) \int_{-\infty}^{-2a} \left(\int_r^0 f(\tilde{r}) d\tilde{r} \right) \left((-a-r)^{-v} - (-r)^{-v} \right) dr. \end{aligned}$$

Since $0 < p < q$ implies $p^{-v} - q^{-v} \leq v(q-p)p^{-v-1}$, we have for $r < -2a < 0$,

$$(-a-r)^{-v} - (-r)^{-v} \leq va(-a-r)^{-v-1}.$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{-2a} \int_{-a}^0 f(r)(s-r)^{-v} ds dr &\leq C(v)a \int_{-\infty}^{-2a} \left(\int_r^0 f(\tilde{r}) d\tilde{r} \right) (-a-r)^{-v-1} dr \\ &\leq C(v)a \mathbb{M}_t f(t) \int_{-\infty}^{-2a} -r(-r/2)^{-v-1} dr \\ &\leq C(v)a^{2-v} \mathbb{M}_t f(t). \end{aligned}$$

(ii) By (2.6), we have

$$\begin{aligned} \left| \frac{d}{d\rho} G_{v,\theta}(\rho) \right| &= \left| -\theta \frac{\phi(\rho^{-2})^v}{\rho^{\theta+1}} - 2v \frac{\phi(\rho^{-2})^{v-1}}{\rho^\theta} \phi'(\rho^{-2}) \rho^{-3} \right| \\ &\leq C(v, \theta) \frac{\phi(\rho^{-2})^v}{\rho^{\theta+1}} = C(v, \theta) G_{v,\theta+1}(\rho). \end{aligned}$$

Thus, (3.23) is proved. Now we prove (3.24). For $0 < v < r^\alpha$, define

$$\mathcal{H}_{v,\theta}(r, v) = \int_v^{2r^\alpha} (\phi^{-1}(l^{-1}))^{\theta/2} l^{-v} dl.$$

Applying the fundamental theorem of calculus and using (2.6),

$$\begin{aligned} \left| \frac{d}{d\rho} H_{v,\theta}(r, \rho) \right| &= C \left| \frac{d}{dv} \mathcal{H}_{v,\theta}(r, v) \Big|_{v=(\phi(\rho^{-2}))^{-1}} (\phi(\rho^{-2}))^{-2} \phi'(\rho^{-2}) \rho^{-3} \right| \\ &\leq C \frac{\phi(\rho^{-2})^v}{\rho^\theta} (\phi(\rho^{-2}))^{-1} \rho^{-1} = C \frac{\phi(\rho^{-2})^{v-1}}{\rho^{\theta+1}}. \end{aligned}$$

By (3.3) with $R = \phi^{-1}(2l^{-1})$ and $r = \rho^{-2}$,

$$\begin{aligned} \frac{\phi(\rho^{-2})^{\nu-1}}{\rho^{\theta+1}} &= \int_{(\phi(\rho^{-2}))^{-1}}^{2(\phi(\rho^{-2}))^{-1}} \frac{\phi(\rho^{-2})^\nu}{\rho^{\theta+1}} dl \\ &\leq C \int_{(\phi(\rho^{-2}))^{-1}}^{2(\phi(\rho^{-2}))^{-1}} (\phi^{-1}(l^{-1}))^{(\theta+1)/2} l^{-\nu} dr \\ &\leq C \int_{(\phi(\rho^{-2}))^{-1}}^{2r^\alpha} (\phi^{-1}(l^{-1}))^{(\theta+1)/2} l^{-\nu} dl \\ &= CH_{\nu, \theta+1}(r, \rho). \end{aligned}$$

Therefore, we have (3.24), and the lemma is proved. □

We will also frequently use the following version of integration by parts formula: if F and G are smooth enough, then for any $0 < \varepsilon < R < \infty$,

$$\begin{aligned} \int_{\varepsilon \leq |z| \leq R} F(z)G(|z|)dz &= - \int_\varepsilon^R G'(\rho) \left[\int_{|z| \leq \rho} F(z)dz \right] d\rho \\ &\quad + G(R) \int_{|z| \leq R} F(z)dz - G(\varepsilon) \int_{|z| \leq \varepsilon} F(z)dz. \end{aligned} \tag{3.25}$$

This is easily obtained using the relations

$$\begin{aligned} \int_{\varepsilon \leq |z| \leq R} F(z)G(|z|) dz &= \int_\varepsilon^R G(\rho) \left(\int_{\partial B_\rho(0)} F(s) dS_\rho \right) d\rho \\ &= \int_\varepsilon^R G(\rho) \frac{d}{d\rho} \left(\int_{B_\rho(0)} F(z) dz \right) d\rho. \end{aligned}$$

and applying standard integration by parts formula to the last term above.

In the following lemmas, Lemmas 3.7–3.11, we estimate the mean oscillation of $\mathcal{T}g$ on Q_b . For this, we consider the following two cases

- g has support in $(-3\kappa(b), \infty) \times \mathbb{R}^d$ (see Lemma 3.8),
- g has support in $(-\infty, -2\kappa(b)) \times \mathbb{R}^d$.

The second case above is further divided into the cases

- g has support in $(-\infty, -2\kappa(b)) \times B_{3b}$ (see Lemma 3.9),
- g has support in $(-\infty, -2\kappa(b)) \times B_{2b}^c$ (see Lemmas 3.10 and 3.11).

Note that, by Jensen’s inequality, for any $c \in \mathbb{R}$ and function h ,

$$\begin{aligned} \left(\int_Q |h(r, z) - h_Q| dr dz \right)^2 &\leq \int_Q |h(r, z) - h_Q|^2 dr dz \\ &= \int_Q \left| \int_Q (h(r, z) - h(s, y)) ds dy \right|^2 dr dz \end{aligned} \tag{3.26}$$

$$\leq 4 \int_Q |h(r, z) - c|^2 dr dz. \tag{3.27}$$

We will consider type (3.26) and type (3.27) below.

Lemma 3.7. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ have a support in $(-3\kappa(b), 3\kappa(b)) \times B_{3b}$. Then for any $(t, x) \in Q_b$,*

$$\int_{Q_b} |\mathcal{T}g(s, y)|^2 ds dy \leq C(\alpha, \beta, d) \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x),$$

where $C = C(\alpha, \beta, d)$.

Proof. By Lemma 3.5,

$$\begin{aligned} \int_{Q_b} |\mathcal{T}g(s, y)|^2 ds dy &\leq C \int_{(-3\kappa(b), 3\kappa(b)) \times B_{3b}} |g(s, y)|_H^2 dy ds \\ &\leq C b^d \int_{-3\kappa(b)}^{3\kappa(b)} \mathbb{M}_x |g|_H^2(s, x) ds \\ &\leq C \kappa(b) b^d \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \end{aligned}$$

This certainly proves the lemma. □

Lemma 3.8. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ have a support in $(-3\kappa(b), \infty) \times \mathbb{R}^d$. Then for any $(t, x) \in Q_b$,*

$$\int_{Q_b} |\mathcal{T}g(s, y)|^2 ds dy \leq C \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x),$$

where $C = C(\alpha, \beta, d, \delta_0, \kappa_0)$.

Proof. Take $\zeta_0 = \zeta_0(t) \in C^\infty(\mathbb{R})$ such that $0 \leq \zeta_0 \leq 1$, $\zeta_0(t) = 1$ for $t \leq 2\kappa(b)$, and $\zeta_0(t) = 0$ for $t \geq 5\kappa(b)/2$. Note that $\mathcal{T}g = \mathcal{T}(g\zeta_0)$ on Q_b , and $|g\zeta_0| \leq |g|$ leads to

$$\mathbb{M}_t \mathbb{M}_x |g\zeta_0|_H^2(t, x) \leq \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x).$$

Therefore, it is enough to assume $f(t, x) = 0$ if $|t| \geq 3\kappa(b)$. Take $\zeta \in C_c^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ in B_{2b} and $\zeta = 0$ outside $B_{5b/2}$. Recall that \mathcal{T} is a sublinear operator, and therefore

$$\mathcal{T}g \leq \mathcal{T}(\zeta g) + \mathcal{T}((1 - \zeta)g).$$

Since $\mathcal{T}(\zeta g)$ can be estimated by Lemma 3.7, we may assume that $g(t, x) = 0$ if $x \in B_{2b}$.

Observe that if $(s, y) \in Q_b$ and $\rho > b$, then

$$|x - y| \leq 2b, \quad B_\rho(y) \subset B_{2b+\rho}(x) \subset B_{3\rho}(x), \tag{3.28}$$

whereas if $\rho \leq b$, then for $z \in B_\rho$, $|y - z| \leq 2b$ and thus $g(r, y - z) = 0$. Hence, by (3.8), (3.25) and (3.28), for $s > r$,

$$\left| \int_{\mathbb{R}^d} q_{\alpha, \beta}^{c_1/2}(s - r, z) g(r, y - z) dz \right|_H \leq \int_{|z| \geq b} |q_{\alpha, \beta}^{c_1/2}(s - r, z)| |g(r, y - z)|_H dz$$

$$\begin{aligned} &\leq \int_b^\infty (s-r)^{\alpha-\beta} \frac{d}{d\rho} G_{c_1/2,d}(\rho) \left[\int_{|z|\leq\rho} |g(r, y-z)|_H dz \right] d\rho \\ &\leq C \int_b^\infty (s-r)^{\alpha-\beta} \frac{d}{d\rho} G_{c_1/2,d}(\rho) \left[\int_{B_{3\rho}(x)} |g(r, z)|_H dz \right] d\rho, \end{aligned}$$

where $G_{c_1/2,d}(\rho)$ is taken from (3.21). Therefore, by (3.23),

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c_1/2}(s-r, z)g(r, y-z)dz \right|_H \\ &\leq C \int_b^\infty (s-r)^{\alpha-\beta} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho^{d+1}} \left[\int_{B_{3\rho}(x)} |g(r, z)|_H dz \right] d\rho \\ &\leq C \int_b^\infty (s-r)^{\alpha-\beta} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho} \mathbb{M}_x |g|_H(r, x) d\rho \\ &\leq C(s-r)^{\alpha-\beta} \phi(b^{-2})^{c_1/2} \mathbb{M}_x |g|_H(r, x). \end{aligned}$$

For the last inequality we use (A.1) with $\phi^{c_1/2}$. Since $(\mathbb{M}_x |g|_H)^2 \leq \mathbb{M}_x |g|_H^2$, we have

$$\begin{aligned} \int_{Q_b} |\mathcal{T}g(s, y)|^2 dsdy &= \int_{Q_b} \int_{-3\kappa(b)}^s \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c_1/2}(s-r, z)g(r, y-z)dz \right|_H^2 drdsdy \\ &\leq C \int_{Q_b} \int_{-3\kappa(b)}^s \left[(s-r)^{2(\alpha-\beta)} \phi(b^{-2})^{c_1} \mathbb{M}_x |g|_H^2(r, x) \right] drdsdy \\ &\leq C\phi(b^{-2})^{c_1} \int_{B_b} \int_{-3\kappa(b)}^0 \left(\int_r^0 (s-r)^{2(\alpha-\beta)} ds \right) \mathbb{M}_x |g|_H^2(r, x) drdy \\ &\leq C\phi(b^{-2})^{c_1} \int_{B_b} \int_{-3\kappa(b)}^0 (-r)^{\alpha c_1} \mathbb{M}_x |g|_H^2(r, x) drdy \\ &\leq C(\kappa(b))^{\alpha c_1} \phi(b^{-2})^{c_1} \\ &\quad \times \int_{B_b} \int_{-3\kappa(b)}^0 \mathbb{M}_x |g|_H^2(r, x) drdy \leq C\kappa(b)b^d \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \end{aligned}$$

The lemma is proved. □

Lemma 3.9. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ have a support in $(-\infty, -2\kappa(b)) \times B_{3b}$. Then for any $(t, x) \in Q_b$,*

$$\int_{Q_b} |\mathcal{T}g(s, y)|^2 dsdy \leq C\mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x),$$

where $C = C(\alpha, \beta, d, c, \delta_0, \kappa_0)$.

Proof. By definition of $\mathcal{T}g$ and Fubini's theorem,

$$\int_{Q_b} |\mathcal{T}g(s, y)|^2 dsdy$$

$$\leq \int_{-\kappa(b)}^0 \int_{-\infty}^{-2\kappa(b)} \int_{B_b} \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c1/2}(s-r,z) g(r,y-z) dz \right|_H^2 dy dr ds.$$

By assumption, for any $y \in B_b$, the function $g(r, y - \cdot)$ vanishes on B_{4b}^c . Therefore, by Minkowski’s inequality, for $s > r$,

$$\begin{aligned} & \int_{B_b} \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c1/2}(s-r,z) g(r,y-z) dz \right|_H^2 dy \\ & \leq \int_{B_b} \left| \int_{B_{4b}} |q_{\alpha,\beta}^{c1/2}(s-r,z)| |g(r,y-z)|_H dz \right|_H^2 dy \\ & \leq \left(\int_{B_{4b}} \left[\int_{B_b} |g(r,y-z)|_H^2 dy \right]^{1/2} |q_{\alpha,\beta}^{c1/2}(s-r,z)| dz \right)^2 \\ & \leq \left(\int_{B_{4b}} \left[\int_{B_{5b}} |g(r,y)|_H^2 dy \right]^{1/2} |q_{\alpha,\beta}^{c1/2}(s-r,z)| dz \right)^2 \\ & \leq Cb^d \mathbb{M}_x |g|_H^2(r,x) \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c1/2}(s-r,z)| dz \right)^2. \end{aligned} \tag{3.29}$$

Therefore, by (3.29), we have

$$\begin{aligned} & \int_{Q_b} |Tg(s,y)|^2 ds dy \\ & \leq Cb^d \int_{-\kappa(b)}^0 \int_{-\infty}^{-2\kappa(b)} \mathbb{M}_x |g|_H^2(r,x) \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c1/2}(s-r,z)| dz \right)^2 dr ds \\ & = Cb^d \int_{-\infty}^{-2\kappa(b)} \int_{-\kappa(b)}^0 \mathbb{M}_x |g|_H^2(r,x) \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c1/2}(s-r,z)| dz \right)^2 ds dr \\ & = Cb^d \int_{-\infty}^{-2\kappa(b)} \mathbb{M}_x |g|_H^2(r,x) \left[\int_{-\kappa(b)-r}^{-r} \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c1/2}(s,z)| dz \right)^2 ds \right] dr \\ & = Cb^d \left(\int_{-m\kappa(b)}^{-2\kappa(b)} \dots dr + \int_{-\infty}^{-m\kappa(b)} \dots dr \right) =: Cb^d (I(b) + II(b)), \end{aligned} \tag{3.30}$$

where m is any fixed integer such that $m \geq 3$ and $(m - 1)\kappa(b) > \kappa(4b)$. Such integer m exists due to (3.4).

Obviously, to finish the proof of the lemma, it suffices to prove

$$I(b) + II(b) \leq C\kappa(b) \mathbb{M}_r \mathbb{M}_x |g|_H^2(t,x). \tag{3.31}$$

To prove this, we first consider the integral inside the square brackets in (3.30). If $-m\kappa(b) < r < -2\kappa(b)$, then by (3.11),

$$[] := \int_{-\kappa(b)-r}^{-r} \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c1/2}(s,z)| dz \right)^2 ds$$

$$\begin{aligned} &\leq \int_{\kappa(b)}^{m\kappa(b)} \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c_1/2}(s, z)| dz \right)^2 ds \\ &\leq \int_{\kappa(b)}^{m\kappa(b)} \left(\int_{\mathbb{R}^d} |q_{\alpha,\beta}^{c_1/2}(s, z)| dz \right)^2 ds \\ &\leq C \int_{\kappa(b)}^{m\kappa(b)} s^{-1} ds = C \ln m \leq C. \end{aligned}$$

Therefore,

$$I(b) \leq C \int_{-m\kappa(b)}^{-2\kappa(b)} \mathbb{M}_x |g|_H^2(r, x) dr \leq C\kappa(b) \mathbb{M}_r \mathbb{M}_x |g|_H^2(t, x). \tag{3.32}$$

Next, we estimate $II(b)$. If $r \leq -m\kappa(b)$, then $-\kappa(b) - r > (m - 1)\kappa(b) > \kappa(4b)$. Therefore, by (3.9),

$$\begin{aligned} [] &:= \int_{-\kappa(b)-r}^{-r} \left(\int_{B_{4b}} |q_{\alpha,\beta}^{c_1/2}(s, z)| dz \right)^2 ds \\ &\leq C \int_{-\kappa(b)-r}^{-r} \left(s^{-\beta} \int_{B_{4b}} \int_{(\phi(|z|^{-2}))^{-1}}^{2s^\alpha} (\phi^{-1}(l^{-1}))^{d/2} l^{-c_1/2} dl dz \right)^2 ds. \end{aligned} \tag{3.33}$$

By Fubini’s theorem, for $s > \kappa(4b)$ (equivalently, $2s^\alpha > (\phi(b^{-2}/16))^{-1}$),

$$\begin{aligned} &s^{-\beta} \int_{B_{4b}} \int_{(\phi(|z|^{-2}))^{-1}}^{2s^\alpha} (\phi^{-1}(l^{-1}))^{d/2} l^{-c_1/2} dl dz \\ &= s^{-\beta} \int_0^{(\phi(b^{-2}/16))^{-1}} \int_{|z| \leq (\phi^{-1}(l^{-1}))^{-1/2}} (\phi^{-1}(l^{-1}))^{d/2} l^{-c_1/2} dz dl \\ &\quad + s^{-\beta} \int_{(\phi(b^{-2}/16))^{-1}}^{2s^\alpha} \int_{B_{4b}} (\phi^{-1}(l^{-1}))^{d/2} l^{-c_1/2} dz dl \\ &\leq C \left(\phi(b^{-2}/16) \right)^{c_1/2-1} s^{-\beta} + C b^d s^{-\beta} \int_{(\phi(b^{-2}/16))^{-1}}^{2s^\alpha} (\phi^{-1}(l^{-1}))^{d/2} l^{-c_1/2} dl \\ &\leq C \left(\phi(b^{-2}/16) \right)^{c_1/2-1} s^{-\beta} + C s^{-\beta} \phi(b^{-2}/16)^{-d/2} \int_{(\phi(b^{-2}/16))^{-1}}^{2s^\alpha} l^{(-d-c_1)/2} dl. \end{aligned} \tag{3.34}$$

The last inequality above is due to (3.4) with $R = b^{-2}/16$ and $r = \phi^{-1}(l^{-1})$. To estimate the integral above, we use the following inequality: for any $c \in \mathbb{R}$ and $\varepsilon > 0$,

$$\int_b^a l^c dl \leq C(c) \left(a^{c+1} + b^{c+1} \right) + C(\varepsilon) 1_{c=-1} a^\varepsilon b^{-\varepsilon}, \quad \forall 0 < b < a.$$

This is obvious if $c \neq -1$, and if $c = -1$ then we use $\ln(a/b) \leq C(\varepsilon)(a/b)^\varepsilon$. Thus we get

$$\int_{(\phi(b^{-2}/16))^{-1}}^{2s^\alpha} l^{(-d-c_1)/2} dl \leq C s^{\alpha(2-d-c_1)/2} + C \left(\phi(b^{-2}/16)\right)^{(-2+d+c_1)/2} + 1_{c_1+d=2} C(\varepsilon) s^{\alpha\varepsilon} \left(\phi(b^{-2}/16)\right)^\varepsilon.$$

Coming back to (3.34) and (3.33), and using the definition of $II(b)$, we get

$$\begin{aligned} II(b) &\leq C \left(\phi(b^{-2}/16)\right)^{c_1-2} \int_{-\infty}^{-m\kappa(b)} \int_{-\kappa(b)-r}^{-r} \mathbb{M}_x |g|_H^2(r, x) s^{-2\beta} ds dr \\ &\quad + C \left(\phi(b^{-2}/16)\right)^{-d} \int_{-\infty}^{-m\kappa(b)} \int_{-\kappa(b)-r}^{-r} \mathbb{M}_x |g|_H^2(r, x) s^{-2\beta+\alpha(2-d-c_1)} ds dr \\ &\quad + C(\varepsilon) 1_{c_1+d=2} \left(\phi(b^{-2}/16)\right)^{-d+2\varepsilon} \\ &\quad \times \int_{-\infty}^{-m\kappa(b)} \int_{-\kappa(b)-r}^{-r} \mathbb{M}_x |g|_H^2(r, x) s^{-2\beta+2\alpha\varepsilon} ds dr \\ &=: II_1(b) + II_2(b) + II_3(b). \end{aligned}$$

Now we fix $\varepsilon > 0$ such that $-2\beta + 2\alpha\varepsilon < -1$ (recall $\beta > 1/2$). Then,

$$-2\beta < -1, \quad -2\beta + \alpha(2-d-c_1) < -1, \quad -2\beta + 2\alpha\varepsilon < -1.$$

Therefore, all of $II_i(b)$, $i = 1, 2, 3$, can be handled by (3.20). For instance, by (3.20) with $\nu = 2\beta - 2\alpha\varepsilon$,

$$\begin{aligned} II_3(b) &\leq C 1_{c_1+d=2} \left(\phi(b^{-2}/16)\right)^{-d+2\varepsilon} \kappa(b)^{2-2\beta+2\alpha\varepsilon} \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x) \\ &\leq C \kappa(b) \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \end{aligned} \tag{3.35}$$

One can handle $II_1(b)$ and $II_2(b)$ in the same way, and get

$$II(b) \leq C \kappa(b) \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x).$$

This together with (3.32) proves (3.31), and the lemma is proved. □

Lemma 3.10. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ have a support in $(-\infty, -2\kappa(b)) \times B_{2b}^c$. Then for any $(t, x) \in Q_b$,*

$$\int_{Q_b} \int_{Q_b} |\mathcal{T}g(s, y_1) - \mathcal{T}g(s, y_2)|^2 ds dy_1 d\tilde{s} dy_2 \leq C \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x),$$

where $C = C(\alpha, \beta, d, \delta_0, \kappa_0)$.

Proof. Since $g(r, z) = 0$ if $|z| \leq 2b$, by the fundamental theorem of calculus, for any $s \in (-\kappa(b), 0)$ and $y_1, y_2 \in B_b$ we have

$$\begin{aligned} &|\mathcal{T}g(s, y_1) - \mathcal{T}g(s, y_2)|^2 \\ &\leq \int_{-\infty}^s \left| \int_{\mathbb{R}^d} \int_0^1 \nabla q_{\alpha, \beta}^{c_1/2}(s-r, \bar{u}-z) \cdot (y_2 - y_1) g(r, z) dudz \right|_H^2 dr \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^s \left| \int_{\mathbb{R}^d} \int_0^1 \nabla q_{\alpha,\beta}^{c_1/2}(s-r, z) \cdot (y_2 - y_1) g(r, \bar{u} - z) dudz \right|_H^2 dr \\
 &\leq 4b^2 \int_{-\infty}^{-2\kappa(b)} \left| \int_{B_b^c} \int_0^1 |\nabla q_{\alpha,\beta}^{c_1/2}(s-r, z)| |g(r, \bar{u} - z)|_H dudz \right|_H^2 dr \tag{3.36}
 \end{aligned}$$

where $\bar{u} = \bar{u}(u, y_1, y_2) = (1 - u)y_1 + uy_2$, and $|\bar{u}| \leq b$.

Note that for $s \in (-\kappa(b), 0)$ and $r < -2\kappa(b)$,

$$s - r > \kappa(b), \quad (\phi^{-1}((s - r)^{-\alpha}))^{-1/2} > b. \tag{3.37}$$

Hence, by Lemma 3.2 (iii),

$$\begin{aligned}
 &\int_{B_b^c} \int_0^1 |\nabla q_{\alpha,\beta}^{c_1/2}(s-r, z)| |g(r, \bar{u} - z)|_H dudz \\
 &= \int_{|z| \geq (\phi^{-1}((s-r)^{-\alpha}))^{-1/2}} \int_0^1 |\nabla q_{\alpha,\beta}^{c_1/2}(s-r, z)| |g(r, \bar{u} - z)|_H dudz \\
 &\quad + \int_{b < |z| < (\phi^{-1}((s-r)^{-\alpha}))^{-1/2}} \int_0^1 |\nabla q_{\alpha,\beta}^{c_1/2}(s-r, z)| |g(r, \bar{u} - z)|_H dudz \\
 &\leq \int_0^1 \int_{|z| \geq (\phi^{-1}((s-r)^{-\alpha}))^{-1/2}} (s-r)^{\alpha-\beta} G_{c_1/2, d+1}(|z|) |g(r, \bar{u} - z)|_H dz du \\
 &\quad + \int_0^1 \int_{b < |z| < (\phi^{-1}((s-r)^{-\alpha}))^{-1/2}} H_{c_1/2, d+1}(s-r, |z|) |g(r, \bar{u} - z)|_H dz du \\
 &=: I(s, r, y_1, y_2) + II(s, r, y_1, y_2), \tag{3.38}
 \end{aligned}$$

where $G_{c_1/2, d+1}(|z|)$ and $H_{c_1/2, d+1}(s-r, |z|)$ are from (3.21) and (3.22), respectively.

By (3.38) and (3.36),

$$\begin{aligned}
 &\int_{Q_b} \int_{Q_b} |\mathcal{T}g(s, y_1) - \mathcal{T}g(s, y_2)|^2 ds dy_1 d\tilde{s} dy_2 \\
 &\leq Cb^2 \int_{Q_b} \int_{Q_b} \int_{-\infty}^{-2\kappa(b)} \left(I^2(s, r, y_1, y_2) + II^2(s, r, y_1, y_2) \right) dr ds dy_1 d\tilde{s} dy_2. \tag{3.39}
 \end{aligned}$$

As in the proof of Lemma 3.8, using (3.25), (3.23) and (3.28) (recall $(\phi^{-1}((s - r)^{-\alpha}))^{-1/2} > b$), we get

$$\begin{aligned}
 &I(s, r, y_1, y_2) \\
 &\leq C \int_{(\phi^{-1}((s-r)^{-\alpha}))^{-1/2}}^{\infty} (s-r)^{\alpha-\beta} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho^{d+2}} \left(\int_{B_{3\rho}(x)} |g(r, z)|_H dz \right) d\rho \\
 &\leq CM_x |g|_H(r, x) \int_{(\phi^{-1}((s-r)^{-\alpha}))^{-1/2}}^{\infty} (s-r)^{\alpha-\beta} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho^2} d\rho.
 \end{aligned}$$

Hence, by (A.1), for $r < -2\kappa(b) < s$,

$$\begin{aligned}
 & I(s, r, y_1, y_2) \\
 & \leq C\mathbb{M}_x |g|_H(r, x) (\phi^{-1}((s-r)^{-\alpha}))^{1/2} \int_{(\phi^{-1}((s-r)^{-\alpha}))^{-1/2}}^{\infty} (s-r)^{\alpha-\beta} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho} d\rho \\
 & \leq C\mathbb{M}_x |g|_H(r, x) (\phi^{-1}((s-r)^{-\alpha}))^{1/2} (s-r)^{\alpha(1-c_1/2)-\beta} \\
 & = C\mathbb{M}_x |g|_H(r, x) (\phi^{-1}((s-r)^{-\alpha}))^{1/2} (s-r)^{-1/2} \\
 & \leq C\mathbb{M}_x |g|_H(r, x) \phi(b^{-2})^{-1/2} b^{-1} (s-r)^{-\alpha/2-1/2}.
 \end{aligned}$$

For the last inequality, we use (3.4) with $r = \phi^{-1}((s-r)^{-\alpha})$ and $R = b^{-2}$. Therefore, by (3.20),

$$\begin{aligned}
 & b^2 \int_{Q_b} \int_{Q_b} \int_{-\infty}^{-2\kappa(b)} I^2(s, r, y_1, y_2) dr ds dy_1 d\tilde{s} dy_2 \\
 & \leq Cb^{2+2d} \kappa(b) \int_{-\kappa(b)}^0 \int_{-\infty}^{s-\kappa(b)} I^2(s, r, y_1, y_2) dr ds \\
 & \leq Cb^{2+2d} \kappa(b) \int_{-\kappa(b)}^0 \int_{-\infty}^{-2\kappa(b)} \mathbb{M}_x |g|_H^2(r, x) \phi(b^{-2})^{-1} b^{-2} (s-r)^{-\alpha-1} dr ds \\
 & \leq C\phi(b^{-2})^{-1} b^{2d} (\kappa(b))^{2-\alpha} \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x) \\
 & = C(\kappa(b))^2 b^{2d} \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \tag{3.40}
 \end{aligned}$$

Similarly, by (3.25), (3.24) and (3.28) (recall $(\phi^{-1}((s-r)^{-\alpha}))^{-1/2} > b$), we have

$$\begin{aligned}
 & II(s, r, y_1, y_2) \\
 & \leq C \int_b^{(\phi^{-1}((s-r)^{-\alpha}))^{-\frac{1}{2}}} H_{c_1/2, d+2}(s-r, \rho) \left(\int_{B_{3\rho}(x)} |g(r, z)|_H dz \right) d\rho \\
 & \quad + CH_{c_1/2, d+1}(s-r, (\phi^{-1}((s-r)^{-\alpha}))^{-\frac{1}{2}}) \left(\int_B |g(r, \bar{u}-z)|_H dz \right) \\
 & =: II_1(s, r) + II_2(s, r, y_1, y_2).
 \end{aligned}$$

By definition of $H_{c_1/2, d+1}$ (see (3.22)),

$$\begin{aligned}
 & II_1(s, r) \\
 & \leq C\mathbb{M}_x |g|_H(r, x) (s-r)^{-\beta} \int_{(\kappa(b))^\alpha}^{2(s-r)^\alpha} \int_b^{(\phi^{-1}(l^{-1}))^{-\frac{1}{2}}} (\phi^{-1}(l^{-1}))^{(d+2)/2} l^{-c_1/2} \rho^d d\rho dl \\
 & \leq C\mathbb{M}_x |g|_H(r, x) (s-r)^{-\beta} \int_{(\kappa(b))^\alpha}^{2(s-r)^\alpha} (\phi^{-1}(l^{-1}))^{1/2} l^{-c_1/2} dl \\
 & \leq C\mathbb{M}_x |g|_H(r, x) (s-r)^{-\beta} \phi(b^{-2})^{-1/2} b^{-1} \int_{(\kappa(b))^\alpha}^{2(s-r)^\alpha} l^{-1/2-c_1/2} dl
 \end{aligned}$$

For the last inequality above we used (3.4) with $R = b^{-2}$ and $r = \phi^{-1}(l^{-1})$.

Also, by (3.22) and (3.4) with $R = \phi^{-1}(l^{-1})$ and $r = \phi^{-1}((s - r)^{-\alpha})$ (recall $(\phi^{-1}((s - r)^{-\alpha}))^{-1/2} > b$),

$$II_2(s, r, y_1, y_2) \leq C\mathbb{M}_x|g|_H(r, x) \int_{(s-r)^\alpha}^{2(s-r)^\alpha} l^{-1/2-c_1/2} (s-r)^{\alpha/2-\beta} (\phi^{-1}((s-r)^{-\alpha}))^{1/2} dl.$$

We use (3.4) again with $R = b^{-2}$ and $r = \phi^{-1}((s - r)^{-\alpha})$, and get

$$\begin{aligned} II_2(s, r, y_1, y_2) &\leq C\phi(b^{-2})^{-1/2}b^{-1}(s-r)^{-\beta}\mathbb{M}_x|g|_H(r, x) \int_{(s-r)^\alpha}^{2(s-r)^\alpha} l^{-1/2-c_1/2} dl \\ &\leq C\phi(b^{-2})^{-1/2}b^{-1}(s-r)^{-\beta}\mathbb{M}_x|g|_H(r, x) \int_{(\kappa(b))^\alpha}^{2(s-r)^\alpha} l^{-1/2-c_1/2} dl. \end{aligned}$$

The second inequality above is due to (3.37). Thus,

$$\begin{aligned} II^2 &\leq 2(II_1(s, r) + II_2(s, r, y_1, y_2))^2 \\ &\leq C\phi(b^{-2})^{-1}b^{-2}(s-r)^{-2\beta}\mathbb{M}_x|g|_H^2(r, x) \left(\int_{(\kappa(b))^\alpha}^{2(s-r)^\alpha} l^{-1/2-c_1/2} dl \right)^2 \\ &\leq C\phi(b^{-2})^{-1}b^{-2}(s-r)^{-2\beta}\mathbb{M}_x|g|_H^2(r, x) \\ &\quad \times \left[(s-r)^{\alpha(1-c_1)} + (\kappa(b))^{\alpha(1-c_1)} + C(\varepsilon)1_{c_1=1}(s-r)^{2\alpha\varepsilon}(\kappa(b))^{-2\alpha\varepsilon} \right], \end{aligned}$$

where $\varepsilon > 0$ is chosen so that $-2\beta + 2\alpha\varepsilon < -1$. Note that

$$-2\beta + \alpha(1 - c_1) = -1 - \alpha < -1, \quad -2\beta < -1, \quad -2\beta + 2\alpha\varepsilon < -1.$$

As is done for (3.35), applying (3.20) three times with $v = 1 + \alpha$, 2β and $2\beta - 2\alpha\varepsilon$, we get

$$\begin{aligned} &b^2 \int_{Q_b} \int_{Q_b} \int_{-\infty}^{-2\kappa(b)} II^2(s, r, y_1, y_2) dr ds dy_1 d\tilde{s} dy_2 \\ &\leq Cb^{2+2d}\kappa(b) \int_{-\kappa(b)}^0 \int_{-\infty}^{-2\kappa(b)} II^2(r, s, y_1, y_2) dr ds \\ &\leq C(\kappa(b))^2 b^{2d} \mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \end{aligned}$$

This, (3.40) and (3.39) prove the lemma. □

Lemma 3.11. *Let $g \in C_c^\infty(\mathbb{R}^{d+1}; H)$ have a support in $(-\infty, -2\kappa(b)) \times B_{2b}^c$. Then, for any $(t, x) \in Q_b$,*

$$\int_{Q_b} \int_{Q_b} |\mathcal{T}g(s_1, y) - \mathcal{T}g(s_2, y)|^2 ds_1 dy ds_2 d\tilde{y} \leq C\mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x),$$

where $C = C(\alpha, \beta, d, \delta_0, \kappa_0)$.

Proof. Since $g(r, z) = 0$ if $|z| \leq 2b$, for $s_1, s_2 \in (-\kappa(b), 0)$, and $y \in B_b$,

$$\begin{aligned} & |\mathcal{T}g(s_1, y) - \mathcal{T}g(s_2, y)| \\ & \leq \left[\int_{-\infty}^{-2\kappa(b)} \left| \int_{\mathbb{R}^d} \int_0^1 q_{\alpha, \beta+1}^{c_1/2}(\bar{s} - r, y - z)(s_1 - s_2)g(r, z)dudz \right|_H^2 dr \right]^{1/2} \\ & \leq \left[\int_{-\infty}^{-2\kappa(b)} \left| \int_{\mathbb{R}^d} \int_0^1 q_{\alpha, \beta+1}^{c_1/2}(\bar{s} - r, z)(s_1 - s_2)g(r, y - z)dudz \right|_H^2 dr \right]^{1/2} \\ & \leq C\kappa(b) \left[\int_{-\infty}^{-2\kappa(b)} \left| \int_{B_b^c} \int_0^1 q_{\alpha, \beta+1}^{c_1/2}(\bar{s} - r, z)g(r, y - z)dudz \right|_H^2 dr \right]^{1/2} \end{aligned}$$

where $\bar{s} = \bar{s}(u, s_1, s_2) = (1 - u)s_1 + us_2$.

Note that for $B_\rho(y) \subset B_{2\rho}(x)$ if $\rho \geq b$, and $x, y \in B_b$. Therefore, by Lemma 3.2 (ii), (3.25), (3.23) and (A.1), for $s_1, s_2 \in (-\kappa(b), 0)$ and $r < -2\kappa(b)$,

$$\begin{aligned} & \left| \int_{B_b^c} q_{\alpha, \beta+1}^{c_1/2}(\bar{s} - r, z)g(r, y - z)dz \right|_H \\ & \leq C \int_b^\infty (\bar{s} - r)^{\alpha-\beta-1} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho^{d+1}} \left(\int_{B_{2\rho}(x)} |g(r, z)|_H dz \right) d\rho \\ & \leq C\mathbb{M}_x |g|_H(r, x) \int_b^\infty (\bar{s} - r)^{\alpha-\beta-1} \frac{\phi(\rho^{-2})^{c_1/2}}{\rho} d\rho \\ & \leq C\mathbb{M}_x |g|_H(r, x) (\bar{s} - r)^{\alpha-\beta-1} \phi(b^{-2})^{c_1/2}. \end{aligned}$$

Since $-\kappa(b) < s_1, s_2 < 0$, and $r < -2\kappa(b)$, we have $\frac{1}{2} \leq \frac{\bar{s}-r}{s-r} \leq 2$ for any $s \in (-\kappa(b), 0)$. Hence, by (3.20) (recall $2\alpha - 2\beta - 2 < -1$),

$$\begin{aligned} & \int_{-\kappa(b)}^0 \int_{-\kappa(b)}^0 \int_{-\infty}^{-2\kappa(b)} \left| \int_{B_b^c} \int_0^1 q_{\alpha, \beta+1}^{c_1/2}(\bar{s} - r, z)g(r, y - z)dudz \right|_H^2 dr ds_1 ds_2 \\ & \leq C\phi(b^{-2})^{c_1} \kappa(b) \int_{-\kappa(b)}^0 \int_{-\infty}^{-2\kappa(b)} \mathbb{M}_x |g|_H^2(r, x) (s - r)^{2\alpha-2\beta-2} dr ds \\ & \leq C\mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \end{aligned}$$

The lemma is proved. □

Proof of Theorem 3.4. Due to Lemma 3.5, we may assume $p > 2$.

First we prove for each $Q = Q_b(t_0, x_0)$ and $(t, x) \in Q$,

$$\int_Q |\mathcal{T}g - (\mathcal{T}g)_Q|^2 ds dy \leq C\mathbb{M}_t \mathbb{M}_x |g|_H^2(t, x). \tag{3.41}$$

Note that for any $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$,

$$\mathcal{T}g(t + t_0, x + x_0)$$

$$\begin{aligned}
 &= \left[\int_{-\infty}^{t+t_0} \left| \phi(\Delta)^{c_1/2} T_{t+t_0-s}^{\alpha,\beta} g(s, \cdot)(x + x_0) \right|_H^2 ds \right]^{1/2} \\
 &= \left[\int_{-\infty}^{t+t_0} \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c_1/2}(t + t_0 - s, x + x_0 - y) g(s, y) dy \right|_H^2 ds \right]^{1/2} \\
 &= \left[\int_{-\infty}^t \left| \int_{\mathbb{R}^d} q_{\alpha,\beta}^{c_1/2}(t - s, x - y) \bar{g}(s, y) dy \right|_H^2 ds \right]^{1/2} \\
 &= \mathcal{T} \bar{g}(t, x),
 \end{aligned}$$

where $\bar{g}(s, y) := g(s + t_0, y + x_0)$. Therefore,

$$\int_{Q_b(t_0, x_0)} |\mathcal{T}g - (\mathcal{T}g)_{Q_b(t_0, x_0)}|^2 ds dy = \int_{Q_b(0, 0)} |\mathcal{T}\bar{g} - (\mathcal{T}\bar{g})_{Q_b(0, 0)}|^2 ds dy.$$

This implies that, to prove (3.41), it suffices to consider the only case $Q = Q_b(0, 0)$.

Now we fix $b > 0$ and take a function $\zeta \in C^\infty(\mathbb{R})$ such that $\zeta = 1$ on $[-7\kappa(b)/3, \infty)$, $\zeta = 0$ on $(-\infty, -8\kappa(b)/3)$, and $0 \leq \zeta \leq 1$. We also choose a function $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta = 1$ on $B_{7b/3}$, $\eta = 0$ outside of $B_{8b/3}$, and $0 \leq \eta \leq 1$. Set

$$g_1(t, x) = g\zeta, \quad g_2 = g(1 - \zeta)\eta, \quad g_3 = g(1 - \zeta)(1 - \eta).$$

We show that for any $c \in \mathbb{R}$,

$$\begin{aligned}
 |\mathcal{T}g(s, y) - c| &\leq |\mathcal{T}g_1(s, y)| + |\mathcal{T}(g_2 + g_3)(s, y) - c| \\
 &\leq |\mathcal{T}g_1(s, y)| + |\mathcal{T}g_2(s, y)| + |\mathcal{T}g_3(s, y) - c|.
 \end{aligned} \tag{3.42}$$

Fix $c \in \mathbb{R}$. If $\mathcal{T}g(s, y) > c$, then due to the sublinearity of \mathcal{T}

$$\begin{aligned}
 \mathcal{T}g(s, y) - c &\leq \mathcal{T}g_1(s, y) + \mathcal{T}g_2(s, y) + \mathcal{T}g_3(s, y) - c \\
 &\leq |\mathcal{T}g_1(s, y)| + |\mathcal{T}g_2(s, y)| + |\mathcal{T}g_3(s, y) - c|
 \end{aligned}$$

Suppose $\mathcal{T}g(s, y) < c$. Again by the sublinearity,

$$\mathcal{T}g \geq -\mathcal{T}g_1 + \mathcal{T}(g_2 + g_3),$$

Therefore,

$$\begin{aligned}
 c - \mathcal{T}g(s, y) &\leq \mathcal{T}g_1(s, y) + c - \mathcal{T}(g_2 + g_3)(s, y) \\
 &\leq |\mathcal{T}g_1(s, y)| + |c - \mathcal{T}g_3(s, y)| + |\mathcal{T}g_3(s, y) - \mathcal{T}(g_2 + g_3)(s, y)| \\
 &\leq |\mathcal{T}g_1(s, y)| + |\mathcal{T}g_2(s, y)| + |\mathcal{T}g_3(s, y) - c|.
 \end{aligned}$$

Thus, (3.42) is proved.

By Lemmas 3.8 and 3.9, we have

$$\int_Q |\mathcal{T}g_1(s, y)|^2 dy ds \leq CM_t M_x |g_1|_H^2 \leq CM_t M_x |g|_H^2, \tag{3.43}$$

$$\int_Q |\mathcal{T}g_2(s, y)|^2 dy ds \leq C M_t M_x |g_2|_H^2 \leq C M_t M_x |g|_H^2. \tag{3.44}$$

Also using Lemmas 3.10 and 3.11, we have

$$\begin{aligned} & \int_Q |\mathcal{T}g_3(s, y) - (\mathcal{T}g_3)_Q|^2 dz dr dy ds \\ & \leq C \int_Q \int_Q |\mathcal{T}g_3(s, y) - \mathcal{T}g_3(s, z)|^2 dz dr dy ds \\ & \quad + C \int_Q \int_Q |\mathcal{T}g_3(s, z) - \mathcal{T}g_3(r, z)|^2 dz dr dy ds \\ & \leq C M_t M_x |g|_H^2(t, x). \end{aligned} \tag{3.45}$$

Therefore, if we take $c = (\mathcal{T}g_3)_Q$, by (3.27), (3.43), (3.44), and (3.45) it follows that

$$\begin{aligned} & \int_Q |\mathcal{T}g - (\mathcal{T}g)_Q|^2 dy ds \\ & \leq C \int_Q |\mathcal{T}g_1(s, y)|^2 dy ds + C \int_Q |\mathcal{T}g_2(s, y)|^2 dy ds \\ & \quad + C \int_Q |\mathcal{T}g_3(s, y) - (\mathcal{T}g_3)_Q|^2 dy ds \\ & \leq C M_t M_x |g|_H^2(t, x), \end{aligned}$$

and thus (3.41) is proved. By (3.41) and Jensen’s inequality,

$$(\mathcal{T}g)^\#(t, x) \leq C \left(M_t M_x |g|_H^2(t, x) \right)^{1/2}, \quad \forall (t, x). \tag{3.46}$$

Therefore, by Fefferman–Stein theorem (e.g., [43, Theorem IV.2.2]) and (3.46),

$$\|\mathcal{T}g\|_{L^p(\mathbb{R}^{d+1})}^p \leq C \|(\mathcal{T}g)^\#\|_{L^p(\mathbb{R}^{d+1})}^p \leq C \|M_t M_x |g|_H\|_{L^{p/2}(\mathbb{R}^{d+1})}^{p/2}.$$

Next, we use Hardy–Littlewood maximal theorem (e.g., [43, Theorem I.3.1]) twice with respect to time and spatial variables in order, and get

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left(M_t M_x |g|_H^2 \right)^{p/2} dt dx & \leq C \int_{\mathbb{R}^d} \int_{\mathbb{R}} \left(M_x |g|_H^2 \right)^{p/2} dt dx \\ & = C \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(M_x |g|_H^2 \right)^{p/2} dx dt \\ & \leq C \int_{\mathbb{R}} \int_{\mathbb{R}^d} (|g|_H^2)^{p/2} dx dt = C \| |g|_H \|_{L^p(\mathbb{R}^{d+1})}^p. \end{aligned}$$

This proves the theorem if $T = \infty$. For $T < \infty$ take $\xi \in C^\infty(\mathbb{R})$ such that $0 \leq \xi \leq 1$, $\xi = 1$ for $t \leq T$ and $\xi = 0$ for $t \geq T + \varepsilon$, $\varepsilon > 0$. Then, it is enough to apply the result for $T = \infty$ with $g\xi$. Since $\varepsilon > 0$ is arbitrary, the theorem is proved. \square

3.2. H^∞ -calculus

First, we provide some definitions related to H^∞ -calculus. For $\eta > 0$, let

$$\Sigma_\eta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \eta\}.$$

By $H^p(\Sigma_\eta)$ ($1 \leq p \leq \infty$), we denote the (complex) Banach space of all holomorphic functions $f : \Sigma_\eta \rightarrow \mathbb{C}$ satisfying

$$\|f\|_{H^p(\Sigma_\eta)} := \sup_{|\nu| < \eta} \|f(e^{i\nu}t)\|_{L_p(\mathbb{R}_+, \frac{dt}{t})} < \infty.$$

Let A be a linear operator on a Banach space X . We say that z is in the resolvent set $\rho(A)$ of A if the range of $A_z := z - A$ is dense in X and A_z has a continuous inverse. Here, for $z \in \rho(A)$, we can define $R(z, A) := (z - A)^{-1}$. We say that a linear operator A is sectorial if there exists $\omega \in (0, \pi)$ such that the spectrum $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is contained in $\overline{\Sigma_\omega}$ and

$$\sup_{z \in (\overline{\Sigma_\omega})^c} \|zR(z, A)\| < \infty.$$

In this case, we say A is ω -sectorial. The infimum of all ω such that A is ω -sectorial is called the angle of sectoriality of A and is denoted by $\omega(A)$.

Let A be a sectorial operator with angle of sectoriality $\omega(A)$. For functions $f \in H^1(\Sigma_\omega)$, denote

$$f(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_\nu} f(z)R(z, A)dz$$

where $\omega(A) < \nu < \sigma$ is chosen arbitrarily. It is well known (see [19, Sect. 10.2]) that the definition of $f(A)$ is independent of the choice of ν . For a constant $\sigma \in (\omega(A), \pi)$, we say that the operator A has a bounded $H^\infty(\Sigma_\sigma)$ -calculus if there exists a constant $C > 0$ such that

$$\|f(A)\| \leq C\|f\|_{L_\infty}, \quad f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma).$$

We define

$$\omega_{H^\infty}(A) := \inf\{\sigma \in (\omega(A), \pi) : A \text{ has a bounded } H^\infty(\Sigma_\sigma)\text{-calculus}\},$$

and we say that A has a bounded H^∞ -calculus of angle $\omega_{H^\infty}(A)$. For instance, the Laplace operator $-\Delta$ has a bounded H^∞ -calculus on $L_p(\mathbb{R}^d)$ of angle 0 (see, e.g., [19, Theorem 10.2.25]).

Now we are ready to prove the following:

Theorem 3.12. *Let ϕ be a Bernstein function. Then, $-\phi(\Delta) = \phi(-\Delta)$ has a bounded H^∞ -calculus on $L_p(\mathbb{R}^d)$ of angle 0.*

Proof. Note that ϕ can be extended to a holomorphic function which maps $\mathbb{C}_+ := \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ into itself and satisfies $\phi(\Sigma_\sigma) \subset \Sigma_\sigma$ for any $\sigma \in [0, \pi)$ (see [1, Proposition 3.3]). Hence, [1, Theorem 1.1] yields that $-\phi(\Delta) = \phi(-\Delta)$ is 0-sectorial.

Next, for $f \in H^\infty(\Sigma_\sigma)$ with $0 < \sigma < \pi$, define

$$\Psi_\Delta(f)\zeta := \mathcal{F}^{-1} \left[f(|\cdot|^2) \hat{\zeta}(\cdot) \right], \quad \zeta \in \mathcal{S}(\mathbb{R}^d).$$

For any multi indices $\alpha \in \{0, 1\}^d$ and $\xi \in \Sigma_\sigma$, one can easily show that

$$|\xi|^{|\alpha|} D^\alpha (f(|\xi|^2)) = |\xi|^{|\alpha|} 2^{|\alpha|} \xi^\alpha (D^\alpha f)(|\xi|^2) = (2\xi/|\xi|)^\alpha (|\xi|^2)^{|\alpha|} (D^\alpha f)(|\xi|^2).$$

Also, by the Cauchy formula,

$$(|\xi|^2)^{|\alpha|} (D^\alpha f)(|\xi|^2) \leq \frac{1}{2\pi} \int_{\Gamma_{\sigma'}} \frac{|\xi|^{2|\alpha|} |f(z)|}{|z - |\xi|^2|^{|\alpha|+1}} |dz| \leq \frac{C}{2\pi} \int_{\Gamma_{\sigma'}} \frac{1}{|z - 1|^{|\alpha|+1}} |dz| < \infty,$$

where $\Gamma_{\sigma'} := \{z \in \mathbb{C} \setminus \{0\} : \text{arg}(z) = \pm\sigma'\}$ with $\sigma' \in (0, \sigma)$. Thus, by Mihlin’s multiplier theorem (see, e.g., [20, Theorem 5.5.10]), $\Psi_\Delta(f)$ is a bounded operator on L_p . Since $f \circ \phi \in H^\infty(\Sigma_\sigma)$, by considering $f \circ \phi$ instead of f , we find that the operator

$$\Psi(f)\zeta := \mathcal{F}^{-1} \left[f(\phi(|\cdot|^2)) \hat{\zeta}(\cdot) \right], \quad \zeta \in \mathcal{S}(\mathbb{R}^d),$$

is a bounded operator on L_p . Moreover, by following the proof of [19, Theorem 10.2.25], one can check that the mapping $f \rightarrow \Psi(f)$ satisfies the assumptions of [19, Theorem 10.2.14]. Thus, by [19, Theorem 10.2.14], $-\phi(\Delta)$ has a bounded $H^\infty(\Sigma_\sigma)$ -calculus. Since $\sigma > 0$ is arbitrary, the theorem is proved. \square

Now we define an operator associated with $(\alpha, \beta, -\phi(\Delta))$. Let

$$T_{\alpha,\beta}(t)v := \frac{1}{2\pi i} \int_{\Gamma_{1,\psi}} e^{\lambda t} (\lambda^\alpha - \phi(\Delta))^{-1} \lambda^{\beta-1} v d\lambda, \quad t > 0,$$

for $v \in L_p$, $\psi \in (\frac{\pi}{2}, \pi)$, and

$$\Gamma_{1,\psi} := \{e^{it} : |t| \leq \psi\} \cup \{\rho e^{i\psi} : 1 < \rho < \infty\} \cup \{\rho e^{-i\psi} : 1 < \rho < \infty\}.$$

We have the following representation for solution (cf. [12]).

Lemma 3.13. *For given $g \in \mathbb{H}_0^\infty(T, l_2)$, the function*

$$u(t, x) := \sum_{k=1}^\infty \int_0^t T_{\alpha,\beta}(t-s) g^k(s, x) dw_s^k. \tag{3.47}$$

is in $\mathbb{H}_p^{\phi,2}(T)$ and satisfies (3.1) in the sense of distributions.

Proof. We first show that $u \in \mathbb{H}_p^{\phi,2}(T)$. By the Burkholder–Davis–Gundy inequality,

$$\begin{aligned} \|u\|_{\mathbb{L}_p(T)}^p &\leq C(p)\mathbb{E}\left\|\left(\int_0^t |T_{\alpha,\beta}(t-s)g(s)(x)|_2^2 ds\right)^{1/2}\right\|_{L_p((0,T)\times\mathbb{R}^d)}^p \\ &\leq C\|g\|_{\mathbb{L}_p(T,l_2)}^p. \end{aligned}$$

For the second inequality above we used [12, Lemma 5.6].

Applying (5.16) in [12], we get

$$\phi(\Delta)u(t,x) = \sum_{k=1}^{\infty} \int_0^t T_{\alpha,\beta}(t-s)\phi(\Delta)g^k(s,x)dw_s^k, \quad \forall t > 0. \tag{3.48}$$

Hence, using [12, Lemma 5.6] again, we have (recall $g \in \mathbb{H}_0^\infty(T, l_2)$)

$$\|\phi(\Delta)u\|_{\mathbb{L}_p(T)} \leq C\|\phi(\Delta)g\|_{\mathbb{L}_p(T,l_2)} < \infty.$$

Therefore, by Lemma 2.1 (iv), we get $u \in \mathbb{H}_p^{\phi,2}(T)$.

Next we show that u satisfies (3.1). By (5.17) in [12],

$$\begin{aligned} u(t,x) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(\Delta)u(s,x)ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta+1)} \sum_{k=1}^{\infty} \int_0^t (t-s)^{\alpha-\beta} g^k(s,x)dw_s^k \end{aligned} \tag{3.49}$$

for all $t > 0$, a.e. on $\mathbb{R}^d \times \Omega$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Multiplying both sides of (3.49) by φ and applying (stochastic) Fubini’s theorem (see, e.g., [31, Lemma 2.7]), one can easily show that u satisfies (3.1) in the sense of distributions. The lemma is proved. \square

Theorem 3.14. *Let $p \in [2, \infty)$, $g \in \mathbb{H}_0^\infty(T, l_2)$, and let $u(t, x)$ be taken from Lemma 3.13. Then we have*

$$\|\phi^{c_1/2}(\Delta)u\|_{\mathbb{L}_p(T)} \leq C\|g\|_{\mathbb{L}_p(T,l_2)}, \tag{3.50}$$

where $c_1 := 2 - (2\beta - 1)/\alpha$ and C is a constant independent of g and T .

Proof. Due to Theorem 3.12, the operator $\phi(\Delta)$ satisfies the assumption in [13, Theorem 3.1]. Thus, (3.50) is a direct consequence of [13, Theorem 3.1] if we take $(\alpha, 1 + \alpha - \beta, 0, c_1/2)$ in place of $(\alpha, \beta, \eta, \theta)$ therein. \square

Remark 3.15. (i) Actually, Theorem 3.12 and [13, Theorem 3.1] together yield L_p -estimates for $D_t^\eta \phi(\Delta)^\theta u$, where $\eta \in (-1, 1) \cap (-\beta + 1/2, \alpha - \beta + 1/2)$ and $\theta := c_1/2 - \eta/\alpha$. This also can be obtained using Krylov’s analytic approach if one considers $q_{\alpha,\beta+\eta}^\theta = q_{\alpha,\beta+\eta}^{c_1/2-\eta/\alpha}$ in place of $q_{\alpha,\beta}^{c_1/2}$.

(ii) As in [33, Section 8G], one can also obtain sharp estimates of $D_t^\eta \phi(\Delta)^\theta u$ in the space $L_q(L_p)$ given with appropriate weights. Here, $p \geq 2, q > 2$.

4. Proof of Theorem 2.12

The following lemma is used to estimate solutions of SPDEs when $\beta < 1/2$.

Lemma 4.1. *Let $\gamma \in \mathbb{R}$, $p \geq 2$, $\beta < \frac{1}{2}$, and $g \in \mathbb{H}_p^{\phi,\gamma}(T, l_2)$. Then, for any $t \in [0, T]$,*

$$\mathbb{E} \int_0^t \left\| \sum_{k=1}^{\infty} \partial_r^\beta \int_0^\cdot g^k(s, \cdot) dw_s^k \right\|_{\mathbb{H}_p^{\phi,\gamma}}^p dr \leq C(d, p, \beta, T) I_t^{1-2\beta} \|g\|_{\mathbb{H}_p^{\phi,\gamma}(\cdot, l_2)}^p(t).$$

In particular,

$$\mathbb{E} \int_0^r \left\| \sum_{k=1}^{\infty} \partial_t^\beta \int_0^\cdot g^k(s, \cdot) dw_s^k \right\|_{\mathbb{H}_p^{\phi,\gamma}}^p dr \leq C \|g\|_{\mathbb{H}_p^{\phi,\gamma}(t, l_2)}^p.$$

Proof. Due to the isometry $(1 - \phi(\Delta))^{\gamma/2} : H_p^{\phi,\gamma} \rightarrow L_p$, it is enough to prove the case $\gamma = 0$. In this case, it is a consequence of [26, Lemma 4.1]. The lemma is proved. \square

Recall

$$c_0 = \frac{(2\beta - 1)^+}{\alpha} + \kappa 1_{\beta=1/2} \in [0, 2),$$

where $\kappa > 0$ is a fixed constant.

Lemma 4.2. *Let $\gamma \in \mathbb{R}$, $p \geq 2$, $\alpha \in (0, 1)$ and $\beta < \alpha + 1/2$. Then for any $g \in \mathbb{H}_p^{\phi,\gamma+c_0}(T, l_2)$, the linear equation*

$$\partial_t^\alpha u = \phi(\Delta)u + \sum_{k=1}^{\infty} \partial_t^\beta \int_0^t g^k dw_s^k, \quad t > 0; \quad u(0, \cdot) = 0 \tag{4.1}$$

has a unique solution $u \in \mathbb{H}_p^{\phi,\gamma+2}(T)$ in the sense of distributions, and for this solution we have

$$\|u\|_{\mathbb{H}_p^{\phi,\gamma+2}(T)} \leq C \|g\|_{\mathbb{H}_p^{\phi,\gamma+c_0}(T, l_2)}, \tag{4.2}$$

where $C = C(\alpha, \beta, d, p, \delta_0, \kappa_0, \gamma, \kappa, T)$. Furthermore, if $\beta > 1/2$ then

$$\|\phi(\Delta)u\|_{\mathbb{H}_p^{\phi,\gamma}(T)} \leq C \|\phi(\Delta)^{c_0/2} g\|_{\mathbb{H}_p^{\phi,\gamma}(T, l_2)}, \tag{4.3}$$

where $C = C(\alpha, \beta, d, p, \delta_0, \kappa_0, \gamma)$ is independent of T .

Proof. Due to Remark 2.7 and Lemma 2.1 (ii), without loss of generality we may assume $\gamma = 0$.

The uniqueness is a consequence of the corresponding result for the deterministic equation, [41, Theorem 8.7 (a)]. Indeed, if u is a solution to the equation with $g = 0$, then for each fixed ω , the function $u(\omega, \cdot, \cdot)$ satisfies the deterministic equation

$$\partial_t^\alpha v = \phi(\Delta)v, \quad t > 0, x \in \mathbb{R}^d; \quad v(0, \cdot) = 0, \tag{4.4}$$

and we conclude $u(\omega, \cdot, \cdot) \equiv 0$. Therefore, it is sufficient to prove the existence result together with estimates (4.2) and (4.3).

Step 1. First, assume $g \in \mathbb{H}_0^\infty(T, l_2)$. Define u by (3.47). Then, by Lemma 3.13, $u \in \mathbb{H}_p^{\phi, 2}(T)$ becomes a solution to equation (4.1). Now we prove the estimates. We divide the proof according to the range of β .

Case 1. $\beta > \frac{1}{2}$.

We first show (4.3). Denote

$$v = \phi(\Delta)^{c_0/2}u, \quad \bar{g} = \phi(\Delta)^{c_0/2}g.$$

By (3.48) and Theorem 3.14,

$$\begin{aligned} \|\phi(\Delta)u\|_{\mathbb{L}_p(T)}^p &= \|\phi(\Delta)^{c_1/2}v\|_{\mathbb{L}_p(T)}^p \leq C\mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\bar{g}|^p dx dt \\ &= C\|\phi(\Delta)^{c_0/2}g\|_{\mathbb{L}_p(T, l_2)}^p. \end{aligned}$$

Thus, (4.3) is proved. Furthermore, by (2.13),

$$\begin{aligned} \|u\|_{\mathbb{L}_p(T)}^p &\leq C \int_0^T (T-s)^{\theta-1} \left(\|\phi(\Delta)u\|_{\mathbb{L}_p(s)}^p + \|g\|_{\mathbb{L}_p(s, l_2)}^p \right) ds \\ &\leq C \int_0^T (T-s)^{\theta-1} \|g\|_{\mathbb{H}_p^{\phi, c_0}(s, l_2)}^p ds \\ &\leq C \|g\|_{\mathbb{H}_p^{\phi, c_0}(T, l_2)}^p \int_0^T (T-s)^{\theta-1} ds \leq C \|g\|_{\mathbb{H}_p^{\phi, c_0}(T, l_2)}^p. \end{aligned}$$

Thus, we have (4.2).

Case 2. $\beta < \frac{1}{2}$.

In this case, $c_0 = 0$. By Lemma 2.4 (iii), u satisfies

$$\partial_t^\alpha u = \phi(\Delta)u + \bar{f},$$

where

$$\bar{f}(t) = \frac{1}{\Gamma(1-\beta)} \sum_{k=1}^\infty \int_0^t (t-s)^{-\beta} g^k(s) dw_s^k.$$

Due to [41, Theorem 8.7 (a)] and Lemma 4.1,

$$\|u\|_{\mathbb{H}_p^{\phi, 2}(T)}^p \leq C\|\bar{f}\|_{\mathbb{L}_p(T)}^p \leq C\|g\|_{\mathbb{L}_p(T, l_2)}^p.$$

Case 3: $\beta = \frac{1}{2}$.

Put $\delta = \frac{\kappa\alpha}{2}$ and $\tilde{\beta} = \frac{1}{2} + \delta$. Then, $0 < \delta < \alpha$ and $\frac{1}{2} < \tilde{\beta} < 2$. Define v by (3.47) with $\tilde{\beta}$ instead of β . By the result from Case 1 with $c_0 = (2\tilde{\beta} - 1)/\alpha = \kappa$, $v \in \mathbb{H}_p^{\phi, 2}(T)$ satisfies

$$\partial_t^\alpha v = \phi(\Delta)v + \sum_{k=1}^\infty \partial_t^{\tilde{\beta}} \int_0^t g^k dw_s^k, \quad ; \quad v(0, \cdot) = 0,$$

and it also holds that

$$\|v\|_{\mathbb{H}_p^{\phi,2}(T)} \leq C \|g\|_{\mathbb{H}_p^{\phi,c_0}(T,l_2)}.$$

Note that $I_t^\delta v$ also satisfies (4.1). Thus, by the uniqueness of solution, we conclude that $I_t^\delta v = u$. Therefore, by the result for the case $\beta > 1/2$ and (2.1),

$$\|u\|_{\mathbb{H}_p^{\phi,2}(T)} = \|I_t^\delta v\|_{\mathbb{H}_p^{\phi,2}(T)} \leq C \|v\|_{\mathbb{H}_p^{\phi,2}(T)} \leq C \|g\|_{\mathbb{H}_p^{\phi,c_0}(T,l_2)}.$$

Thus, the lemma is proved if $g \in \mathbb{H}_0^\infty(T, l_2)$.

Step 2. General case. For given $g \in \mathbb{H}_p^{\phi,c_0}(T, l_2)$, we take a sequence $g_n \in \mathbb{H}_0^\infty(T, l_2)$ so that $g_n \rightarrow g$ in $\mathbb{H}_p^{\phi,c_0}(T, l_2)$. Define u_n using (3.47) with g_n in place of g . Then,

$$\begin{aligned} \|u_n\|_{\mathbb{H}_p^{\phi,2}(T)} &\leq C \|g_n\|_{\mathbb{H}_p^{\phi,c_0}(T,l_2)}, \\ \|u_n - u_m\|_{\mathbb{H}_p^{\phi,2}(T)} &\leq C \|g_n - g_m\|_{\mathbb{H}_p^{\phi,c_0}(T,l_2)}. \end{aligned}$$

Thus, u_n converges to a function $u \in \mathbb{H}_p^{\phi,2}(T)$. Considering (2.11) corresponding to u_n and using Lemma 2.4 (iv), we conclude that u satisfies equation (4.1) in the sense of distributions. The estimates of u also easily follow. The lemma is proved. \square

Remark 4.3. The uniqueness result in Lemma 4.2 yields that two representations (3.14) and (3.47) coincide under Assumption 3.1.

Now we are ready to prove Theorem 2.12.

Proof of Theorem 2.12. Step 1 (linear equation). Let f and g be independent of u .

As before, due to Remark 2.7 and Lemma 2.1 (ii), we may assume $\gamma = 0$. Also, by the uniqueness of deterministic equation (4.4), we only need to prove the existence result and estimates of the solution.

Case I: Let $g \equiv 0$. Then, roughly speaking, using [27, Theorem 2.8], for each fixed ω one can solve the deterministic equation

$$\partial_t^\alpha u^\omega(t, x) = \phi(\Delta)u^\omega(t, x) + f(\omega, t, x), \quad t > 0, x \in \mathbb{R}^d; \quad u^\omega(0, x) = 0, \quad (4.5)$$

and can define $u(\omega, t, x) = u^\omega(t, x)$ so that u solves the equation

$$\partial_t^\alpha u = \phi(\Delta)u + f, \quad t > 0; \quad u(0, x) = 0$$

on $\Omega \times [0, T]$. However, this method may leave the measurability issue. Therefore, we argue as follows. First, assume f is sufficiently smooth, that is, let f be of the type

$$f(t, x) = \sum_{i=1}^{n(k)} 1_{(\tau_{i-1}, \tau_i]}(t)h^i(x), \quad h^i \in C_c^\infty(\mathbb{R}^d)$$

where $0 \leq \tau_0 \leq \tau_1 \leq \dots \leq \tau_n$ are bounded stopping times. Define

$$u_1(t, x) = \int_0^t \int_0^\infty P_r^0 f(s, \cdot) \varphi_{\alpha,1}(t, r) dr ds,$$

where $\varphi_{\alpha,1}$ is defined in (3.5) and $\{P_t^0; t \geq 0\}$ is the strongly continuous semi-group generated by $\phi(\Delta)$ (cf. [8]). Due to (A.8), the above integral is well defined. Thus, we have the desired measurability of u_1 . Moreover, [8, Proposition 2.3] implies that u_1 becomes a solution to (4.5) in the sense of distributions and $u_1 \in L_\infty(\Omega \times [0, T], \mathcal{P}; H_p^{\phi,2}) \subset \mathbb{H}_p^{\phi,2}(T)$. Also, estimate (2.15) for this u_1 follows from [41, Theorem 8.7 (a)].

For general f , one can use the standard approximation argument as in the proof of Lemma 4.2 (see Step 2 there).

Case 2: Let $g \not\equiv 0$. Take $u_1 \in \mathbb{H}_p^{\phi,2}(T)$ from Case 1. Also take $u \in \mathbb{H}_p^{\phi,2}(T)$ from Lemma 4.2. Then, thanks to the linearity of the equations, $v := u_1 + u$ satisfies

$$\partial_t^\alpha v = \phi(\Delta)v + f + \sum_{k=1}^\infty \partial_t^\beta \int_0^t g^k dw_s^k, \quad t > 0; \quad v(0, \cdot) = 0,$$

and estimate (2.15) for v follows from those for u_1 and u . Therefore, the theorem is proved for the linear equation.

Step 2 (nonlinear equation).

We first prove the uniqueness result of the equation.

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \sum_{k=1}^\infty g^k(u)dw_t^k, \quad t > 0; \quad u(0, \cdot) = 0.$$

Let $u_1, u_2 \in \mathbb{H}_p^{\phi,\gamma+2}(T)$ be two solutions to the equation. Then, $\tilde{u} := u_1 - u_2$ satisfies

$$\partial_t^\alpha \tilde{u} = \phi(\Delta)\tilde{u} + f(u_1) - f(u_2) + \sum_{k=1}^\infty (g^k(u_1) - g^k(u_2))dw_t^k, \quad t > 0; \quad \tilde{u}(0, \cdot) = 0.$$

By the continuity of f and g (Assumption 2.11 with $\varepsilon = 1$), for each $t \leq T$,

$$\begin{aligned} & \|\phi(\Delta)(u_1 - u_2)\|_{\mathbb{H}_p^{\phi,\gamma}(t)} + \|f(u_1) - f(u_2)\|_{\mathbb{H}_p^{\phi,\gamma}(t)} \\ & + \|g(u_1) - g(u_2)\|_{\mathbb{H}_p^{\phi,\gamma+c_0}(t,l_2)} \leq C \|u_1 - u_2\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)}. \end{aligned} \tag{4.6}$$

Also, by the result for the linear case and Assumption 2.11, any $\varepsilon > 0$ and $t \leq T$, we have

$$\begin{aligned} & \|u_1 - u_2\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)}^p \\ & \leq C (\|f(u_1) - f(u_2)\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p + \|g(u_1) - g(u_2)\|_{\mathbb{H}_p^{\phi,\gamma+c_0}(t,l_2)}^p) \\ & \leq C \varepsilon^p \|u_1 - u_2\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)}^p + CN_0(\varepsilon) \|u_1 - u_2\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p \end{aligned}$$

$$\leq C\varepsilon^p \|u_1 - u_2\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p + CN_0(\varepsilon) \int_0^t (t-s)^{\theta-1} \|u_1 - u_2\|_{\mathbb{H}_p^{\phi, \gamma+2}(s)}^p ds.$$

For the last inequality above we used (2.13) and (4.6). Now we take $\varepsilon > 0$ so that $C\varepsilon^p < 1/2$, and we conclude $u_1 = u_2$ due to the fractional Gronwall lemma (see [46, Corollary 1]). The uniqueness is proved.

Next we prove the existence result. Let $u^0 \in \mathbb{H}_p^{\phi, \gamma+2}(T)$ denote the solution obtained in Step 1 corresponding to the inhomogeneous terms $f(0)$ and $g(0)$. For $n \geq 0$, using the result of Step 1, we define $u^{n+1} \in \mathbb{H}_p^{\phi, \gamma+2}(T)$ as the solution to the equation

$$\partial_t^\alpha u^{n+1} = \phi(\Delta)u^{n+1} + f(u^n) + \partial_t^\beta \sum_{k=1}^\infty \int_0^t g^k(u^n)dw_s^k, \quad t > 0; \quad u(0) = 0. \tag{4.7}$$

Then $\tilde{u}^n := u^{n+1} - u^n$ satisfies

$$\partial_t^\alpha \tilde{u}^n = \phi(\Delta)\tilde{u}^n + f(u^n) - f(u^{n-1}) + \sum_{k=1}^\infty (g^k(u^n) - g^k(u^{n-1}))dw_t^k, \quad t > 0,$$

with $\tilde{u}^n(0, \cdot) = 0$. By Step 1 and Assumption 2.11, for any $\varepsilon > 0$ and $t \leq T$, we have

$$\begin{aligned} & \|u^{n+1} - u^n\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p \\ & \leq C \left(\|f(u^n) - f(u^{n-1})\|_{\mathbb{H}_p^{\phi, \gamma}(t)}^p + \|g(u^n) - g(u^{n-1})\|_{\mathbb{H}_p^{\phi, \gamma+c_0}(t, l_2)}^p \right) \\ & \leq C\varepsilon^p \|u^n - u^{n-1}\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p + CN_0(\varepsilon) \|u^n - u^{n-1}\|_{\mathbb{H}_p^{\phi, \gamma}(t)}^p. \end{aligned} \tag{4.8}$$

In particular, taking $\varepsilon = 1$, for any $n \geq 1$,

$$\|u^{n+1} - u^n\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p \leq C \|u^n - u^{n-1}\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p. \tag{4.9}$$

Note that $(u^n - u^{n-1})(0, \cdot) = 0$. Thus, by (2.13) and (4.9),

$$\begin{aligned} & \|u^n - u^{n-1}\|_{\mathbb{H}_p^{\phi, \gamma}(t)}^p \\ & \leq C \int_0^t (t-s)^{\theta-1} \left(\|\phi(\Delta)(u^n - u^{n-1}) + f(u^{n-1}) - f(u^{n-2})\|_{\mathbb{H}_p^{\phi, \gamma}(s)}^p \right. \\ & \quad \left. + \|g(u^{n-1}) - g(u^{n-2})\|_{\mathbb{H}_p^{\phi, \gamma}(s, l_2)}^p \right) ds \\ & \leq C \int_0^t (t-s)^{\theta-1} \|u^{n-1} - u^{n-2}\|_{\mathbb{H}_p^{\phi, \gamma+2}(s)}^p ds. \end{aligned} \tag{4.10}$$

Plugging (4.9) and (4.10) into (4.8), we get

$$\|u^{n+1} - u^n\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p \leq C\varepsilon^p \|u^{n-1} - u^{n-2}\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p$$

$$+CN_0 \int_0^t (t-s)^{\theta-1} \|u^{n-1} - u^{n-2}\|_{\mathbb{H}_p^{\phi, \gamma+2}(s)}^p ds,$$

where $N_0 = N_0(\varepsilon)$. Considering $\varepsilon C^{-1/p}$ in place of ε , and repeating the above argument one more time, we get for $t \leq T$,

$$\begin{aligned} & \|u^{n+1} - u^n\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p \\ & \leq \varepsilon^p \left(\varepsilon^p \|u^{n-3} - u^{n-4}\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p + N_1 \int_0^t (t-s)^{\theta-1} \|u^{n-3} - u^{n-4}\|_{\mathbb{H}_p^{\phi, \gamma+2}(s)}^p ds \right) \\ & \quad + N_1 \int_0^t (t-s)^{\theta-1} \left(\varepsilon^p \|u^{n-3} - u^{n-4}\|_{\mathbb{H}_p^{\phi, \gamma+2}(s)}^p \right. \\ & \quad \left. + N_1 \int_0^s (s-r)^{\theta-1} \|u^{n-3} - u^{n-4}\|_{\mathbb{H}_p^{\phi, \gamma+2}(r)}^p dr \right) ds. \end{aligned}$$

Therefore, by using the identity

$$\begin{aligned} & \frac{\Gamma(\theta)^n}{\Gamma(n\theta + 1)} t^{n\theta} \\ & = \int_0^t (t-s_1)^{\theta-1} \int_0^{s_1} (s_1-s_2)^{\theta-1} \dots \int_0^{s_{n-1}} (s_{n-1}-s_n)^{\theta-1} ds_n \dots ds_1 \end{aligned}$$

and repeating above inequality, for $n \in \mathbb{N}_0$ we get

$$\begin{aligned} & \|u^{2n+1} - u^{2n}\|_{\mathbb{H}_p^{\phi, \gamma+2}(T)}^p \\ & \leq \sum_{k=0}^n \binom{n}{k} \varepsilon^{(n-k)p} (T^\theta N_1)^k \frac{\Gamma(\theta)^k}{\Gamma(k\theta + 1)} \|u^1 - u^0\|_{\mathcal{H}_p^{\phi, \gamma+2}(T)}^p \\ & \leq 2^n \varepsilon^{np} \max_k \left(\frac{(\varepsilon^{-1} T^\theta N_1 \Gamma(\theta))^k}{\Gamma(k\theta + 1)} \right) \|u^1 - u^0\|_{\mathbb{H}_p^{\phi, \gamma+2}(T)}^p. \end{aligned}$$

Now fix $\varepsilon < 1/8$. Note that the above maximum is finite and is independent of n . This and (4.9) imply

$$\sum_{n=1}^\infty \|u^{n+1} - u^n\|_{\mathbb{H}_p^{\phi, \gamma+2}(T)}^p < \infty,$$

and therefore u^n is a Cauchy sequence in $\mathbb{H}_p^{\phi, \gamma+2}(T)$. Now let u denote the its limit in $\mathbb{H}_p^{\phi, \gamma+2}(T)$. Then, taking $n \rightarrow \infty$ from (4.7) and using the continuity of f and g , we easily find that u is a solution to (2.14) in the sense of distributions.

Finally we prove estimate (2.15) for the solution u obtained above. Obviously,

$$\|u\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)} \leq \|u - u^0\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)} + \|u^0\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}.$$

Also, by Step 1 and Assumption 2.11, for any $\varepsilon > 0$ and $t \leq T$,

$$\|u - u^0\|_{\mathbb{H}_p^{\phi, \gamma+2}(t)}^p$$

$$\begin{aligned}
 &\leq C \|f(u) - f(0)\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p + C \|g(u) - g(0)\|_{\mathbb{H}_p^{\phi,\gamma+c_0}(t,l_2)}^p \\
 &\leq C\varepsilon^p \|u\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)}^p + C(\varepsilon) \|u\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p \\
 &\leq C\varepsilon^p \|u\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)}^p + C(\varepsilon) \|u - u^0\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p + C(\varepsilon) \|u^0\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)}^p. \tag{4.11}
 \end{aligned}$$

Recall $(u - u^0)(0, \cdot) = 0$. Thus, by (2.13) and the continuity of f and g ,

$$\begin{aligned}
 \|u - u^0\|_{\mathbb{H}_p^{\phi,\gamma}(t)}^p &\leq C \int_0^t (t-s)^{\theta-1} \left(\|\phi(\Delta)u - \phi(\Delta)u^0 + f(u) - f(0)\|_{\mathbb{H}_p^{\phi,\gamma}(s)}^p \right. \\
 &\quad \left. + \|g(u) - g(0)\|_{\mathbb{H}_p^{\phi,\gamma}(s,l_2)}^p \right) ds \\
 &\leq C \int_0^t (t-s)^{\theta-1} \|u\|_{\mathbb{H}_p^{\phi,\gamma+2}(s)}^p ds + C \|u^0\|_{\mathbb{H}_p^{\phi,\gamma+2}(T)}^p.
 \end{aligned}$$

Using this and (4.11), and taking $\varepsilon > 0$ sufficiently small, we get for $t \leq T$

$$\|u\|_{\mathbb{H}_p^{\phi,\gamma+2}(t)} \leq C \|u^0\|_{\mathbb{H}_p^{\phi,\gamma+2}(T)} + C \int_0^t (t-s)^{\theta-1} \|u\|_{\mathbb{H}_p^{\phi,\gamma+2}(s)} ds.$$

Since the estimate of u^0 is obtained in Step 1, the desired estimate follows from the fractional Gronwall lemma. The theorem is proved. □

5. Proof of Theorem 2.19

Recall $H_p^{\phi,1} \subset H_p^{\delta_0}$ for $\delta_0 \in (1/4, 1]$, and

$$\beta < \left(1 - \frac{1}{4\delta_0}\right) \alpha + \frac{1}{2}, \quad d < 2\delta_0 \left(2 - \frac{(2\beta - 1)^+}{\alpha}\right) =: d_0.$$

Remark 5.1. From the relation $H_p^{\phi,1} \subset H_p^{\delta_0}$, one can deduce $H_p^{\phi,\gamma} \subset H_p^{\delta_0\gamma}$ for $\gamma \geq 0$, and $H_p^{\delta_0\gamma} \subset H_p^{\phi,\gamma}$ for $\gamma \leq 0$. Indeed, first, by [14, Theorem 2.4.6] we have $H_p^{\phi,\gamma} \subset H_p^{\delta_0\gamma}$ for $\gamma \in (0, 1)$. Also, for $\gamma \in (1, 2]$,

$$\begin{aligned}
 \|u\|_{H_p^{\delta_0\gamma}} &= \|(1 - \Delta)^{\delta_0\gamma/2} u\|_{L_p} = \|(1 - \Delta)^{(\delta_0\gamma - \delta_0)/2} (1 - \Delta)^{\delta_0/2} u\|_{L_p} \\
 &\leq N \|(1 - \phi(\Delta))^{(\gamma-1)/2} (1 - \Delta)^{\delta_0/2} u\|_{L_p} \\
 &= N \|(1 - \Delta)^{\delta_0/2} (1 - \phi(\Delta))^{(\gamma-1)/2} u\|_{L_p} \\
 &\leq N \|(1 - \phi(\Delta))^{\gamma/2} u\|_{L_p} = \|u\|_{H_p^{\phi,\gamma}}.
 \end{aligned}$$

Repeating above argument, we get $H_p^{\phi,\gamma} \subset H_p^{\delta_0\gamma}$ for $\gamma \geq 0$. Second, by the duality theorem [14, Theorem 2.2.10], we have $H_p^{\delta_0\gamma} \subset H_p^{\phi,\gamma}$ for $\gamma \leq 0$.

Also recall that the equation

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^{\beta-1}h(u)\dot{W}, \quad t > 0; \quad u(0, \cdot) = 0,$$

can be written as

$$\partial_t^\alpha u = \phi(\Delta)u + f(u) + \partial_t^\beta \sum_{k=1}^\infty \int_0^t g^k(u)dw_t^k, \quad t > 0; \quad u(0, \cdot) = 0,$$

where $g^k(t, x, u) = h(t, x, u)\eta^k(x)$. Therefore, to prove the theorem, it suffices to prove that f and g satisfy conditions in Assumption 2.11. To check this, we first prove some auxiliary results below.

Note that by definition, for any $\gamma > 0$ and smooth function φ , we have

$$\begin{aligned} \mathcal{F}\{(1 - \Delta)^{-\gamma/2}\varphi\}(\xi) &= (1 + |\xi|^2)^{-\gamma/2}\mathcal{F}\{\varphi\}(\xi) \\ &= c\hat{\varphi}(\xi) \int_0^\infty t^{\gamma/2}e^{-t}e^{-t|\xi|^2} \frac{1}{t} dt, \end{aligned} \tag{5.1}$$

where $c = c(\gamma) > 0$. Set

$$R_{\gamma,d}(x) := \int_0^\infty t^{\gamma/2}e^{-t} \bar{p}(t, x) \frac{1}{t} dt,$$

where $\bar{p}(t, x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$. Then,

$$\int_{\mathbb{R}^d} R_{\gamma,d}(x)dx = \int_0^\infty t^{\gamma/2-1}e^{-t} \int_{\mathbb{R}^d} \bar{p}(t, x)dx dt = \int_0^\infty t^{\gamma/2-1}e^{-t} dt < \infty.$$

Therefore, by Fubini's theorem,

$$\mathcal{F}\{R_{\gamma,d}\}(\xi) = c(\gamma, d) \int_0^\infty t^{\gamma/2}e^{-t}e^{-t|\xi|^2} \frac{1}{t} dt.$$

Hence, from (5.1) for any $\gamma > 0$ we have

$$(1 - \Delta)^{-\gamma/2}\varphi = c(\gamma, d) \int_{\mathbb{R}^d} R_{\gamma,d}(x - y)\varphi(y)dy. \tag{5.2}$$

It is known that $R_{\gamma,d}$ decays exponentially at infinity and is comparable to $|x|^{-d+\gamma}$ near $x = 0$ (see [28] or [29]). Thus, for $\gamma < d$ and $2r < \frac{d}{d-\gamma}$, we have $R_{\gamma,d} \in L_{2r}$.

Lemma 5.2. *Assume*

$$k_0 \in \left(\frac{d}{2\delta_0}, \frac{d}{\delta_0} \right), \quad 2 \leq 2r \leq p, \quad 2r < \frac{d}{d - k_0\delta_0}.$$

Let $h = h(x, u)$ be a function of (x, u) and $\xi = \xi(x)$ a function of x such that

$$|h(x, u) - h(x, v)| \leq \xi(x)|u - v|, \quad \forall x \in \mathbb{R}^d, u, v \in \mathbb{R}.$$

If we set $g^k(u) = h(u)\eta^k$ and $g = (g^1, g^2, \dots)$, then for $u, v \in L_p$, we have

$$\|g(u) - g(v)\|_{H_p^{\phi, -k_0}(L_2)} \leq C \|\xi\|_{L_{2s}} \|u - v\|_{L_p},$$

where $s = r/(r - 1)$, and $C = C(r) < \infty$. In particular, if $r = 1$, and $\xi \in L_\infty$, then

$$\|g(u) - g(v)\|_{H_p^{\phi, -k_0}(L_2)} \leq C \|u - v\|_{L_p}.$$

Proof. By Remark 5.1, (5.2) and Parseval’s identity in Hilbert space $L_2(\mathbb{R}^d)$, we have

$$\begin{aligned} \|g(u) - g(v)\|_{H_p^{\phi, -k_0}(L_2)} &\leq \|g(u) - g(v)\|_{H_p^{-k_0\delta_0}(L_2)} \\ &= C \left\| \left(\int_{\mathbb{R}^d} |R_{k_0\delta_0, d}(\cdot - y)|^2 |h(y, u(y)) - h(y, v(y))|^2 dy \right)^{1/2} \right\|_{L_p} \\ &\leq C \left\| \left(\int_{\mathbb{R}^d} |R_{k_0\delta_0, d}(\cdot - y)|^2 |\xi(y)|^2 |u(y) - v(y)|^2 dy \right)^{1/2} \right\|_{L_p}. \end{aligned}$$

Note that $R_{k_0\delta_0} \in L_{2r}$ since $k_0\delta_0 < d$ and $2r < d/(d - k_0\delta_0)$. Hence, by Hölder’s inequality and Minkowski’s inequality, we have

$$\begin{aligned} \|g(u) - g(v)\|_{H_p^{\phi, -k_0}(L_2)} &\leq C \left\| \left(\int_{\mathbb{R}^d} |R_{k_0\delta_0, d}(\cdot - y)|^2 |\xi(y)|^2 |u(y) - v(y)|^2 dy \right)^{1/2} \right\|_{L_p} \\ &\leq C \|\xi\|_{L_{2s}} \left\| \left(\int_{\mathbb{R}^d} |R_{k_0\delta_0, d}(\cdot - y)|^{2r} |u(y) - v(y)|^{2r} dy \right)^{1/2r} \right\|_{L_p} \\ &\leq C \|\xi\|_{L_{2s}} \|R_{k_0\delta_0, d}\|_{L_{2r}} \|u - v\|_{L_p} \leq C \|\xi\|_{L_{2s}} \|u - v\|_{L_p}. \end{aligned}$$

The lemma is proved. □

Proof of Theorem 2.19. As mentioned above, it suffices to prove that the conditions in Theorem 2.12 hold with $\gamma = -k_0 - c_0$. By [14, Theorem 2.4.6], if $v_1 < v_2 < v_3$, then for any $\varepsilon > 0$

$$\|u\|_{H_p^{\phi, v_2}} \leq \varepsilon \|u\|_{H_p^{\phi, v_3}} + N(\varepsilon) \|u\|_{H_p^{\phi, v_1}}, \tag{5.3}$$

where $N(\varepsilon)$ depends on ε, v_1, v_2 and v_3 . Due to (2.19) one can choose κ small enough such that $\gamma + 2 > 0$. Since $\gamma < 0$ and $\gamma + 2 > 0$, by the assumption of f and (5.3), we have for any $\varepsilon > 0$

$$\begin{aligned} \|f(u) - f(v)\|_{H_p^{\phi, \gamma}} &\leq C \|f(u) - f(v)\|_{L_p} \leq C \|u - v\|_{L_p} \\ &\leq \varepsilon \|u - v\|_{H_p^{\phi, \gamma+2}} + N(\varepsilon) \|u - v\|_{H_p^{\phi, \gamma}}. \end{aligned}$$

Therefore it remains to check the conditions for $g(u)$. Let $r = s/(s - 1)$. Then $2r < d/(d - k_0\delta_0)$ due to the assumption on s . Since $\gamma + c_0 = -k_0$, by Lemma 5.2 and (5.3), for any $\varepsilon > 0$, we have

$$\begin{aligned} \|g(u) - g(v)\|_{H_p^{\phi, \gamma+c_0}(l_2)} &\leq C \|\xi\|_{L_{2s}} \|u - v\|_{L_p} \\ &\leq \varepsilon \|u - v\|_{H_p^{\phi, \gamma+2}} + N(\varepsilon) \|u - v\|_{H_p^{\phi, \gamma}}. \end{aligned}$$

Hence, the condition for g is also fulfilled. Furthermore, by inspecting the proof of Lemma 5.2, one can easily check

$$\|g(0)\|_{\mathbb{H}_p^{-k_0}(T, l_2)} \leq C \|h(0)\|_{\mathbb{L}_p(T)}.$$

Therefore, we finish the proof of the theorem. □

Acknowledgements

The authors are very grateful to the referee for valuable comments and suggestions. We could considerably improve the early version of this article due to the referee.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

A. Auxiliary results

In this section, we obtain some sharp upper bounds of space-time fractional derivatives of the fundamental solution $q(t, x)$ related to the equation

$$\partial_t^\alpha u = \phi(\Delta)u, \quad t > 0; \quad u(0, \cdot) = u_0.$$

First we record some elementary facts on Bernstein functions.

Lemma A.1. *Let ϕ be a Bernstein function satisfying Assumption 3.1.*

(i) *There exists a constant $c = c(\gamma, \kappa_0, \delta_0)$ such that for any $\lambda > 0$,*

$$\int_{\lambda^{-1}}^\infty r^{-1} \phi(r^{-2}) dr \leq c \phi(\lambda^2). \tag{A.1}$$

(ii) *For any $\gamma \in (0, 1)$, the function $\phi^\gamma := (\phi(\cdot))^\gamma$ is also a Bernstein function with no drift, and it satisfies Assumption 3.1 with $\gamma\delta_0$ and κ_0^γ , in place of δ_0 and κ_0 , respectively.*

(iii) Let μ_γ be the Lévy measure of ϕ^γ (i.e., $\phi^\gamma(\lambda) = \int_{(0,\infty)}(1 - e^{-\lambda t})\mu_\gamma(dt)$), and set

$$j_{\gamma,d}(r) := \int_0^\infty (4\pi t)^{-d/2} \exp(-r^2/4t)\mu_\gamma(dt), \quad r > 0. \tag{A.2}$$

Then

$$j_{\gamma,d}(r) \leq c(d) r^{-d} \phi(r^{-2})^\gamma, \quad \forall r > 0, \tag{A.3}$$

and for any $f \in C_b^2(\mathbb{R}^d)$ and $r > 0$, it holds that

$$\phi(\Delta)^\gamma f(\cdot)(x) = \int_{\mathbb{R}^d} (f(x + y) - f(x) - \nabla f(x) \cdot y \mathbf{1}_{|y| \leq r}) j_{\gamma,d}(|y|) dy \tag{A.4}$$

Proof. (i) By (3.4) this and the change of variables,

$$\begin{aligned} \int_{\lambda^{-1}}^\infty r^{-1} \phi(r^{-2}) dr &= \int_1^\infty r^{-1} \phi(\lambda^2 r^{-2}) dr = \int_1^\infty r^{-1} \phi(\lambda^2 r^{-2}) \frac{\phi(\lambda^2)}{\phi(\lambda^2)} dr \\ &\leq c \int_1^\infty r^{-1-2\delta_0} dr \phi(\lambda^2) = c \phi(\lambda^2). \end{aligned}$$

(ii) ϕ^γ is a Bernstein function due to [42, Corollary 3.8 (iii)], and (3.3) easily yields

$$\kappa_0^\gamma \left(\frac{R}{r}\right)^{\gamma\delta_0} \leq \frac{\phi(R)^\gamma}{\phi(r)^\gamma}, \quad 0 < r < R < \infty.$$

If we denote drift of ϕ^γ by b_γ (see (2.4)), it follows that

$$\lim_{\lambda \rightarrow \infty} \frac{\phi(\lambda)}{\lambda} = b, \quad \lim_{\lambda \rightarrow \infty} \frac{\phi(\lambda)^\gamma}{\lambda} = b_\gamma.$$

Hence, we have

$$b_\gamma = \lim_{\lambda \rightarrow \infty} \frac{\phi(\lambda)^\gamma}{\lambda} = \lim_{\lambda \rightarrow \infty} \left(\frac{\phi(\lambda)}{\lambda}\right)^\gamma \lambda^{\gamma-1} = 0.$$

(iii) (A.3) follows from [24, Lemma 3.3] (recall that ϕ^γ is a Bernstein function with no drift), and the second assertion is a consequence of (2.7). \square

Recall that $p(t, x)$ is the transition density of the subordinate Brownian motion X_t with characteristic exponent $\phi(|\xi|^2)$. Also, for any $t > 0$, and $x \in \mathbb{R}^d$,

$$\begin{aligned} p(t, x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-t\phi(|\xi|^2)} d\xi \\ &= \int_{(0,\infty)} (4\pi s)^{-d/2} \exp\left(-\frac{|x|^2}{4s}\right) \eta_t(ds) \end{aligned} \tag{A.5}$$

where $\eta_t(ds)$ is the distribution of S_t (see [2, Sect. 5.3.1]). Thus X_t is rotationally invariant.

Lemma A.2. (i) *There exists a constant $C = C(d, \delta_0, \kappa_0)$ such that for $(t, x) \in (0, \infty) \times \mathbb{R}^d$,*

$$p(t, x) \leq C \left((\phi^{-1}(t^{-1}))^{d/2} \wedge t \frac{\phi(|x|^{-2})}{|x|^d} \right).$$

(ii) *For any $m \in \mathbb{N}$, there exists a constant $C = C(d, \delta_0, \kappa_0, m)$ so that for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$,*

$$|D_x^m p(t, x)| \leq C \sum_{m-2n \geq 0, n \in \mathbb{N}_0} |x|^{m-2n} \left((\phi^{-1}(t^{-1}))^{d/2+m-n} \wedge t \frac{\phi(|x|^{-2})}{|x|^{d+2(m-n)}} \right).$$

Proof. See [27, Lemma 3.4, Lemma 3.6]. □

The following lemma is an extension of [24, Lemma 4.2]. The main difference is that our estimate holds for all $t > 0$. Such result is needed for us to prove estimates of solutions to SPDEs (see, e.g., (3.2)).

Lemma A.3. *Let $\gamma \in (0, 1)$ and $m \in \mathbb{N}_0$. Then for any $(t, x) \in (0, \infty) \times \mathbb{R}^d$,*

$$\begin{aligned} & |\phi(\Delta)^\gamma D_x^m p(t, \cdot)(x)| \\ & \leq C(d, \delta_0, \kappa_0, m, \gamma) \left(t^{-\gamma} (\phi^{-1}(t^{-1}))^{(d+m)/2} \wedge \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} \right). \end{aligned}$$

Proof. Note first that for any given $a > 0$,

$$s^a e^{-s} \leq c(a) e^{-s/2}, \quad \forall s > 0.$$

Also note that by (3.3), if $a^2 \geq \phi^{-1}(t^{-1})$, then

$$\kappa_0 \left(\frac{a^2}{\phi^{-1}(t^{-1})} \right)^{\delta_0} \leq \frac{\phi(a^2)}{t^{-1}} = t\phi(a^2).$$

Therefore, by (A.5),

$$\begin{aligned} |\phi(\Delta)^\gamma D_x^m p(t, x)| &= \left| \mathcal{F}^{-1} [\mathcal{F}(\phi(\Delta)^\gamma D_x^m p(t, \cdot))](\xi) \right|(x) \\ &\leq C \int_{\mathbb{R}^d} \left| t^{-\gamma} \left(t\phi(|\xi|^2) \right)^\gamma e^{-t\phi(|\xi|^2)} \right| |\xi|^m d\xi \\ &\leq C t^{-\gamma} \left(\int_{|\xi|^2 > \phi^{-1}(t^{-1})} |\xi|^m e^{-t\frac{\phi(|\xi|^2)}{2}} d\xi + \int_{|\xi|^2 \leq \phi^{-1}(t^{-1})} |\xi|^m d\xi \right) \\ &\leq C t^{-\gamma} \left(\int_{|\xi|^2 > \phi^{-1}(t^{-1})} |\xi|^m e^{-\frac{\kappa_0}{2} \left(\frac{|\xi|^2}{\phi^{-1}(t^{-1})} \right)^{\delta_0}} d\xi + \left(\phi^{-1}(t^{-1}) \right)^{\frac{d+m}{2}} \right) \\ &\leq C t^{-\gamma} \left(\phi^{-1}(t^{-1}) \right)^{\frac{d+m}{2}} \left(\int_{|\xi|^2 > 1} e^{-\frac{\kappa_0}{4} |\xi|^{2\delta_0}} d\xi + 1 \right) \end{aligned}$$

$$\leq C t^{-\gamma} \left(\phi^{-1}(t^{-1}) \right)^{\frac{d+m}{2}}.$$

Hence, to finish the proof, we may assume $t^{-\gamma} (\phi^{-1}(t^{-1}))^{\frac{d+m}{2}} \geq \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}$ (equivalently, $t\phi(|x|^{-2}) \leq 1$) and prove

$$|\phi(\Delta)^\gamma D_x^m p(t, \cdot)(x)| \leq C \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}.$$

By (A.4) with $r = |x|/2$,

$$\begin{aligned} & |\phi(\Delta)^\gamma D_x^m p(t, \cdot)(x)| \\ &= \left| \int_{\mathbb{R}^d} \left(D_x^m p(t, x+y) - D_x^m p(t, x) - \nabla D_x^m p(t, x) \cdot y \mathbf{1}_{|y| \leq \frac{|x|}{2}} \right) j_{\gamma,d}(|y|) dy \right| \\ &\leq |D_x^m p(t, x)| \int_{|y| > |x|/2} j_{\gamma,d}(|y|) dy \\ &\quad + \left| \int_{|y| > |x|/2} D_x^m p(t, x+y) j_{\gamma,d}(|y|) dy \right| \\ &\quad + \int_{|x|/2 > |y|} \int_0^1 \left| D_x^{m+1} p(t, x+sy) - D_x^{m+1} p(t, x) \right| |y| j_{\gamma,d}(|y|) ds dy \\ &=: |D_x^m p(t, x)| \times I + II + III. \end{aligned}$$

By (A.3), (A.1) with ϕ^γ , and (3.4), we have

$$I \leq C \int_{r > |x|/2} r^{-1} \phi(r^{-2})^\gamma dr \leq C \phi(4|x|^{-2})^\gamma \leq C \phi(|x|^{-2})^\gamma.$$

This together with Lemma A.2 (ii) yields (recall we assume $t\phi(|x|^{-2}) \leq 1$)

$$|D_x^m p(t, x)| \times I \leq C t \frac{\phi(|x|^{-2})^{1+\gamma}}{|x|^{d+m}} \leq C \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}.$$

For III, by the fundamental theorem of calculus and Lemma A.2 (ii),

$$\begin{aligned} III &\leq C(d) \int_{|x|/2 > |y|} \int_0^1 \int_0^1 \left| D_x^{m+2} p(t, x+usy) \right| |y|^2 j_{\gamma,d}(|y|) dudsd y \\ &\leq C \int_{|x|/2 > |y|} \int_0^1 \int_0^1 t \frac{\phi(|x+usy|^{-2})}{|x+usy|^{d+m+2}} |y|^2 j_{\gamma,d}(|y|) dudsd y \\ &\leq C t \frac{\phi(|x|^{-2})}{|x|^{d+m+2}} \int_{|x|/2 > |y|} |y|^2 j_{\gamma,d}(|y|) dy. \end{aligned}$$

For the last inequality above, we used $|x+usy| \geq |x|/2$. By (A.3), and (3.4) with $r = |x|^{-2}$ and $R = \rho^{-2}$,

$$\int_{|x|/2 > |y|} |y|^2 j_{\gamma,d}(|y|) dy \leq C \int_0^{|x|} \rho \phi(\rho^{-2})^\gamma d\rho$$

$$\begin{aligned} &\leq C|x|^{2\gamma} \phi(|x|^{-2})^\gamma \int_0^{|x|} \rho^{1-2\gamma} d\rho \\ &\leq C|x|^2 \phi(|x|^{-2})^\gamma. \end{aligned}$$

Therefore, it follows that for $t\phi(|x|^{-2}) \leq 1$,

$$III \leq Ct \frac{\phi(|x|^{-2})^{1+\gamma}}{|x|^{d+m}} \leq C \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}.$$

Now we estimate II . By using the integration by parts m -times, we have

$$\begin{aligned} II \leq &\sum_{k=0}^{m-1} \int_{|y|=\frac{|x|}{2}} \left| \left(\frac{d^k}{d\rho^k} j_{\gamma,d} \right) (|y|) D_x^{m-1-k} p(t, x+y) \right| dS \\ &+ \int_{|y|>\frac{|x|}{2}} \left| \left(\frac{d^m}{d\rho^m} j_{\gamma,d} \right) (|y|) p(t, x+y) \right| dy. \end{aligned}$$

Differentiating $j_{\gamma,d}(\rho)$, and then using (A.2) and (A.3), for $k \in \mathbb{N}_0$ we get

$$\left| \frac{d^k}{d\rho^k} j_{\gamma,d}(\rho) \right| \leq C \sum_{k-2l \geq 0, l \in \mathbb{N}_0} \rho^{k-2l} |j_{\gamma,d+2(k-l)}(\rho)| \leq C\rho^{-d-k} \phi(\rho^{-2})^\gamma.$$

This and Lemma A.2 (ii) yield that

$$\begin{aligned} II \leq &C \sum_{k=0}^{m-1} \left(|x|^{d-1} |x|^{-d-k} |x|^{-d-m+1+k} t \phi(|x|^{-2})^{\gamma+1} \right) + C \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} \\ &\leq C \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}. \end{aligned}$$

for $t\phi(|x|^{-2}) \leq 1$. Hence, the lemma is proved. □

Below we provide the proof of Lemma 3.2. The proof is mainly based on [27, Lemma 3.7, Lemma 3.8].

Proof of Lemma 3.2. (i) See [27, Lemma 3.7 (iii)].

(ii) See [27, Lemma 3.8] for (3.6) with arbitrary β and for (3.7) when $\beta \notin \mathbb{N}$. Hence, we only prove (3.7) when $\beta \in \mathbb{N}$. Let $\beta \in \mathbb{N}$, then by [27, Lemma 3.8], we have

$$\begin{aligned} |q_{\alpha,\beta}(t, x)| &\leq C \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} r t^{-\alpha-\beta} dr \\ &\leq C \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} t^{-\beta} dr. \end{aligned}$$

For the last inequality, we used $rt^{-\alpha} \leq 2$ whenever $r \leq 2t^\alpha$.

(iii) We follow the proof of [27, Lemma 3.8]. By [27, Lemma 3.7], there exist constants $c, C > 0$ depending only on α, β such that

$$|\varphi_{\alpha,\beta}(t, r)| \leq Ct^{-\beta} e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} \tag{A.6}$$

for $rt^{-\alpha} \geq 1$, and

$$|\varphi_{\alpha,\beta}(t, r)| \leq \begin{cases} Crt^{-\alpha-\beta} & : \beta \in \mathbb{N} \\ Ct^{-\beta} & : \beta \notin \mathbb{N} \end{cases} \tag{A.7}$$

for $rt^{-\alpha} \leq 1$. Therefore, we have

$$\int_0^\infty |\varphi_{\alpha,\beta}(t, r)| dr < \infty. \tag{A.8}$$

Let $x \in \mathbb{R}^d \setminus \{0\}$. Then for any $r > 0$ and $y \neq 0$ sufficiently close to x , we have

$$|\phi(\Delta)^\gamma D^\sigma p(r, y)| \leq C(\phi, x, d, m, \gamma), \quad |\sigma| \leq m$$

due to Lemma A.3. Using (A.8) and the dominated convergence theorem, we get

$$D_x^m q_{\alpha,\beta}^\gamma(t, x) = \int_0^\infty \phi(\Delta)^\gamma D_x^m p(r, x) \varphi_{\alpha,\beta}(t, r) dr.$$

Hence, by (A.6) and (A.7) (also recall $rt^{-\alpha} \leq 2$ whenever $r \leq 2t^\alpha$),

$$\begin{aligned} |D_x^m q_{\alpha,\beta}^\gamma(t, x)| &\leq C \int_0^{t^\alpha} |\phi(\Delta)^\gamma D_x^m p(r, x)| t^{-\beta} dr \\ &\quad + C \int_{t^\alpha}^\infty |\phi(\Delta)^\gamma D_x^m p(r, x)| t^{-\beta} e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} dr \\ &=: I + II. \end{aligned} \tag{A.9}$$

By Lemma A.3,

$$I \leq C \int_0^{t^\alpha} t^{-\beta} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} dr \leq Ct^{\alpha-\beta} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}.$$

Also, by the change of variables $rt^{-\alpha} \rightarrow r$,

$$\begin{aligned} II &\leq Ct^{-\beta} \int_{t^\alpha}^\infty \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} dr \\ &\leq Ct^{\alpha-\beta} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} \int_1^\infty e^{-cr^{1/(1-\alpha)}} dr \\ &\leq Ct^{\alpha-\beta} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}}. \end{aligned}$$

Hence, (3.8) is proved.

Next we prove (3.9). Assume $t^\alpha \phi(|x|^{-2}) \geq 1$. Again we consider I and II defined in (A.9). For I we have

$$I = t^{-\beta} \int_0^{(\phi(|x|^{-2}))^{-1}} |\phi(\Delta)^\gamma D_x^m p(r, x)| dr$$

$$\begin{aligned}
 &+ t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{t^\alpha} |\phi(\Delta)^\gamma D_x^m p(r, x)| t^{-\beta} dr \\
 &=: I_1 + I_2.
 \end{aligned}$$

By Lemma A.3 (recall that $t^\alpha \phi(|x|^{-2}) \geq 1$),

$$\begin{aligned}
 I_1 &\leq C t^{-\beta} \int_0^{(\phi(|x|^{-2}))^{-1}} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} dr \\
 &= C t^{-\beta} \frac{\phi(|x|^{-2})^{\gamma-1}}{|x|^{d+m}} = C t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{2(\phi(|x|^{-2}))^{-1}} \frac{\phi(|x|^{-2})^\gamma}{|x|^{d+m}} dr
 \end{aligned}$$

Note that if $r \leq 2(\phi(|x|^{-2}))^{-1}$, then by (3.4)

$$|x|^{-2} \leq \phi^{-1}(2r^{-1}) \leq \left(\frac{2}{k_0}\right)^{1/\delta_0} \phi^{-1}(r^{-1}). \tag{A.10}$$

Thus, using $r \leq 2(\phi(|x|^{-2}))^{-1}$, we get

$$\begin{aligned}
 I_1 &\leq C t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{2(\phi(|x|^{-2}))^{-1}} (\phi^{-1}(r^{-1}))^{(d+m)/2} r^{-\gamma} dr \\
 &\leq C t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} r^{-\gamma} dr.
 \end{aligned}$$

We also get, by Lemma A.3,

$$I_2 \leq C t^{-\beta} \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} r^{-\gamma} dr.$$

Thus, I is handled. Next we estimate II . By (3.4), we find that

$$\phi^{-1}(r^{-1}) \leq t^\alpha r^{-1} \phi^{-1}(t^{-\alpha}), \quad t^\alpha \leq r.$$

Therefore, by Lemma A.3 and the change of variables $rt^{-\alpha} \rightarrow r$,

$$\begin{aligned}
 II &\leq C t^{-\beta} \int_{t^\alpha}^\infty r^{-\gamma} (\phi^{-1}(r^{-1}))^{(d+m)/2} e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} dr \\
 &\leq C t^{-\beta} \int_{t^\alpha}^\infty r^{-\gamma} (t^\alpha r^{-1} \phi^{-1}(t^{-\alpha}))^{(d+m)/2} e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} dr \\
 &= C t^{(1-\gamma)\alpha-\beta} (\phi^{-1}(t^{-\alpha}))^{(d+m)/2} \int_1^\infty r^{-\gamma-(d+m)/2} e^{-cr^{1/(1-\alpha)}} dr \\
 &\leq C t^{(1-\gamma)\alpha-\beta} (\phi^{-1}(t^{-\alpha}))^{(d+m)/2}. \tag{A.11}
 \end{aligned}$$

As (A.10), if $r \leq 2t^\alpha$, then by (3.4)

$$\phi^{-1}(t^{-\alpha}) \leq \phi^{-1}(2r^{-1}) \leq \left(\frac{2}{\kappa_0}\right)^{1/\delta_0} \phi^{-1}(r^{-1}).$$

Therefore,

$$\begin{aligned}
 t^{(1-\gamma)\alpha-\beta}(\phi^{-1}(t^{-\alpha}))^{(d+m)/2} &\leq C \int_{t^\alpha}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} r^{-\gamma} t^{-\beta} dr \\
 &\leq C \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2} r^{-\gamma} t^{-\beta} dr.
 \end{aligned}$$

This and (A.11) take care of II, and consequently (3.9) is proved.

(iv) See [27, Corollary 3.9] for (3.10). We prove (3.11). By (3.8), (3.9), Fubini’s theorem, and (A.1),

$$\begin{aligned}
 \int_{\mathbb{R}^d} |q_{\alpha,\beta}^\gamma(t, x)| dx &= \int_{|x| \geq (\phi^{-1}(t^{-\alpha}))^{-\frac{1}{2}}} |q_{\alpha,\beta}^\gamma(t, x)| dx \\
 &\quad + \int_{|x| < (\phi^{-1}(t^{-\alpha}))^{-\frac{1}{2}}} |q_{\alpha,\beta}^\gamma(t, x)| dx \\
 &\leq C \int_{|x| \geq (\phi^{-1}(t^{-\alpha}))^{-\frac{1}{2}}} t^{\alpha-\beta} \frac{\phi(|x|^{-2})^\gamma}{|x|^d} dx \\
 &\quad + C \int_{|x| < (\phi^{-1}(t^{-\alpha}))^{-\frac{1}{2}}} \int_{(\phi(|x|^{-2}))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{d/2} r^{-\gamma} t^{-\beta} dr dx \\
 &\leq C \int_{r \geq (\phi^{-1}(t^{-\alpha}))^{-\frac{1}{2}}} t^{\alpha-\beta} \frac{\phi(r^{-2})^\gamma}{r} dr \\
 &\quad + C \int_0^{2t^\alpha} \int_{(\phi(|x|^{-2}))^{-1} \leq r} (\phi^{-1}(r^{-1}))^{d/2} r^{-\gamma} t^{-\beta} dx dr \\
 &\leq Ct^{(1-\gamma)\alpha-\beta} + C \int_0^{2t^\alpha} r^{-\gamma} t^{-\beta} dr \leq Ct^{(1-\gamma)\alpha-\beta}.
 \end{aligned}$$

(v) By (3.24) in [27] (or see (34) of [18]),

$$\int_0^\infty e^{-sr} \varphi_{\alpha,\beta}(t, r) dr = t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}(-st^\alpha).$$

Hence, by Fubini’s theorem and (A.5) for $\gamma \in (0, 1)$,

$$\begin{aligned}
 \mathcal{F}_d(q_{\alpha,\beta}^\gamma)(t, \xi) &= \int_0^\infty \varphi_{\alpha,\beta}(t, r) \left[\int_{\mathbb{R}^d} e^{-ix \cdot \xi} \phi(\Delta)^\gamma p(r, x) dx \right] dr \\
 &= - \int_0^\infty \varphi_{\alpha,\beta}(t, r) \phi(|\xi|^2)^\gamma e^{-r\phi(|\xi|^2)} dr \\
 &= -t^{\alpha-\beta} \phi(|\xi|)^\gamma E_{\alpha,1-\beta+\alpha}(-t^\alpha \phi(|\xi|^2)).
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 \mathcal{F}_d(q_{\alpha,\beta})(t, \xi) &= \int_0^\infty \varphi_{\alpha,\beta}(t, r) \left[\int_{\mathbb{R}^d} e^{-ix \cdot \xi} p(r, x) dx \right] dr \\
 &= t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}(-t^\alpha \phi(|\xi|^2)).
 \end{aligned}$$

Thus (v) is also proved. □

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Accepted: 8 May 2022