A SOBOLEV SPACE THEORY FOR THE TIME-FRACTIONAL STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES

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ABSTRACT. We present an L_p-theory ($p \geq 2$) for semi-linear time-fractional stochastic partial differential equations driven by Lévy processes of the type

$$\partial_t^\alpha u = \sum_{i,j=1}^d a^{ij} u_{x^i x^j} + f(u) + \sum_{k=1}^\infty \partial_t^\beta \int_0^t (\sum_{i=1}^d \mu^{ik} u_{x^i} + g^k(u)) dZ_s^k$$

given with nonzero intial data. Here ∂_t^α and ∂_t^β are the Caputo fractional derivatives,

$$0 < \alpha < 2, \qquad \beta < \alpha + 1/p,$$

and $\{Z_t^k : k = 1, 2, \cdots\}$ is a sequence of independent Lévy processes. The coefficients are random functions depending on (t, x). We prove the uniqueness and existence results in Sobolev spaces, and obtain the maximal regularity of the solution.

As an application, we also obtain an L_p-regularity theory of the equation

$$\partial_t^\alpha u = \sum_{i,j=1}^d a^{ij} u_{x^i x^j} + f(u) + \partial_t^\beta \int_0^t h(u) d\mathcal{Z}_s,$$

where $\dot{\mathcal{Z}}_t$ is a multi-dimensional Lévy space-time white noise with the space dimension $d < 4 - \frac{2(2\beta - 2/p)^+}{\alpha}$. In particular, if $\beta < \alpha/4 + 1/p$ then one can take $d = 1, 2, 3$.

1. INTRODUCTION

Let $\{W_t^k : k \in 1, 2, \cdots\}$ and $\{Z_t^k : k = 1, 2, \cdots\}$ be sequences of independent one dimensional Brownian motions and d_1-dimensional Lévy processes respectively. In this article we present an L_p-theory ($p \geq 2$) for the time-fractional stochastic partial differential equation (SPDE) defined on $\mathbb{R}^d$:

$$\begin{aligned} \partial_t^\alpha u =& a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u) \\ &+ \partial_t^{\beta_1} \int_0^t \left(\mu^{ik} u_{x^i} + \nu^k u + g^k(u) \right) dW_s^k \\ &+ \partial_t^{\beta_2} \int_0^t \left(\bar{\mu}^{irk} u_{x^i} + \bar{\nu}^{rk} u + h^{rk}(u) \right) dZ_s^{rk}, \quad t > 0, \\ & u(0, \cdot) = u_0, \quad 1_{\alpha > 1} \partial_t u(0, \cdot) = 1_{\alpha > 1} v_0. \end{aligned} \tag{1.1}$$

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Here $\partial_t^\alpha, \partial_t^{\beta_1}, \partial_t^{\beta_2}$ are the Caputo fractional derivatives,

$$\alpha \in (0,2), \quad \beta_1 < \alpha + 1/2, \quad \beta_2 < \alpha + 1/p, \tag{1.2}$$

and Einstein's summation convention is used in (1.1) for the repeated indices $i, j \in \{1, 2, \cdots, d\}$, $r \in \{1, 2, \cdots, d_1\}$ and $k = 1, 2, \cdots$. The coefficients depend on (ω, t, x) and initial data depend on (ω, x). The conditions on β_1 and β_2 in (1.2) are necessary and will be discussed later (see Remark 2.9).

Equation (1.1) is understood by its integral form (see Defintion 2.7), and this type of SPDE naturally arises, for instance, when one describes the random effects on transport of particles subject to sticking and trapping or particles in medium with thermal memory. See [4] for a detailed derivation of such equations. Note that if $\alpha = \beta_1 = \beta_2 = 1$, then (1.1) becomes the classical second-order parabolic SPDE.

This article is a natural continuation of [10], which deals with the equation driven by Wiener processes. We extend the result of [10] to more general equation, that is, equation (1.1). Furthermore, unlike in [10], we impose non-zero initial data. Actually, even for deterministic initial-value proble

$$\partial_t^\alpha u = \Delta u, \quad t > 0\,; \quad u(0,\cdot) = u_0, \ 1_{\alpha>1}\partial_t u(0) = 1_{\alpha>1} v_0,$$

our result is partially new and an extension of [23, Theorem 3.1], which is based on the semi-group approach and requires some extra algebraic conditions such as $\alpha \notin \{\frac{1}{p}, 1 + \frac{1}{p}\}$. Our approach is based on Littlewood-Paley theory.

To explain a technical difference between the equation with Wiener processes and the equation with Lévy processes, let us consider the model equation

$$\partial_t^\alpha u = \Delta u + \partial_t^\beta \int_0^t h(s)\, dX_t, \ \ t > 0 \quad ; \quad u(0) = 1_{\alpha>1} u_t(0) = 0.$$

It turns out that if X_t is a Wiener process, then for any $n \geq 0$ and $p \geq 2$, we have

$$\|D_x^n u\|_{\mathbb{L}_p(T)}^p \leq N \left\| \left(\int_0^t \left| \left(D_x^n D_t^{\beta-\alpha} p(t-s,\cdot)\right) *_x h(s,\cdot) \right|^2 ds \right)^{1/2} \right\|_{\mathbb{L}_p(T)}^p, \tag{1.3}$$

where $\mathbb{L}_p = L_p(\Omega \times [0,T]; L_p(\mathbb{R}^d))$, and $p(t,x)$ is the fundamental solution to the fractional heat equation $\partial_t^\alpha v = \Delta v$. On the other hand, if X_t is a Lévy process, then we have

$$\|D_x^n u\|_{\mathbb{L}_p(T)}^p \leq N \left\| \int_0^t \left| \left(D_x^n D_t^{\beta-\alpha} p(t-s,\cdot)\right) *_x h(s,\cdot) \right|^p ds \right\|_{L_1(\Omega\times[0,T];L_1(\mathbb{R}^d))}. \tag{1.4}$$

A sharp estimate of the right hand side of (1.3) is introduced in [10], and in this article we obtain a sharp upper bound of the right hand side of (1.4) with the help of Littlewood-Paley theory in harmonic analysis.

Below we introduce some related results. To the best of our knowledge, the regularity result for the time fractional SPDE was firstly introduced in [5, 6, 7]. The authors in [5, 6, 7] applied H^∞-functional calculus technique to obtain a maximal regularity for the mild solution to the integral equation

$$U(t) + \int_0^t (t-s)^{\alpha-1} AU(s) ds = \int_0^t (t-s)^{\beta-1} G(s) dW_s, \tag{1.5}$$

where W_t is a Brownian motion, and A is the generator of a bounded analytic semigroup and is assumed to admit a bounded H^∞-calculus on L_p. Quite recently, non-linear SPDE of type (1.5) with non-linear term $A(U)$ in place of AU was studied in [15] in the Gelfand triple setting with the restriction $\alpha < 1$ and $\beta < (\alpha + 1/2) \vee 1$.

With regard to equation (1.1), an L_2-theory was introduced in [4] for the equation driven only by Wiener processes, and the result of [4] was extended in [10] for $p \geq 2$. The zero initial condition is assumed in both [4] and [10].

Actually equation (1.1) can be written in the integral form like (1.5), and it is much general than (1.5) in the sense that it involves multiplicative noises and random operators depending also on (t,x) together with non-zero initial data. We do not impose unnecessary algebraic conditions on α, β_1, β_2, and most importantly our equation is driven by more general processes, that is Lévy processes. However our results do not cover those in [5, 6, 7, 15] because the operator A can belong to quite large class of operators.

For the deterministic counterpart of our result we refer e.g to [8, 11, 23]. We also refer to [3, 12, 14] for the classical case $\alpha = \beta_1 = \beta_2 = 1$.

This article is organized as follows. In Section 2 we introduce stochastic calculus related to Lévy processes, preliminary results on the fractional calculus, and some properties of the solution space, and we present our main result, Theorem 2.15. In Section 3 we use Littlewood-Paley theory to obtain key estimates for solutions. In Section 4 we prove our main result. In Section 5 we give an application to SPDEs driven by Lévy space-time white noise.

Finally we introduce notation used in this article. We use ":=" to denote a definition. As usual, $\mathbb{R}^d$ stands for the d-dimensional Euclidean space of points $x = (x^1, \ldots, x^d)$. $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_+ = \{0\} \cup \mathbb{N}$. For $i = 1, 2, \cdots, d$ and multi-index $\mathfrak{a} = (\mathfrak{a}_1, \cdots, \mathfrak{a}_d)$, where $\mathfrak{a}_i \in \mathbb{N}_+$, we set

$$D_i u = u_{x^i} = \frac{\partial}{\partial x^i} u, \quad D^{\mathfrak{a}} u = D_1^{\mathfrak{a}_1} \ldots D_d^{\mathfrak{a}_d} u, \quad |\mathfrak{a}| = \mathfrak{a}_1 + \cdots + \mathfrak{a}_d.$$

We also use D_x^m or D^m to denote arbitarry m-th order partial derivative with respect to x. For $a, b \in \mathbb{R}$, $a \vee b := \max(a,b)$ and $a^+ := a \vee 0$. By $\mathcal{F}(f)$ or $\hat{f}$ we denote the Fourier transform of f. $C_c^\infty(\mathbb{R}^d)$ denotes the set of infinitely differentiable functions with compact support in $\mathbb{R}^d$, $\mathcal{S}(\mathbb{R}^d)$ is the class of Schwartz functions on $\mathbb{R}^d$, and $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ is the class of tempered distributions. For $p \in [1, \infty]$, a measure space $(X, \mathcal{A}, \mu)$, a normed vector space B with norm $\|\cdot\|_B$, $L_p(X, \mathcal{A}; B)$ is the set of B-valued $\bar{\mathcal{A}}$-measurable functions f satisfying

$$\|f\|_{L_p(X,\mathcal{A};B)} = \left(\int_X \|f(x)\|_B^p d\mu \right)^{1/p},$$

where $\bar{\mathcal{A}}$ is the completion of $\mathcal{A}$ with respect to μ. We say X is a version of Y in B if $\|X - Y\|_B = 0$. If we write $N = N(a, b, \cdots)$, this means that the constant N depends only on $a, b, \cdots$. Throughout the article, for functions depending on (ω, t, x), the argument $\omega \in \Omega$ will be usually omitted.

2. MAIN RESULTS

First we introduce some definitions and facts related to the fractional calculus. For more detail, see e.g. [1, 17, 19, 20]. For $\alpha > 0$ and $\varphi \in L_1((0,T))$, the Riemann-Liouville fractional integral of order α is defined by

$$I_t^\alpha \varphi(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds, \quad t \leq T.$$

For any $p \in [1, \infty]$, we easily have

$$\|I^\alpha \varphi\|_{L_p((0,T))} \leq N(\alpha, p, T) \|\varphi\|_{L_p((0,T))}. \tag{2.1}$$

It is also easy to check that if $\varphi \in L_p((0,T);B)$ and $\alpha > 1/p$ then $I_t^\alpha \varphi(t)$ is a continuous function satisfying $I_t^\alpha \varphi(0) = 0$. In particular if φ is bounded, then $I_t^\alpha \varphi(t)$ is continuous for any $\alpha > 0$. The similar statements hold if $\varphi(t)$ is an $L_p(\mathbb{R}^d)$-valued (or Banach space-valued) function.

Let n be the integer such that $n-1 \leq \alpha < n$. If φ is $(n-1)$-times differentiable, and $(\frac{d}{dt})^{n-1} I_t^{n-\alpha}\varphi$ is absolutely continuous on $[0,T]$, then the Riemann-Liouville fractional derivative $D_t^\alpha \varphi$ and the Caputo fractional derivative $\partial_t^\alpha \varphi$ are defined by

$$D_t^\alpha \varphi := \left(\frac{d}{dt}\right)^n \left(I_t^{n-\alpha}\varphi\right),$$

$$\partial_t^\alpha \varphi := D_t^\alpha \left(\varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!}\varphi^{(k)}(0)\right). \tag{2.2}$$

One can easily check that for any $\alpha, \beta \geq 0$,

$$I^{\alpha+\beta}\varphi(t) = I^\alpha I^\beta \varphi(t), \quad D^\alpha D^\beta \varphi = D^{\alpha+\beta}\varphi, \tag{2.3}$$

and

$$D^\alpha I^\beta \varphi = \begin{cases} D^{\alpha-\beta}\varphi & \text{if } \alpha > \beta \\ I^{\beta-\alpha}\varphi & \text{if } \alpha \leq \beta \end{cases}. \tag{2.4}$$

For $p > 1$ and $\gamma \in \mathbb{R}$, let $H_p^\gamma = H_p^\gamma(\mathbb{R}^d)$ denote the class of all tempered distributions u on $\mathbb{R}^d$ such that

$$\|u\|_{H_p^\gamma} := \|(1-\Delta)^{\gamma/2}u\|_{L_p} < \infty, \tag{2.5}$$

where

$$(1-\Delta)^{\gamma/2}u = \mathcal{F}^{-1}\left((1+|\xi|^2)^{\gamma/2}\mathcal{F}(u)\right).$$

The action of u on $\phi \in \mathcal{S}(\mathbb{R}^d)$, which is denoted by (u,ϕ), is defined by

$$(u,\phi) := ((1-\Delta)^{\gamma/2}u, (1-\Delta)^{-\gamma/2}\phi). \tag{2.6}$$

It is well-known that if $\gamma = 0, 1, 2, \cdots$, then

$$H_p^\gamma = W_p^\gamma := \{u : D^{\mathfrak{a}}u \in L_p(\mathbb{R}^d), \ |\mathfrak{a}| \leq \gamma\}, \qquad H_p^{-\gamma} = (H_{p/(p-1)}^\gamma)^*.$$

Let l_2 denote the set of all sequences $a = (a^1, a^2, \cdots)$ such that

$$|a|_{l_2} := \left(\sum_{k=1}^\infty |a^k|^2\right)^{1/2} < \infty.$$

By $H_p^\gamma(l_2) = H_p^\gamma(\mathbb{R}^d, l_2)$ we denote the class of all l_2-valued tempered distributions $v = (v^1, v^2, \cdots)$ on $\mathbb{R}^d$ such that

$$\|v\|_{H_p^\gamma(l_2)} := \||(1-\Delta)^{\gamma/2}v|_{l_2}\|_{L_p} < \infty.$$

Also we write $h = (h^1, \ldots, h^{d_1}) \in H_p^\gamma(l_2, d_1)$ if

$$\|h\|_{H_p^\gamma(l_2,d_1)} := \sum_{r=1}^{d_1} \|h^r\|_{H_p^\gamma(l_2)} < \infty.$$

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and $\{\mathscr{F}_t, t \geq 0\}$ be an increasing filtration of σ-fields $\mathscr{F}_t \subset \mathscr{F}$, each of which contains all $(\mathscr{F}, P)$-null sets. We assume that independent families of one-dimensional Wiener processes $\{W_t^k\}_{k\in\mathbb{N}}$ and d_1-dimensional Lévy processes $\{Z_t^k\}_{k\in\mathbb{N}}$ relative to the filtration $\{\mathscr{F}_t, t \geq 0\}$

are given on Ω. By $\mathcal{P}$ we denote the predictable σ-field generated by $\mathscr{F}_t$, i.e., $\mathcal{P}$ is the smallest σ-field containing sets of the type $A\times(s,t]$, where $s<t$ and $A\in\mathscr{F}_s$.

For $p>1$ and $\gamma\in\mathbb{R}$ denote

$$\mathbb{H}_p^\gamma(T):=L_p\left(\Omega\times(0,T),\mathcal{P};H_p^\gamma\right),\quad \mathbb{L}_p(T)=\mathbb{H}_p^0(T),$$
$$\mathbb{H}_p^\gamma(T,l_2):=L_p\left(\Omega\times(0,T),\mathcal{P};H_p^\gamma(l_2)\right),\quad \mathbb{L}_p(T,l_2)=\mathbb{H}_p^0(T,l_2),$$
$$\mathbb{H}_p^\gamma(T,l_2,d_1):=L_p\left(\Omega\times(0,T),\mathcal{P};H_p^\gamma(l_2,d_1)\right),\quad \mathbb{L}_p(T,l_2,d_1)=\mathbb{H}_p^0(T,l_2,d_1).$$

Also, for l_2-valued functions h, we write $h\in\mathbb{H}_c^\infty(T,l_2)$ if $h^k=0$ for all large k, and each h^k is of the type

$$h^k(t,x)=\sum_{i=1}^n 1_{(\tau_{i-1},\tau_i]}(t)g^{ik}(x),$$

where τ_i are bounded stopping times, $\tau_i\leq\tau_{i+1}$, and $g^{ik}\in C_c^\infty(\mathbb{R}^d)$. The space $\mathbb{H}_c^\infty(T,l_2,d_1)$ is defined similarly. By [14, Theorem 3.10], $\mathbb{H}_c^\infty(T,l_2)$ is dense in $\mathbb{H}_p^\gamma(T,l_2)$. Similarly, the space $\mathbb{L}_c$ of the functions g of the form

$$g(\omega,x)=\sum_{i=1}^n 1_{A_i}(\omega)g_i(x),\quad A_i\in\mathscr{F}_0,\ g_i\in C_c^\infty(\mathbb{R}^d) \tag{2.7}$$

is dense in $L_p(\Omega,\mathscr{F}_0;H_p^\gamma)$.

For $t\geq 0$ and $A\in\mathcal{B}(\mathbb{R}^{d_1}\setminus\{0\})$, denote

$$N^k(t,A):=\#\{0\leq s\leq t:\Delta Z_s^k:=Z_s^k-Z_{s-}^k\in A\}$$
$$\tilde{N}^k(t,A):=N^k(t,A)-t\nu^k(A),$$

where $\nu^k(A):=\mathbb{E}N^k(1,A)$ is the Lévy measure of Z_t^k. Set

$$m_p(k):=\left(\int_{\mathbb{R}^{d_1}}|z|^p\nu^k(dz)\right)^{1/p}.$$

If $m_2(k)<\infty$, then by the Lévy-Itô decomposition, there exist a vector $a^k=(a^{1k},\dots,a^{d_1k})$, a non-negative definite $d_1\times d_1$ matrix b^k, and d_1-dimensional Wiener process $\tilde{W}_t^k$ such that

$$Z_t^k=a^kt+b^k\tilde{W}_t^k+\int_{\mathbb{R}^{d_1}}z\tilde{N}^k(t,dz)=:a^kt+b^k\tilde{W}_t^k+\tilde{Z}_t^k$$

(i.e. $Z_t^{rk}=a^{rk}t+\sum_{l=1}^{d_1}b^{rlk}\tilde{W}_t^l+\int_{\mathbb{R}^{d_1}}z^r\tilde{N}^k(t,dz)$).

In this article we assume the following.

Assumption 2.1. (i) $p\in[2,\infty)$ and

$$m_p:=\sup_k\left(m_2(k)\vee m_p(k)\right)<\infty.$$

(ii) For each k, $Z_t^k=(Z^{1k},\dots,Z^{d_1k})$ is a pure jump process with no diffusion and drift parts (i.e. $a^k=0$ and $b^k=0$).

Remark 2.2. One can also consider equation (1.1) with general Z_t^k, without Assumption 2.1 (ii), by rewriting it into the form of the equation driven by a set of Brownian motions $\{W_t^k\}\cup\{\tilde{W}_t^k\}$ and Lévy processes $\tilde{Z}_t^k$. See (2.1) in [12] for the detail.

Remark 2.3. If one only wants to prove the uniqueness and existence of $H_p^{\gamma+2}$-valued path-wise solution u, then Assumption 2.1 (i) can be replaced by the weaker condition that there is an integer $k_0 \geq 1$ so that $\sup_{k\geq k_0} m_p(k) < \infty$. In particular, it can be completely dropped if only finitely many Lévy processes appear in (1.1). However, under this conditon we may have $\mathbb{E}\int_0^T \|u\|^p_{H_p^{\gamma+2}} dt = \infty$. See the proof of [13, Theorem 4.9].

Due to the assumption $m_2(k) < \infty$, Z_t^k is a square integrable martingale, and the stochastic integral against Z_t^{rk} $(r = 1, \cdots, d_1)$ can be easily understood as follows. For functions h of the type $h = \sum_{i=1}^m a_i 1_{(\tau_i,\tau_{i+1}]}(t)$, where τ_i are bounded stopping times, $\tau_i \leq \tau_{i+1}$, and a_i are bounded $\mathcal{F}_{\tau_i}$-measurable random variables, we define

$$(\Lambda h)_t := \int_0^t h\, dZ_s^{rk} := \sum_{i=1}^m a_i (Z^{rk}_{\tau_{i+1}\wedge t} - Z^{rk}_{\tau_i \wedge t}).$$

Then Λh becomes a square integrable martingale with cádlág sample paths, and one can easily check

$$\mathbb{E}\sup_{t\leq T} |(\Lambda h)_t|^2 \leq c_2(k)\|h\|^2_{L_2(\Omega\times[0,T])}.$$

Therefore, the stochastic integral can be continuously extended to all $h \in L_2(\Omega \times [0,T], \mathcal{P}; \mathbb{R})$, and $\int_0^t h\, dZ_t^{rk}$ becomes a square integrable martingale with cádlág sample paths. Furthermore, if $h_1 = h_2$ in $L_2(\Omega \times [0,T], \mathcal{P}; \mathbb{R})$, then

$$\int_0^t h_1\, dZ_t^{rk} = \int_0^t h_2\, dZ_t^{rk}, \qquad \forall\ t \leq T\ (a.s.).$$

This is because both are cádlág processes.

Remark 2.4. For any $h = (h^1, \ldots, h^{d_1}) \in L_2(\Omega\times[0,T], \mathcal{P}; \mathbb{R}^{d_1})$ with a predictable version $\bar{h}$,

$$M_t^k = \int_0^t h \cdot dZ_s^k := \sum_{r=1}^{d_1} \int_0^t h^r dZ_t^{rk} = \sum_{r=1}^{d_1} \int_0^t \bar{h}^r dZ_t^{rk}$$

is a square integrable martingale with the quadratic variation (see e.g. [18])

$$[M^k]_t = \sum_{r,l=1}^{d_1} \int_0^t \int_{\mathbb{R}^{d_1}} z^r z^l \bar{h}^r_s \bar{h}^l_s N^k(ds, dz). \tag{2.8}$$

By [3, Lemma 2.5] (or see [16, Lemma 1]) we have

$$\begin{aligned} &\mathbb{E}\left[\left(\sum_{k=1}^\infty \int_0^T \int_{\mathbb{R}^{d_1}} |z|^2 |\bar{h}^k(s)|^2 N^k(ds,dz)\right)^{p/2}\right] \\ &\quad \leq N(p, m_p)\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^\infty |h^k(s)|^2 ds\right)^{p/2} + \int_0^T \sum_{k=1}^\infty |h^k(s)|^p ds\right], \end{aligned} \tag{2.9}$$

where $|h^k(s)|^2 = \sum_{r=1}^{d_1} |h^{rk}(s)|^2$. Since

$$\sum_{k=1}^\infty |a_k|^p \leq \left(\sum_{k=1}^\infty |a_k|^2\right)^{p/2}, \quad \left(\int_0^t |h| ds\right)^{p/2} \leq t^{p/2-1}\int_0^t |h|^{p/2} ds,$$

(recall that $p \geq 2$), we have

$$\mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^T\int_{\mathbb{R}^{d_1}}|z|^2|\bar{h}^k(s)|^2N^k(ds,dz)\right)^{p/2}\right] \leq N\sum_{r=1}^{d_1}\mathbb{E}\|h^r\|^p_{L_p([0,T];l_2)}, \quad (2.10)$$

where $N = N(p, m_p, d_1, T)$. Therefore by the Burkholder-Davis-Gundy inequality, (2.8), and (2.10),

$$\mathbb{E}\left[\sup_{s\leq t}\left|\sum_{k=1}^{\infty}M_s^k\right|^p\right] \leq N\sum_{r=1}^{d_1}\mathbb{E}\|h^r\|^p_{L_p([0,T];l_2)}. \quad (2.11)$$

Remark 2.5. (i) If $g \in \mathbb{H}_p^\gamma(T, l_2)$, and $h \in \mathbb{H}_p^\gamma(T, l_2, d_1)$, then the series

$$\sum_{k=1}^{\infty}\int_0^t(g^k(s,\cdot),\phi)dW_s^k, \quad \sum_{k=1}^{\infty}\int_0^t(h^k(s,\cdot),\phi)\cdot dZ_s^k$$

are well-defined due to Assumption 2.1 and Remark 2.4. Indeed, using Remark 2.4 one can show (see [14, Remark 3.2] for detail)

$$\sum_{r=1}^{d_1}\sum_{k=1}^{\infty}\int_0^T(h^{rk},\phi)^2ds \leq N(\phi, m_p, d_1, T)\|h\|^p_{\mathbb{H}_p^\gamma(T,l_2,d_1)}.$$

Therefore, the series

$$\sum_{k=1}^{\infty}\int_0^t(h^k(s,\cdot),\phi)\cdot dZ_s^k$$

converges in probability uniformly on $[0, T]$, and it is a square integrable martingale on $[0, T]$, which is cádlág. The same argument holds for $\sum_{k=1}^{\infty}\int_0^t(g^k(s,\cdot),\phi)dW_s^k$, which is a continuous martingale on $[0, T]$.

(ii) The argument in (i) shows that if, for instance, $h_n \to h$ in $\mathbb{H}_p^\gamma(T, l_2, d_1)$, then as $n \to \infty$,

$$\sum_{k=1}^{\infty}\int_0^t(h_n^k(s,\cdot),\phi)\cdot dZ_s^k \quad \to \quad \sum_{k=1}^{\infty}\int_0^t(h^k(s,\cdot),\phi)\cdot dZ_s^k$$

in probability uniformly on $[0, T]$.

We say that $X_t = Y_t$ for almost all $t \leq T$ at once if

$$P\big(\{\omega : X_t(\omega) = Y_t(\omega),\ a.e.\, t \leq T\}\big) = 1,$$

and $X_t = Y_t$ for all $t \leq T$ at once if

$$P\big(\{\omega : X_t(\omega) = Y_t(\omega), \forall\, t \leq T\}\big) = 1.$$

Lemma 2.6. *Let $X_t^k = W_t^k$ or $X_t^k = Z_t^{rk}$, $r \in \{1, \cdots, d_1\}$, and $h \in L_2(\Omega \times [0, T], \mathcal{P}; l_2)$.*

(i) Let $\alpha > 0$ and $h = (h^1, h^2, \cdots) \in \mathbb{L}_2(T, l_2)$. Then

$$I^\alpha\left(\sum_{k=1}^{\infty}\int_0^\cdot h^k(s)dX_s^k\right)(t) \quad = \quad \sum_{k=1}^{\infty}I^\alpha\left(\int_0^\cdot h^k(s)dX_s^k\right)(t)$$

for almost all $t \leq T$ at once.

(ii) Under the assumptions in (i),

$$\sum_{k=1}^{\infty} I^{\alpha}\left(\int_0^{\cdot} h^k(s)dX^k_s\right)(t) = \frac{1}{\alpha\Gamma(\alpha)}\sum_{k=1}^{\infty}\int_0^t (t-s)^{\alpha} h^k(s)dX^k_s$$

a.e. on $\Omega\times[0,T]$.

(iii) If $\alpha<1/2$, *then*

$$\begin{aligned} D^{\alpha}_t\left(\sum_{k=1}^{\infty}\int_0^{\cdot} h^k(s)dX^k_s\right)(t) &= \sum_{k=1}^{\infty} D^{\alpha}_t\left(\int_0^{\cdot} h^k(s)dX^k_s\right)(t) \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{\infty}\int_0^t (t-s)^{-\alpha} h^k(s)dX^k_s \end{aligned}$$

a.e. on $\Omega\times[0,T]$.

Proof. See Lemmas 3.1 and 3.3 in [4] for (i) and (iii). Actually, the case $X^k_t = W^k_t$ is proved in [4], and the same argument works for the general case for $X^k_t = Z^{rk}_t$.

(ii) easily follows from the Stochastic Fubuni theorem (see [18, Chapter IV, Theorem 65]). □

Fix a small constant $\kappa>0$. Set

$$\begin{aligned} c_0 &:= 1_{\beta_1>1/2}\frac{(2\beta_1-1)}{\alpha} + \kappa 1_{\beta_1=1/2}, \\ \bar{c}_0 &:= 1_{\beta_2>1/p}\frac{(2\beta_2-2/p)}{\alpha} + \kappa 1_{\beta_2=1/p}. \end{aligned} \tag{2.12}$$

Note that $0\le c_0,\bar{c}_0<2$, $c_0=0$ if $\beta_1<1/2$, and $\bar{c}_0=0$ if $\beta_2<1/p$. Also set

$$U^{\gamma+2}_p = L_p(\Omega,\mathscr{F}_0; H^{\gamma+(2-2/\alpha p)^+}_p),$$

and

$$V^{\gamma+2}_p = \begin{cases} L_p(\Omega,\mathscr{F}_0; H^{\gamma+2-2/\alpha-2/\alpha p}_p) & \alpha>1+1/p \\ L_p(\Omega,\mathscr{F}_0; H^{\gamma+2-2/\alpha}_p) & 1<\alpha\le 1+1/p. \end{cases} \tag{2.13}$$

Note that if $\alpha>1+1/p$, then $2-2/\alpha-2/\alpha p>0$, and $2-2/\alpha>0$ for any $\alpha>1$.

Definition 2.7. Let $p\ge 2$ and $\gamma\in\mathbb{R}$. We write $u\in\mathcal{H}^{\gamma+2}_p(T)$ if $u\in\mathbb{H}^{\gamma+2}_p(T)$ and there exist $f\in\mathbb{H}^{\gamma}_p(T), g\in\mathbb{H}^{\gamma+c_0}_p(T,l_2), h\in\mathbb{H}^{\gamma+\bar{c}_0}_p(T,l_2,d_1), u_0\in U^{\gamma+2}_p$, and $v_0\in V^{\gamma+2}_p$ such that u satisfies

$$\begin{aligned} &\partial^{\alpha}_t u(t,x) = f(t,x) + \partial^{\beta_1}_t\sum_{k=1}^{\infty}\int_0^t g^k(s,x)dW^k_s + \partial^{\beta_2}_t\sum_{k=1}^{\infty}\int_0^t h^k(s,x)\cdot dZ^k_s, \quad t\in(0,T] \\ &\quad u(0,\cdot)=u_0, \quad 1_{\alpha>1}\partial_t u(0,\cdot) = 1_{\alpha>1}v_0 \end{aligned} \tag{2.14}$$

in the sense of distributions. In other words, for any $\phi\in\mathcal{S}(\mathbb{R}^d)$, the equality

$$\begin{aligned} (u(t)-u_0-tv_0 1_{\alpha>1},\phi) = I^{\alpha}_t(f,\phi) &+ \sum_{k=1}^{\infty} I^{\alpha-\beta_1}_t\int_0^t (g^k(s),\phi)dW^k_s \\ &+ \sum_{k=1}^{\infty} I^{\alpha-\beta_2}_t\int_0^t (h^k(s),\phi)\cdot dZ^k_s \end{aligned} \tag{2.15}$$

holds a.e. on $\Omega\times[0,T]$, (here $I^{\alpha-\beta_i}_t := D^{\beta_i-\alpha}_t$ if $\beta_i>\alpha$).

Remark 2.8. Note that, since $\beta_1 < \alpha + 1/2$ and $\beta_2 < \alpha + 1/p$, the right hand side of (2.15) makes sense due to Lemma 2.6.

Remark 2.9. If $\beta_2 > \alpha + 1/p$ then (2.15) does not make sense. For simplicity, let $u_0 = v_0 = 0, f = 0$ and $g = 0$. Then taking $I_t^{\beta_2-\alpha}$ to (2.15) we get

$$I_t^{\beta_2-\alpha}(u(t), \phi) = \sum_{k=1}^{\infty} \int_0^t (h^k(s), \phi) \cdot dZ_s^k.$$

Since $(u(t), \phi) \in L_p([0,T])$ (a.s.) and $\beta_2 - \alpha > 1/p$, the left hand side above is a continuous process. However, the right hand side is only cádlág process. The necessity of condition $\beta_1 < \alpha + 1/2$ can be derived similarly, and is explained in detail in [4].

Due to Lemma 2.6 (iii), if $\beta_1 < 1/2$ or $\beta_2 < 1/2$, then the the expression in (2.14) is not unique, that is, there can be other triple (f, g, h) such that (2.14) holds in the sense of distributions.

To define a norm in $\mathcal{H}_p^{\gamma+2}(T)$, we introduce the space

$$\mathbb{F}_p^{\gamma}(T) := \mathbb{H}_p^{\gamma}(T) \times \mathbb{H}_p^{\gamma+c_0'}(T, l_2) \times \mathbb{H}_p^{\gamma+\bar{c}_0'}(T, l_2, d_1),$$

and for a triple $(f, g, h) \in \mathbb{F}_p^{\gamma}(T)$, we define

$$\|(f, g, h)\|_{\mathbb{F}_p^{\gamma}(T)} = \|f\|_{\mathbb{H}_p^{\gamma}(T)} + \|g\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)} + \|h\|_{\mathbb{H}_p^{\gamma+\bar{c}_0'}(T,l_2,d_1)}.$$

Definition 2.10. For $u \in \mathcal{H}_p^{\gamma+2}(T)$, we define

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)} = \|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|u(0)\|_{U_p^{\gamma+2}} + 1_{\alpha>1}\|\partial_t u(0)\|_{V_p^{\gamma+2}} + \inf\left\{\|(f, g, h)\|_{\mathbb{F}_p^{\gamma}(T)}\right\},$$

where the infimum is taken for all triples $(f, g, h) \in \mathbb{F}_p^{\gamma}(T)$ such that (2.14) holds in the sense of distributions.

In the following proposition, we address that our definition for (2.14) is equivalent to that of [10, Definition 2.5].

Proposition 2.11. *Let $u \in \mathbb{H}_p^{\gamma+2}(T)$, $u_0 \in U_p^{\gamma+2}$, $v_0 \in V_p^{\gamma+2}$, and $(f, g, h) \in \mathbb{F}_p^{\gamma}(T)$. Then the following are equivalent;*

(i) $u \in \mathcal{H}_p^{\gamma+2}(T)$ and (2.14) holds with u_0, v_0, and triple (f, g, h) in the sense of Definition 2.7.

(ii) For any constant Λ such that

$$\Lambda \geq \max(\alpha, \beta_1, \beta_2) \quad \textit{and} \quad \Lambda > \frac{1}{p},$$

$I_t^{\Lambda-\alpha}u$ has an H_p^{γ}-valued cádlág version in $\mathbb{H}_p^{\gamma}(T)$, still denoted by $I_t^{\Lambda-\alpha}u$, such that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, the equality

$$\begin{aligned}(I_t^{\Lambda-\alpha}u - I_t^{\Lambda-\alpha}(u_0 + tv_0 1_{\alpha>1}), \phi)\\ = I_t^{\Lambda}(f, \phi) + \sum_{k=1}^{\infty} I_t^{\Lambda-\beta_1} \int_0^t (g^k(s,\cdot), \phi) dW_s^k + \sum_{k=1}^{\infty} I_t^{\Lambda-\beta_2} \int_0^t (h^k(s,\cdot), \phi) \cdot dZ_s^k\end{aligned} \tag{2.16}$$

holds for all $t \in [0,T]$ at once. Moreover, in this case it holds that

$$\mathbb{E}\sup_{t\le T}\|I_t^{\Lambda-\alpha}u\|_{H_p^\gamma}^p \le N\Big(\mathbb{E}\|u_0\|_{H_p^\gamma}^p + 1_{\alpha>1}\mathbb{E}\|v_0\|_{H_p^\gamma}^p + \|f\|_{\mathbb{H}_p^\gamma(T)}^p + \|g\|_{\mathbb{H}_p^\gamma(T,l_2)}^p + \|h\|_{\mathbb{H}_p^\gamma(T,l_2,d_1)}^p\Big), \tag{2.17}$$

where the constant N depends only on $\alpha, \beta_1, \beta_2, d, d_1, p, \gamma, \Lambda$ and T.

Proof. Considering $(1-\Delta)^{\gamma/2}u$ in place of u, we may assume $\gamma = 0$.

(i) Suppose (2.16) holds for all t at once. Then by applying $D_t^{\Lambda-\alpha}$ to (2.16) and using (2.3), (2.15), and Lemma 2.6, we find that (2.14) holds for a.e. on $\Omega\times[0,T]$.

(ii) Suppose (2.14) holds a.e. on $\Omega\times[0,T]$. Note that $I_t^{\Lambda-\alpha}(u_0+1_{\alpha>1}tv_0)$ is a continuous L_p-valued process, and it satisfies

$$\mathbb{E}\sup_{t\le T}\|I_t^{\Lambda-\alpha}(u_0+1_{\alpha>1}tv_0)\|_{L_p}^p \le N(T)(\|u_0\|_{L_p}^p + 1_{\alpha>1}\|v_0\|_{L_p}^p).$$

Hence we may assume $u_0 = v_0 = 0$.

Take a nonnegative funtion $\zeta \in C_c^\infty(\mathbb{R}^d)$ with unit integral. For each $n > 0$, define $\zeta_n(x) = n^{-d}\zeta(nx)$. For any tempered distribution v, define $v^{(n)}(x) := v * \zeta_n(x)$. Then $v^{(n)}$ is infinitely differentiable function with respect to x. Plugging $\phi = \zeta_n(\cdot - x)$ in (2.14) and applying $I_t^{\Lambda-\alpha}$ to both sides of (2.14), for each x we get

$$(I_t^{\Lambda-\alpha}(u)^{(n)})(t,x) = (I_t^\Lambda f^{(n)})(t,x) + \sum_{k=1}^\infty I_t^{\Lambda-\beta_1}\int_0^t (g^k)^{(n)}(s,x)dW_s^k + \sum_{k=1}^\infty I_t^{\Lambda-\beta_2}\int_0^t (h^k)^{(n)}(s,x)\cdot dZ_s^k \tag{2.18}$$

a.e. on $\Omega\times[0,T]$. Note that since $\Lambda > 1/p$, $I_t^\Lambda f^{(n)}$ is a continuous L_p-valued process. Also, the stochastic integrals

$$\sum_{k=1}^\infty \int_0^t (g^k)^{(n)}(s,x)dW_s^k, \quad \sum_{k=1}^\infty \int_0^t (h^k)^{(n)}(s,x)\cdot dZ_s^k$$

are L_p-valued cádlág processes, and in particular they are bounded on $[0,T]$ (a.s.). Therefore, the right hand side of (2.18) is an L_p-valued cádlág process, and consequently the left hand side has an L_p-valued cádlág version, still denoted by $I_t^{\Lambda-\alpha}(u)^{(n)}$.

By (2.1) with $p=\infty$ and (2.11),

$$\begin{aligned}
&\mathbb{E}\sup_{t\le T}\left\|I_t^{\Lambda-\beta_2}\sum_{k=1}^\infty\int_0^t (h^k)^{(n)}(s,\cdot)\cdot dZ_s^k\right\|_{L_p}^p \\
&\le N\int_{\mathbb{R}^d}\mathbb{E}\sup_{t\le T}\left|I_t^{\Lambda-\beta_2}\sum_{k=1}^\infty\int_0^t (h^k)^{(n)}(s,x)\cdot dZ_s^k\right|^p dx \\
&\le N\int_{\mathbb{R}^d}\mathbb{E}\sup_{t\le T}\left|\sum_{k=1}^\infty\int_0^t (h^k)^{(n)}(s,x)\cdot dZ_s^k\right|^p dx \le N\mathbb{E}\int_0^T \|h^{(n)}(s,\cdot)\|_{L_p(l_2,d_1)}^p ds.
\end{aligned}$$

We handle two other terms on the right hand side of (2.18) similarly, and get

$$\begin{aligned}&\mathbb{E}\sup_{t\le T}\left\|I^{\Lambda-\alpha}(u)^{(n)}(t,\cdot)\right\|^p_{L_p}\\&\le N\big(\|f^{(n)}\|^p_{\mathbb{L}_p(T)}+\|g^{(n)}\|^p_{\mathbb{L}_p(T,l_2)}+\|h^{(n)}\|^p_{\mathbb{L}_p(T,l_2,d_1)}\big).\end{aligned}\tag{2.19}$$

Considering (2.19) corresponding to $I_t^{\Lambda-\alpha}(u)^{(n)}-I_t^{\Lambda-\alpha}(u)^{(m)}$, we find that $I_t^{\Lambda-\alpha}(u)^{(n)}$ is a Cauchy sequence in $L_p(\Omega;D([0,T];L_p))$, where $D([0,T];L_p)$ is a space of L_p-valued cádlág functions. Let w denote the limit in this space. Then since $I_t^{\Lambda-\alpha}(u)^{(n)}\to I^{\Lambda-\alpha}u$ in $\mathbb{L}_p(T)$, we conclude $w=I_t^{\Lambda-\alpha}u$ a.e on $\Omega\times[0,T]$, and w is an L_p-valued cádlág version of $I_t^{\Lambda-\alpha}u$. This proves that (2.16) holds for all t at once because both sides are read-valued cádlág processes. Also we easily obtain (2.17) from (2.19) and the lemma is proved. □

Theorem 2.12. *Let $p\ge 2,\gamma\in\mathbb{R}$ and $T\in(0,\infty)$.*

(i) For any $\nu\in\mathbb{R}$, the map $(1-\Delta)^{\nu/2}:\mathcal{H}_p^{\gamma+2}(T)\to\mathcal{H}_p^{\gamma-\nu+2}(T)$ is an isometry.

(ii) $\mathcal{H}_p^{\gamma+2}(T)$ is a Banach space with the norm in Definition 2.10.

(iii) Suppose that $u\in\mathcal{H}_p^{\gamma+2}(T)$ satisfies (2.14) with a triple $(f,g,h)\in\mathbb{F}_p^{\gamma}(T)$. Then for any $t\le T$,

$$\begin{aligned}\|u\|^p_{\mathbb{H}_p^{\gamma}(t)}\ \le\ & N\int_0^t(t-s)^{\theta-1}\Big(\|f\|^p_{\mathbb{H}_p^{\gamma}(s)}+\|g\|^p_{\mathbb{H}_p^{\gamma}(s,l_2)}+\|h\|^p_{\mathbb{H}_p^{\gamma}(s,l_2,d_1)}\Big)ds\\&+N(\mathbb{E}\|u_0\|^p_{H_p^{\gamma}}+1_{\alpha>1}\mathbb{E}\|v_0\|^p_{H_p^{\gamma}}),\end{aligned}\tag{2.20}$$

where $\theta:=\min\{\alpha,2(\alpha-\beta_1)+1,p(\alpha-\beta_2)+2\}$, and the constant N depends only on $\alpha,\beta_1,\beta_2,d,d_1,p$ and T.

Proof. (i) This easily follows from the fact that $(1-\Delta)^{\gamma/2}:H_p^{\mu}\to H_p^{\mu-\gamma}$ is an isometry.

(ii) We only prove the completeness. Suppose that u_n is a Cauchy sequence in $\mathcal{H}_p^{\gamma+2}(T)$ with $u_n(0)=u_0^n$, and $1_{\alpha>1}\partial_t u_n(0)=1_{\alpha>1}v_0^n$. Since it is enough to prove there exists a convergent subsequence, by taking suitable subsequence we may assume that $\|u_{n+1}-u_n\|_{\mathcal{H}_p^{\gamma+2}(T)}\le 2^{-n}$ for each $n\in\mathbb{N}$. By the definition, for each $n\in\mathbb{N}$, there exists $(\tilde f^{n+1},\tilde g^{n+1},\tilde h^{n+1})\in\mathbb{F}_p^{\gamma}(T)$ with which $u_{n+1}-u_n$ (in place of u) satisfies (2.14), and

$$\begin{aligned}&\|u_{n+1}-u_n\|_{\mathbb{H}_p^{\gamma+2}(T)}+\|u_0^{n+1}-u_0^n\|_{U_p^{\gamma+2}}+1_{\alpha>1}\|v_0^{n+1}-v_0^n\|_{V_p^{\gamma+2}}\\&+\|(\tilde f^n,\tilde g^n,\tilde h^n)\|_{\mathbb{F}_p^{\gamma}(T)}\le\|u_{n+1}-u_n\|_{\mathcal{H}_p^{\gamma+2}(T)}+2^{-n}\le 2^{-n+1}.\end{aligned}\tag{2.21}$$

We take a triple $(\tilde f^1,\tilde g^1,\tilde h^1)\in\mathbb{F}_p^{\gamma}(T)$ such that u_1 satisfies (2.14) with this triple, and define

$$(f_n,g_n,h_n)=\sum_{k=1}^{n}(\tilde f^k,\tilde g^k,\tilde h^k),\qquad (f,g,h)=\sum_{k=1}^{\infty}(\tilde f^k,\tilde g^k,\tilde h^k),$$

$$u=\sum_{k=1}^{\infty}(u_{j+1}-u_j)+u_1.$$

Then, it is obvious that u_n satisfies (2.14) with the triple (f_n, g_n, h_n), and

$$\begin{aligned} &\|u-u_n\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|u_0 - u_0^n\|_{U_p^{\gamma+2}} + 1_{\alpha>1}\|v_0 - v_0^n\|_{V_p^{\gamma+2}} \\ &\quad + \|(f-f_n, g-g_n, h-h_n)\|_{\mathbb{F}_p^{\gamma}(T)} \\ &\leq \sum_{k=n+1}^{\infty} \Big(\|u_k - u_{k-1}\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|u_0^k - u_0^{k-1}\|_{U_p^{\gamma+2}} \\ &\qquad\qquad + 1_{\alpha>1}\|v_0^k - v_0^{k-1}\|_{V_p^{\gamma+2}} + \|(\tilde{f}^k, \tilde{g}^k, \tilde{h}^k)\|_{\mathbb{F}_p^{\gamma}(T)} \Big) \\ &\leq \sum_{k=n+1}^{\infty} 2^{-k+1}. \end{aligned}$$

Hence, to prove u_n converges to u in $\mathcal{H}_p^{\gamma+2}(T)$, it is enough to show u satisfies (2.14) with the triple (f, g, h). This can be easily proved using Proposition 2.11 and Remark 2.5 (ii).

(iii) We repeat the proof of [10, Theorem 2.1] which treats the case $h = 0$. Note first that by the result of (i) we may assume $\gamma = 0$.

We take notation from the proof of Proposition 2.11. Then, from (2.15), for each $x \in \mathbb{R}^d$ we get

$$\begin{aligned} u^{(n)}(t,x) &= (u_0)^{(n)}(x) + 1_{\alpha>1} t (v_0)^{(n)}(x) \\ &\quad + I_t^{\alpha} f^{(n)}(t,x) + \sum_{k=1}^{\infty} I_t^{\alpha-\beta_1} \int_0^t (g^k)^{(n)}(s,x)\, dW_s^k \\ &\quad + \sum_{k=1}^{\infty} I_t^{\alpha-\beta_2} \int_0^t (h^k)^{(n)}(s,x) \cdot dZ_s^k \end{aligned} \tag{2.22}$$

a.e. on $\Omega \times [0,T]$. Since $u^{(n)} \to u$ in $\mathbb{L}_p(T)$, to prove (2.20), it is enough to estimate $\|u^{(n)}\|_{\mathbb{L}_p(t)}$. For this, we only estimate the last term in (2.23) because other terms are estimated in the proof of [10, Theorem 2.1]. By Lemma 2.6, for each $x \in \mathbb{R}^d$ we have

$$\left(\sum_{k=1}^{\infty} I_t^{\alpha-\beta_2} \int_0^{\cdot} (h^k)^{(n)}(r,x) \cdot dZ_r^k \right)(s) = c(\alpha,\beta_2) \sum_{k=1}^{\infty} \int_0^s (s-r)^{\alpha-\beta_2} (h^k)^{(n)}(r,x) \cdot dZ_r^k$$

a.e. on $\Omega \times [0,t]$. Let $\bar{h}$ be a predictable version of h, then by the Burkerholder-Davis-Gundy inequality, (2.10) and Fubini's theorem, we have

$$\begin{aligned} &\left\| \sum_{k=1}^{\infty} I_t^{\alpha-\beta_2} \int_0^{\cdot} (h^k)^{(n)}(s,x) \cdot dZ_s^k \right\|_{\mathbb{L}_p(t)}^p \\ &\leq N \int_{\mathbb{R}^d} \int_0^t \mathbb{E} \left[\left(\sum_{k=1}^{\infty} \int_0^s \int_{\mathbb{R}} |z|^2 |(s-r)^{\alpha-\beta_2} (\bar{h}^k)^{(n)}(r,x)|^2 N^k(dr,dz) \right)^{p/2} \right] ds dx \\ &\leq N \int_0^t \int_0^s (s-r)^{p(\alpha-\beta_2)} \|h^{(n)}(r)\|_{\mathbb{L}_p(l_2,d_1)}^p dr ds \\ &\leq N \int_0^t (t-s)^{p(\alpha-\beta_2)+1} \|h(s)\|_{\mathbb{L}_p(s,l_2,d_1)}^p ds \leq N \int_0^t (t-s)^{\theta-1} \|h(s)\|_{\mathbb{L}_p(s,l_2,d_1)}^p ds. \end{aligned}$$

Other terms in the right hand side of (2.23) can be handled similarly, and these yield inequality (2.20) with $u^{(n)}$. This is enough because $u^{(n)} \to u$ in $\mathbb{L}_p(t)$. $\square$

Take $\kappa' \in (0,1)$, and for $r \geq 0$, set

$$B^r = \begin{cases} L_\infty(\mathbb{R}^d) & \text{if} \quad r = 0 \\ C^{r-1,1}(\mathbb{R}^d) & \text{if} \quad r = 1,2,3,\ldots \\ C^{r+\kappa'}(\mathbb{R}^d) & \text{otherwise,} \end{cases}$$

where $C^{r+\kappa'}(\mathbb{R}^d)$ and $C^{r-1,1}(\mathbb{R}^d)$ are Hölder space and Zygmund space respectively. We use $B^r(l_2)$ for l_2-valued analogue. It is known (see e.g. [14, Lemma 5.2]) that for $u \in H_p^\gamma$, and $v \in H_p^\gamma(l_2)$

$$\begin{aligned} \|au\|_{H_p^\gamma} &\leq N(d,p,\kappa',\gamma)|a|_{B^{|\gamma|}}\|u\|_{H_p^\gamma}, \\ \|bv\|_{H_p^\gamma(l_2)} &\leq N(d,p,\kappa',\gamma)|b|_{B^{|\gamma|}(l_2)}\|v\|_{H_p^\gamma(l_2)}. \end{aligned} \tag{2.23}$$

Assumption 2.13. (i) All the coefficients are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions.

(ii) The coefficients $\mu^i, \nu, \bar{\mu}^{ir}, \bar{\nu}^r$ are l_2-valued functions, where $i = 1,2,\cdots,d$ and $r = 1,2,\cdots,d_1$.

(iii) There exists a constant $0 < \delta < 1$ so that for any (ω,t,x)

$$\delta|\xi|^2 \leq a^{ij}(t,x)\xi^i\xi^j \leq \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d. \tag{2.24}$$

(iv) The coefficients $a^{ij}(\omega,t,x)$ is uniformly continuous in (t,x), uniformly on Ω.

(v) For each ω, t, i, j, r,

$$\begin{aligned} &|a^{ij}(t,\cdot)|_{B^{|\gamma|}} + |b^i(t,\cdot)|_{B^{|\gamma|}} + |c(t,\cdot)|_{B^{|\gamma|}} + |\mu^i(t,\cdot)|_{B^{|\gamma+c_0|}(l_2)} \\ &+|\nu(t,\cdot)|_{B^{|\gamma+c_0|}(l_2)} + |\bar{\mu}^{ir}(t,\cdot)|_{B^{|\gamma+\bar{c}_0|}(l_2)} + |\bar{\nu}^r(t,\cdot)|_{B^{|\gamma+\bar{c}_0|}(l_2)} \leq \delta^{-1}. \end{aligned}$$

(v) $\mu^i = 0$ if $\beta_1 \geq 1/2 + \alpha/2$, and $\bar{\mu}^{ir} = 0$ if $\beta_2 \geq 1/p + \alpha/2$.

Below we use notation $f(u), g(u)$, and $h(u)$ to denote $f(\omega,t,x,u), g(\omega,t,x,u)$, and $h(\omega,t,x,u)$ respectively.

Assumption 2.14. (i)f, g and h are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^{d+1})$ measurable, and for any $u \in \mathbb{H}_p^{\gamma+2}(T)$,

$$f(u) \in \mathbb{H}_p^\gamma(T), \quad g(u) \in \mathbb{H}_p^{\gamma+c_0'}(T,l_2), \quad h(u) \in \mathbb{H}_p^{\gamma+\bar{c}_0'}(T,l_2,d_1).$$

(ii) For any $\varepsilon > 0$, there exists a constant $K = K(\varepsilon) > 0$ so that

$$\begin{aligned} &\|f(t,u) - f(t,v)\|_{H_p^\gamma} + \|g(t,u) - g(t,v)\|_{H_p^{\gamma+c_0'}(l_2)} + \|h(t,u) - f(t,v)\|_{H^{\gamma+\bar{c}_0'}_{p(l_2,d_1)}} \\ &\quad \leq \varepsilon\|u-v\|_{H_p^{\gamma+2}} + K\|u-v\|_{H_p^\gamma} \end{aligned}$$

for any ω, t, and $u, v \in H_p^{\gamma+2}$.

Here is the main result of this article.

Theorem 2.15. *Let $\gamma \in \mathbb{R}$, $p \geq 2$, and $T < \infty$. Suppose Assumption 2.13 and Assumption 2.14 hold and*

$$\alpha \in (0,2), \quad \beta_1 < \alpha + 1/2, \quad \beta_2 < \alpha + 1/p.$$

Then for any $u_0 \in U_p^{\gamma+2}$, $v_0 \in V_p^{\gamma+2}$, equation (1.1) *has a unique solution u in the class $\mathcal{H}_p^{\gamma+2}(T)$ in the sense of Definition 2.7. Moreover,*

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\gamma+2}(T)} \leq N\Big(& \|u_0\|_{U_p^{\gamma+2}} + 1_{\alpha>1}\|v_0\|_{V_p^{\gamma+2}} + \|f(0)\|_{\mathbb{H}_p^{\gamma}(T)} \\ & + \|g(0)\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)} + \|h(0)\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(T,l_2,d_1)}\Big), \end{aligned} \tag{2.25}$$

where the constant N depends only on $\alpha, \beta_1, \beta_2, d, d_1, p, \delta, \gamma, \kappa$, and T.

Remark 2.16. If $\alpha \in (0,1]$ then Assumption 2.13 (iv) can be relaxed and replaced by the uniformly continuity in x, uniformly on $\Omega \times [0,T]$. Assumption 2.13 (iv) is inherited from a result on the deterministic equation, [11, Theorem 2.10]. However, if $\alpha \in (0,1)$ then the continuity in t can be completely dropped for the deterministic equation (see [8]).

3. Key estimates

In this section we study the convolution operators of the type

$$((-\Delta)^a D_t^b p) * f,$$

where $a, b \in \mathbb{R}$, $p(t,x)$ is the fundamental solution to the time fractional heat equation $\partial_t^\alpha u = \Delta u$, and $(-\Delta)^a$ is the fractional Laplacian of order a defined by

$$(-\Delta)^a f(x) = \mathcal{F}^{-1}\{|\cdot|^{2a}\mathcal{F}(f)(\cdot)\}(x).$$

To explain the necessity of such study, let us consider

$$\partial_t^\alpha u = \Delta u + \partial_t^\beta \int_0^t h(s)\, dZ_t, \;\; t > 0 \quad ; \quad u(0) = 1_{\alpha>1}u_t(0) = 0,$$

where Z_t is a Lévy process. It turns out that for the solution u and $c \geq 0$ we have

$$\|(-\Delta)^{c/2}u\|_{\mathbb{L}_p(T)}^p \leq N\Big\| \int_0^t \Big|\Big((-\Delta)^{c/2} D_t^{\beta-\alpha} p\Big) * h(s)\Big|^p ds\Big\|_{L_1(\Omega\times[0,T];L_1(\mathbb{R}^d))}.$$

Thus, for the estimations of solutions, we need to handle the right hand side of the above inequality. If non-zero initial condition is given, this also leads to the similar situation.

Below, to state our main theorems of this section, we introduce the Besov space. We fix $\Psi \in \mathcal{S}(\mathbb{R}^d)$ such that its Fourier transform $\hat{\Psi}(\xi)$ has support in a strip $\{\xi \in \mathbb{R}^d | \frac{1}{2} \leq |\xi| \leq 2\}$, $\hat{\Psi}(\xi) > 0$ for $\frac{1}{2} < |\xi| < 2$, and

$$\sum_{j\in\mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1 \quad \text{for} \quad \xi \neq 0.$$

Define

$$\hat{\Psi}_j(\xi) = \hat{\Psi}(2^{-j}\xi), \quad j = \pm 1, \pm 2, \ldots,$$

$$\hat{\Psi}_0(\xi) = 1 - \sum_{j=1}^\infty \hat{\Psi}_j(\xi).$$

For distributions (or functions) f, we denote $f_j := \Psi_j * f$.

It is known that if $u \in H_p^\gamma$, then

$$\|u\|_{H_p^\gamma} \sim \left(\|u_0\|_{L_p} + \|(\sum_{j=1}^\infty 2^{\gamma j}|u_j|^2)^{1/2}\|_{L_p} \right). \tag{3.1}$$

For $1 < p < \infty$, $s \in \mathbb{R}$, we define Besov space $B_p^s = B_p^s(\mathbb{R}^d)$ as the collection of all tempered distributions u such that

$$\|u\|_{B_p^s} := \|u_0\|_{L_p} + \left[\sum_{j=1}^{\infty} 2^{spj}\|u_j\|_{L_p}^p\right]^{1/p} < \infty.$$

Remark 3.1. It is well known (e.g. [2, 21]) that $C_c^\infty(\mathbb{R}^d)$ is dense in B_p^s, the inclusion $B_p^{s_2} \subset B_p^{s_1}$ holds for $s_1 \leq s_2$, and

$$H_p^s \subset B_p^s, \quad 2 \leq p < \infty.$$

Furthermore, $(-\Delta)^{\frac{\gamma}{2}}$ is a bounded operator from $B_p^{s+\gamma}$ to B_p^s, and $(1-\Delta)^{\gamma/2}$ is an isometry from $B_p^{s+\gamma}$ to B_p^s and from $H_p^{s+\gamma}$ to H_p^s.

Now, let $0 < \alpha < 2$ and $p(t,x)$ be the fundamental solution to the equation

$$\partial_t^\alpha u = \Delta u, \quad u(0,x) = u_0(x), \quad 1_{\alpha>1}\partial_t u(0,x) = 0. \tag{3.2}$$

That is, $p(t,x)$ is the function so that, under appropriate smoothness assumption on u_0, $u = p(t,\cdot) * u_0$ is the solution to (3.2). For $\beta < \alpha + \frac{1}{2}$, we define

$$q_{\alpha,\beta}(t,x) = \begin{cases} I_t^{\alpha-\beta} p(t,x) & \alpha \geq \beta, \\ D_t^{\beta-\alpha} p(t,x) & \alpha < \beta. \end{cases}$$

Below we list some properties of p and $q_{\alpha,\beta}$.

Lemma 3.2. *Let $0 < \alpha < 2$, $\beta < \alpha + \frac{1}{2}$, and $\gamma \in [0,2)$.*

(i) For any $t > 0$, and $x \neq 0$,

$$\partial_t^\alpha p(t,x) = \Delta p(t,x), \quad \frac{\partial}{\partial t} p(t,x) = \Delta q_{\alpha,1}(t,x), \tag{3.3}$$

and $\frac{\partial}{\partial t} p(t,x) \to 0$ as $t \downarrow 0$. Moreover, $\frac{\partial}{\partial t} p(t,\cdot)$ is integrable in $\mathbb{R}^d$ uniformly on $t \in [\varepsilon, T]$ for any $\varepsilon > 0$.

(ii) For $f \in C_c^\infty(\mathbb{R}^d)$, the convolution

$$\int_{\mathbb{R}^d} p(t,x-y) f(y) dy$$

converges to $f(x)$ uniformly as $t \downarrow 0$.

(iii) For any $m \in \mathbb{N}_+$, there exist constants $c = c(\alpha,d,m)$ and $N = N(\alpha,d,m)$ such that if $R := |x|^2 t^{-\alpha} \geq 1$, then

$$|D_x^m p(t,x)| \leq N t^{-\frac{\alpha(d+m)}{2}} \exp\{-ct^{-\frac{\alpha}{2-\alpha}}|x|^{\frac{2}{2-\alpha}}\}, \tag{3.4}$$

and if $R < 1$, then

$$|D_x^m p(t,x)| \leq N|x|^{-d-m}(R + R|\log R| 1_{d=2,m=0} + R^{1/2} 1_{d=1,m=0}). \tag{3.5}$$

(iv) It holds that

$$\mathcal{F}\{D_t^\sigma q_{\alpha,\beta}(t,\cdot)\} = t^{\alpha-\beta-\sigma} E_{\alpha,1+\alpha-\beta-\sigma}(-t^\alpha|\xi|^2), \tag{3.6}$$

where $E_{a,b}$, $a > 0$ is the Mittag-Leffler function defined as

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+b)}, \quad z \in \mathbb{C}.$$

(v) For any $\sigma \geq 0$, there exists a constant $N = N(\alpha, \beta, \sigma, \gamma, d)$ such that

$$|D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}(1,x)| + |D_t^\sigma(-\Delta)^{\gamma/2}\partial_t q_{\alpha,\beta}(1,x)| \leq N(|x|^{-d+2-\gamma} \wedge |x|^{-d-\gamma}) \quad (3.7)$$

if $d \geq 2$, and

$$\begin{aligned} &|D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}(1,x)| + |D_t^\sigma(-\Delta)^{\gamma/2}\partial_t q_{\alpha,\beta}(1,x)| \\ &\leq N(|x|^{1-\gamma}(1+\log|x| 1_{\gamma=1}) \wedge |x|^{-1-\gamma}) \end{aligned} \quad (3.8)$$

if $d = 1$.

(vi) For any $\sigma \geq 0$, the following scaling property holds:

$$D_t^\sigma(-\Delta)^{\gamma/2} q_{\alpha,\beta}(t,x) = t^{-\sigma-\frac{\alpha(d+\gamma)}{2}+\alpha-\beta}(-\Delta)^{\gamma/2} q_{\alpha,\beta}(1, t^{-\frac{\alpha}{2}}x). \quad (3.9)$$

Proof. For (i), (iv), (v), and (vi), see [10, Lemma 3.1]. Also see [11, Lemma 3.1] for (iii), and see [10, Corollary 3.2] for (ii). □

Lemma 3.3. *Let $0 < a < 2$ and $b < a + 1$. Then there exist constants*

$$\eta_1 > 0, \quad \eta_2 \in \mathbb{R}, \quad \eta_3 \in (-1, 1)$$

which depend only on a such that for any $v > 0$,

$$E_{a,b}(-v) = \frac{1}{\pi a}\int_0^\infty \frac{r^{\frac{1-b}{a}} \exp(-r^{1/a}\eta_1)[r\sin(\psi - \eta_2 a) + v\sin(\psi)]}{r^2 + 2rv\eta_3 + v^2} dr, \quad (3.10)$$

where $\psi = \psi(r) = r^{1/a}\sin(\eta_2) + (\eta_2(a+1-b))$.

Proof. The proof is based on [9, Chapter 4]. Since $0 < a < 2$, we can choose a constant η satisfying $\frac{a}{2}\pi < \eta < (\pi \wedge a\pi)$. Then by using formula (4.7.13) in [9], for any $v > 0$ and for any $0 < \lambda < v$, we have

$$\begin{aligned} E_{a,b}(-v) =& \frac{1}{\pi a}\int_\lambda^\infty \frac{r^{\frac{1-b}{a}} \exp(r^{1/a}\cos(\eta/a))[r\sin(\psi-\eta) + v\sin(\psi)]}{r^2 + 2rv\cos(\eta) + v^2} dr \\ &+ \int_{-\eta}^{\eta} G(a, b, \lambda, \phi, v) d\phi, \end{aligned} \quad (3.11)$$

where

$$\psi = \psi(r) = r^{1/a}\sin(\eta/a) + (\eta(a+1-b)/a),$$

$$G = \frac{\lambda^{1+(1-b)/a}}{2\pi a} \frac{\exp(\lambda^{1/a}\cos(\phi/a)) e^{i\nu}}{\lambda e^{i\phi} + v},$$

and $\nu = \lambda^{1/a}\sin(\phi/a) + \phi(1 + (1-b)/a)$. Since $b - 1 < a$, by the dominated convergence theorem, if we let $\lambda \downarrow 0$, the second integral in (3.11) goes to zero. Also since $\frac{a}{2}\pi < \eta < (\pi \wedge a\pi)$, $\cos(\eta/a)$ has negative value, and $|\cos(\eta)| \neq 1$. Therefore, as λ goes to zero, the first integral in (3.11) converges to the integral over positive real line with the same integrand by the dominated convergence theorem. Therefore, to finish the proof, it is enough to take $\eta_1 = -\cos(\eta/a), \eta_2 = \eta/a$ and $\eta_3 = \cos(\eta)$. □

Remark 3.4. If $b = 1$, then we have

$$E_{a,1}(-v) = \frac{\sin a\pi}{\pi}\int_0^\infty \frac{r^{a-1}}{r^{2a} + 2r^a\cos(a\pi) + 1} \exp(-rv^{1/a}) r dr. \quad (3.12)$$

(see e.g. [9, Exercise 3.9.5]).

Lemma 3.5. *Let $\alpha \in (0,2)$ and $\beta < \alpha + 1/2$. Then there exist constants N and*

$$m_1 > 0, \quad m_2 \in \mathbb{R}, \quad m_3 \in \mathbb{R}, \quad m_4 \in \mathbb{R}, \quad m_5 \in (-1,1),$$

depending only on α, β, such that for any $\mu \in \mathbb{R}$

$$\begin{aligned}&\mathcal{F}\{(-\Delta)^{\mu/2} q_{\alpha,\beta}(t,\cdot)\}(\xi)\\ &\quad = N|\xi|^{\mu+\frac{2\beta-2\alpha}{\alpha}} \int_0^\infty \frac{\exp\left(-m_1 t|\xi|^{\frac{2}{\alpha}} r\right)[r^\alpha \sin(\tilde{\psi}+m_3) + \sin(\tilde{\psi}+m_4)]}{r^{2\alpha} - 2r^\alpha m_5 + 1} r^{\beta-1} dr,\end{aligned} \tag{3.13}$$

where $\tilde{\psi} = \tilde{\psi}(r) = m_2 t|\xi|^{\frac{2}{\alpha}} r$.

Proof. By the definition of fractional Laplacian and (3.6), we have

$$\mathcal{F}\{(-\Delta)^{\mu/2} q_{\alpha,\beta}(t,\cdot)\}(\xi) = t^{\alpha-\beta}|\xi|^\mu E_{\alpha,1+\alpha-\beta}(-t^\alpha|\xi|^2).$$

By (3.10) with $a = \alpha, b = 1+\alpha-\beta$, and the change of variables $r \to vr$, for any $v > 0$ we have

$$\begin{aligned}&E_{\alpha,1+\alpha-\beta}(-v)\\ &\quad = \frac{1}{\pi\alpha}\int_0^\infty \frac{r^{\frac{\beta-\alpha}{\alpha}} \exp\left(-r^{1/\alpha}\eta_1\right)[r\sin(\psi-\eta_2\alpha) + v\sin(\psi)]}{r^2 + 2rv\eta_3 + v^2} dr\\ &\quad = \frac{1}{\pi\alpha}\int_0^\infty \frac{v^{\frac{\beta-\alpha}{\alpha}} r^{\frac{\beta-\alpha}{\alpha}} \exp\left(-v^{1/\alpha} r^{1/\alpha}\eta_1\right)[r\sin(\psi_1-\eta_2\alpha) + \sin(\psi_1)]}{r^2 + 2r\eta_3 + 1} dr,\end{aligned}$$

where $\psi_1 = \psi_1(r) = v^{1/\alpha} r^{1/a} \sin(\eta_2) + \eta_2\beta$. By the change of variables $r \to r^\alpha$,

$$\begin{aligned}&E_{\alpha,1+\alpha-\beta}(-v)\\ &\quad = \frac{1}{\pi\alpha}\int_0^\infty \frac{v^{\frac{\beta-\alpha}{\alpha}} r^{\beta-\alpha} \exp\left(-v^{1/\alpha} r\eta_1\right)[r^\alpha\sin(\psi''-\eta_2\alpha) + \sin(\psi')]}{r^{2\alpha} + 2r^\alpha\eta_3 + 1} \alpha r^{\alpha-1} dr\\ &\quad = N\int_0^\infty v^{\frac{\beta-\alpha}{\alpha}} \frac{\exp\left(-v^{1/\alpha} r\eta_1\right)[r^\alpha\sin(\psi_2-\eta_2\alpha) + \sin(\psi_2)]}{r^{2\alpha} + 2r^\alpha\eta_3 + 1} r^{\beta-1} dr,\end{aligned}$$

where $\psi_2 = \psi_2(r) = v^{1/\alpha} r \sin(\eta_2) + \eta_2\beta$. Putting $v = t^\alpha|\xi|^2$, we have

$$\begin{aligned}&\mathcal{F}\{(-\Delta)^{\mu/2} q_{\alpha,\beta}(t,\cdot)\}(\xi)\\ &\quad = N|\xi|^{\mu+\frac{2\beta-2\alpha}{\alpha}} \int_0^\infty \frac{\exp\left(-\eta_1 t|\xi|^{\frac{2}{\alpha}} r\right)[r^\alpha\sin(\psi_3-\eta_2\alpha) + \sin(\psi_3)]}{r^{2\alpha} + 2r^\alpha\eta_3 + 1} r^{\beta-1} dr,\end{aligned}$$

due to (3.6) and (3.15), where $\psi_3 = \psi_3(r) = \sin(\eta_2) t|\xi|^{\frac{2}{\alpha}} r + \eta_2\beta$.

Finally, for (3.13) we take

$$m_1 = \eta_1, \quad m_2 = \sin(\eta_2), \quad m_4 = \eta_2\beta, \quad m_3 = m_4 - \alpha\eta_2, \quad m_5 = \eta_3.$$

The lemma is proved. □

For each $j = 0,1,\ldots$ and $c > 0$, denote

$$\begin{aligned}q^{c,j}_{\alpha,\beta}(t,x) &= \Psi_j * (-\Delta)^{\frac{c}{2}} q_{\alpha,\beta}(t,x)\\ &= \mathcal{F}^{-1}\{\hat{\Psi}(2^{-j}\cdot)\mathcal{F}\{(-\Delta)^{\frac{c}{2}} q_{\alpha,\beta}\}(t,\cdot)\}(x)\\ &= 2^{jd}\mathcal{F}^{-1}\{\hat{\Psi}(\cdot)\mathcal{F}\{(-\Delta)^{\frac{c}{2}} q_{\alpha,\beta}\}(t,2^j\cdot)\}(2^j x)\\ &=: 2^{jd}\bar{q}^{c,j}_{\alpha,\beta}(t,2^j x).\end{aligned} \tag{3.14}$$

Lemma 3.6. *Assume*

$$p \geq 2, \quad 0 < \alpha < 2, \quad \frac{1}{p} < \beta < \alpha + \frac{1}{p}, \tag{3.15}$$

denote $c_1 := \frac{2(\alpha+1/p-\beta)}{\alpha} > 0$*. Then for any constants* ε, δ *satisfying*

$$\frac{1}{p} < \beta - \frac{\alpha}{2}\varepsilon, \quad \beta - \alpha < \frac{1}{p} - \delta < \frac{1}{p}, \tag{3.16}$$

we have

$$\|q^{c_1+\varepsilon,j}_{\alpha,\beta}(t,\cdot)\|_{L_1} \leq N(2^{\frac{2\delta}{\alpha}j+\varepsilon j}t^{-\frac{1}{p}+\delta} \wedge t^{-\frac{1}{p}-\frac{\alpha\varepsilon}{2}}), \tag{3.17}$$

where $N = N(\alpha, \beta, d, p, \varepsilon, \delta)$*.*

Proof. Put $c_2 = c_1 + \varepsilon$. Then by (3.16), $0 < c_2 < 2$. Due to (3.9), we easily get

$$\|(-\Delta)^{\frac{c_2}{2}} q_{\alpha,\beta}(t,\cdot)\|_{L_1} \leq N t^{-\frac{\alpha c_2}{2}+\alpha-\beta}. \tag{3.18}$$

Recall that the convolution operator is bounded in L_p for any $p \geq 1$, that is $\|f * g\|_{L_p} \leq \|f\|_{L_1}\|g\|_{L_p}$. Thus, (3.18) together with the first equality in (3.14) yields

$$\|q^{c_2,j}_{\alpha,\beta}(t,\cdot)\|_{L_1} \leq N t^{-\frac{1}{p}-\frac{\alpha\varepsilon}{2}}.$$

This and the equality $\|q^{c_2,j}_{\alpha,\beta}(t,\cdot)\|_{L_1} = \|\bar{q}^{c_2,j}_{\alpha,\beta}(t,\cdot)\|_{L_1}$ show that it remains to prove

$$\|\bar{q}^{c_2,j}_{\alpha,\beta}(t,\cdot)\|_{L_1} \leq N 2^{\frac{2\delta}{\alpha}j+\varepsilon j} t^{-\frac{1}{p}+\delta}.$$

By definition (see (3.14))

$$\mathcal{F}(\bar{q}^{c_2,j}_{\alpha,\beta})(t,\xi) = \hat{\Psi}(\xi)\mathcal{F}\{(-\Delta)^{\frac{c_2}{2}} q_{\alpha,\beta}\}(t, 2^j\xi). \tag{3.19}$$

Thus

$$\begin{aligned} |\mathcal{F}(\bar{q}^{c_2,j}_{\alpha,\beta})(t,\xi)| &= |\hat{\Psi}(\xi)||\mathcal{F}\{(-\Delta)^{\frac{c_2}{2}} q_{\alpha,\beta}(t,\cdot)\}(2^j\xi)| \\ &\leq N 1_{\frac{1}{2}\leq|\xi|\leq 2}|\mathcal{F}\{(-\Delta)^{\frac{c_2}{2}} q_{\alpha,\beta}(t,\cdot)\}(2^j\xi)|. \end{aligned} \tag{3.20}$$

By (3.13) with $\mu = \frac{c_2}{2}$,

$$\begin{aligned} &|\mathcal{F}\{(-\Delta)^{\frac{c_2}{2}} q_{\alpha,\beta}(t,\cdot)\}(2^j\xi)| \\ &\leq N|2^j\xi|^{\frac{2}{\alpha p}+\varepsilon} \int_0^\infty \frac{\exp(-m_1 t|2^j\xi|^{\frac{2}{\alpha}} r)(|r^\alpha \sin(\psi+m_3)| + |\sin(\psi+m_4)|)}{r^{2\alpha} - 2r^\alpha m_5 + 1} r^{\beta-1} dr, \end{aligned} \tag{3.21}$$

where $\psi = m_2 t|2^j\xi|^{\frac{2}{\alpha}} r$. Note that for any polynomial Q of degree m and $c > 0$, there exists a constant $N(c,m)$ such that

$$Q(r)\exp(-cr) \leq N r^{-\frac{1}{p}+\delta} \quad r > 0. \tag{3.22}$$

Applying this inequality with $Q(r) = 1$ and $c = m_1$ to (3.21), we have

$$\begin{aligned} |\mathcal{F}(\bar{q}^{c_2,j}_{\alpha,\beta})(t,\xi)| &\leq N 1_{\frac{1}{2}\leq|\xi|\leq 2}|2^j\xi|^{\frac{2}{\alpha p}+\varepsilon} \\ &\quad \times \left(\int_0^1 (t|2^j\xi|^{\frac{2}{\alpha}} r)^{-\frac{1}{p}+\delta} r^{\beta-1} dr + \int_1^\infty (t|2^j\xi|^{\frac{2}{\alpha}} r)^{-\frac{1}{p}+\delta} r^{\beta-1} r^{-2\alpha} dr\right) \\ &\leq N 2^{\frac{2\delta}{\alpha}j+\varepsilon j} t^{-\frac{1}{p}+\delta} 1_{\frac{1}{2}\leq|\xi|\leq 2}. \end{aligned}$$

For the second inequality above we used $\beta - \alpha < \frac{1}{p} - \delta < \beta$.

Similarly, using (3.19), (3.13) and following the above computations, for any multi-index γ we get

$$|D^{\gamma}_{\xi}\mathcal{F}(\bar{q}^{c_2,j}_{\alpha,\beta})(t,\xi)| \leq N2^{\frac{2\delta}{\alpha}j+\varepsilon j}t^{-\frac{1}{p}+\delta}1_{\frac{1}{2}\leq|\xi|\leq 2}.$$

Hence, we have

$$\begin{aligned}\|\bar{q}^{c_2,j}_{\alpha,\beta}(t,\cdot)\|_{L_1} &= \int_{\mathbb{R}^d}(1+|x|^{2d})^{-1}(1+|x|^{2d})|\bar{q}^{c_2,j}_{\alpha,\beta}(t,x)|dx \\ &\leq N\int_{\mathbb{R}^d}(1+|x|^{2d})^{-1}\sup_{\xi}|(1+\Delta^d_{\xi})\mathcal{F}(\bar{q}^{c_2,j}_{\alpha,\beta})(t,\xi))|dx \\ &\leq N2^{\frac{2\delta}{\alpha}j+\varepsilon j}t^{-\frac{1}{p}+\delta}.\end{aligned}$$

For the first inequality above we used the fact that if $\mathcal{F}(f)$ has compact support, then

$$|f(x)| = |\mathcal{F}^{-1}(\mathcal{F}(f))(x)| \leq \|\mathcal{F}(f)\|_{L_1} \leq N\sup_{\xi}|\mathcal{F}(f)|.$$

The lemma is proved. □

The following result will be used later to study the regularity relation between the solutions and free terms in stochastic parts.

Theorem 3.7. *Let* (3.15) *and* (3.16) *hold, and denote* $c_1 := \frac{2(\alpha+1/p-\beta)}{\alpha}$. *Then there exists a constant* N *depdending only on* $\alpha,\beta,d,p,\varepsilon,\delta$, *and* T *such that for any* $g\in C^{\infty}_c((0,\infty)\times\mathbb{R}^d)$

$$\int_0^T\int_0^t\int_{\mathbb{R}^d}\left|(-\Delta)^{\frac{c_1+\varepsilon}{2}}q_{\alpha,\beta}(t-s,x)*g(s)(x)\right|^p dxdsdt \leq N\int_0^T\|g(t,\cdot)\|^p_{B^{\varepsilon}_p}dt. \tag{3.23}$$

Proof. Denote $c_2 = c_1+\varepsilon$ and $Q(t,x) := (-\Delta)^{\frac{c_2}{2}}q_{\alpha,\beta}(t,x)$. By (3.1),

$$\begin{aligned}&\int_0^T\int_0^t\int_{\mathbb{R}^d}|Q(t-s)*g(s)(x)|^p dxdsdt \\ &\quad\leq N\int_0^T\int_0^t\int_{\mathbb{R}^d}|\Psi_0*(Q(t-s)*g(s))(x)|^p \\ &\qquad\qquad+\left(|\sum_{j=1}^{\infty}|\Psi_j*(Q(t-s)*g(s))(x)|^2\right)^{p/2}dxdsdt.\end{aligned}$$

Note that

$$\begin{aligned}\hat{\Psi}_j &= \hat{\Psi}_j(\hat{\Psi}_{j-1}+\hat{\Psi}_j+\hat{\Psi}_{j+1}), \quad j=1,2,\dots, \\ \hat{\Psi}_0 &= \hat{\Psi}_0(\hat{\Psi}_0+\hat{\Psi}_1).\end{aligned} \tag{3.24}$$

Using this and the relation $\mathcal{F}(f_1*f_2)=\mathcal{F}(f_1)\mathcal{F}(f_2)$, we get

$$\sum_{j=1}^{\infty}|\Psi_j*(Q(t-s)*g(s))(x)|^2 = \sum_{j=1}^{\infty}|\sum_{i=j-1}^{j+1}Q_i(t-s)*g_j(s)(x)|^2,$$

$$|\Psi_0*(Q(t-s)*g(s))(x)| = |q^{c_2,0}_{\alpha,\beta}(t-s)*g_0(s)(x)+q^{c_2,1}_{\alpha,\beta}(t-s,\cdot)*g_0(s)(x)|.$$

Therefore,

$$\begin{aligned}&\int_0^T\int_0^t\int_{\mathbb{R}^d}|Q(t-s)*g(s)(x)|^p dxdsdt\\ \leq\ & N\int_0^T\int_0^t\int_{\mathbb{R}^d}|q^{c_2,0}_{\alpha,\beta}(t-s)*g_0(s)(x)|^p dxdsdt\\ &+N\int_0^T\int_0^t\int_{\mathbb{R}^d}|q^{c_2,1}_{\alpha,\beta}(t-s)*g_0(s)(x)|^p dxdsdt\\ &+N\int_0^T\int_0^t\int_{\mathbb{R}^d}\big(\sum_{j=1}^{\infty}|\sum_{i=j-1}^{j+1}q^{c_2,i}_{\alpha,\beta}(t-s)*g_j(s)(x)|^2\big)^{p/2}dxdsdt. \quad (3.25)\end{aligned}$$

By (3.17), the first two integrals on the right hand side of (3.25) are bounded by

$$N\int_0^T\int_0^t(t-s)^{-1+\delta p}\|g_0(s,\cdot)\|^p_{L_p}dtds\leq N(T)\int_0^T\|g_0(t,\cdot)\|^p_{L_p}dt. \quad (3.26)$$

By Minkowski's inequality and Fubini's thoerem, the third integral is bounded by

$$N\int_0^T\int_s^T\big(\sum_{j=1}^{\infty}|K_j(t-s)|^2\|g_j(s,\cdot)\|^2_{L_p}\big)^{p/2}dtds,$$

where $K_j(t-s)=(2^{\frac{2\delta}{\alpha}j+\varepsilon j}(t-s)^{-\frac{1}{p}+\delta}\wedge(t-s)^{-\frac{1}{p}-\frac{\alpha}{2}\varepsilon})$.

If $p=2$ then

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_0^T\int_s^T|K_j(t-s)|^2\|g_j(s,\cdot)\|^2_{L_2}dtds\\ \leq\ & N\int_0^T\sum_{j=1}^{\infty}\int_s^{s+2^{-\frac{2}{\alpha}j}}2^{\frac{4\delta j}{\alpha}+2\varepsilon j}(t-s)^{-1+2\delta}\|g_j(s,\cdot)\|^2_{L_2}dtds\\ &+N\int_0^T\sum_{j=1}^{\infty}\int_{s+2^{-\frac{2}{\alpha}j}}^{\infty}(t-s)^{-1-\alpha\varepsilon}\|g_j(s,\cdot)\|^2_{L_2}dtds\\ =\ & N\int_0^T\sum_{j=1}^{\infty}2^{2\varepsilon j}\|g_j(s,\cdot)\|^2_{L_2}ds.\end{aligned}$$

This proves the theorem if $p=2$.

If $p>2$, then

$$\begin{aligned}&\int_0^T\int_s^T\big(\sum_{j=1}^{\infty}|K_j(t-s)|^2\|g_j(s,\cdot)\|^2_{L_p}\big)^{p/2}dtds\\ \leq\ & N\int_0^T\int_s^T\big(\sum_{j=1}^{\infty}1_J(t,s,j)|K_j(t-s)|^2\|g_j(s,\cdot)\|^2_{L_p}\big)^{p/2}dtds\\ &+\int_0^T\int_s^T\big(\sum_{j=1}^{\infty}1_{J^c}(t,s,j)|K_j(t-s)|^2\|g_j(s,\cdot)\|^2_{L_p}\big)^{p/2}dtds,\end{aligned}$$

where $J=\{(t,s,j)|2^j(t-s)^{\frac{\alpha}{2}}\leq 1\}$. By (3.16), if $(t,s,j)\in J$, then $K_j(t-s)=2^{\frac{2\delta j}{\alpha}+\varepsilon j}(t-s)^{-\frac{1}{p}+\delta}$. Therefore, by Hölder's inequality, we have

$$\begin{aligned}
&\sum_{j=1}^{\infty}1_J|K_j(t-s)|^2\|g_j(s,\cdot)\|_{L_p}^2\\
&=\sum_{j=1}^{\infty}1_J2^{aj}2^{-aj}2^{\frac{4\delta j}{\alpha}+2\varepsilon j}(t-s)^{-\frac{2}{p}+2\delta}\|g_j(s,\cdot)\|_{L_p}^2\\
&\leq (t-s)^{-\frac{2}{p}+2\delta}\Big(\sum_{j\in J(t,s)}2^{aqj}\Big)^{1/q}\Big(\sum_{j\in J(t,s)}2^{-\frac{apj}{2}}2^{\frac{2\delta pj}{\alpha}+p\varepsilon j}\|g_j(s,\cdot)\|_{L_p}^p\Big)^{2/p},
\end{aligned}$$

where $q=\frac{p}{p-2}$, $a\in(0,\frac{4\delta}{\alpha})$, and $J(t,s)=\{j=1,2,\dots|(t,s,j)\in J\}$. Note that

$$\Big(\sum_{j\in J(t,s)}2^{aqj}\Big)^{1/q}\leq N(p)(t-s)^{-\frac{\alpha a}{2}}.$$

Thus we get

$$\begin{aligned}
&\int_0^T\int_s^T\Big(\sum_{j=1}^{\infty}1_J(t,s,j)|K_j(t-s)|^2\|g_j(s,\cdot)\|_{L_p}^2\Big)^{p/2}dtds\\
\leq\ & N\int_0^T\sum_{j=1}^{\infty}\int_s^{s+2^{-\frac{2j}{\alpha}}}(t-s)^{-1+p\delta-\frac{p\alpha a}{4}}2^{-\frac{apj}{2}}2^{\frac{2\delta pj}{\alpha}+p\varepsilon j}\|g_j(s,\cdot)\|_{L_p}^p dtds\\
\leq\ & N\int_0^T\sum_{j=1}^{\infty}2^{p\varepsilon j}\|g_j(t,\cdot)\|_{L_p}^p dt. \qquad (3.27)
\end{aligned}$$

Next we consider the remaining part:

$$\begin{aligned}
\sum_{j=1}^{\infty}1_{J^c}|K_j(t-s)|^2\|g_j(s,\cdot)\|_{L_p}^2&=\sum_{j=1}^{\infty}1_{J^c}2^{bj}2^{-bj}(t-s)^{-\frac{2}{p}-\alpha\varepsilon}\|g_j(s,\cdot)\|_{L_p}^2\\
&\leq (t-s)^{-\frac{2}{p}-\alpha\varepsilon}\Big(\sum_{j\notin J(t,s)}2^{bqj}\Big)^{1/q}\Big(\sum_{j\notin J(t,s)}2^{-\frac{bpj}{2}}\|g_j(s,\cdot)\|_{L_p}^p\Big)^{2/p},
\end{aligned}$$

where $q=\frac{p}{p-2}$, and $b\in(-2\varepsilon,0)$. Note that

$$\Big(\sum_{j\notin J(t,s)}2^{bqj}\Big)^{1/q}\leq N(p)(t-s)^{-\frac{\alpha b}{2}}.$$

Therefore it follows that

$$\begin{aligned}
&\int_0^T\int_s^T\Big(\sum_{j=1}^{\infty}1_{J^c}(t,s,j)|K_j(t-s)|^2\|g_j(s,\cdot)\|_{L_p}^2\Big)^{p/2}dtds\\
\leq\ & N\int_0^T\sum_{j=1}^{\infty}\int_{s+2^{-\frac{2j}{\alpha}}}^{\infty}(t-s)^{-1-\frac{\alpha bp}{4}-\frac{\alpha\varepsilon p}{2}}2^{-\frac{bpj}{2}}\|g_j(s,\cdot)\|_{L_p}^p dtds\\
\leq\ & N\int_0^T\sum_{j=1}^{\infty}2^{p\varepsilon j}\|g_j(t,\cdot)\|_{L_p}^p dt. \qquad (3.28)
\end{aligned}$$

Combining (3.26), (3.27) and (3.28) we get (3.23) for $p>2$. Hence, the theorem is proved. $\square$

The next part of this section is related to the non-zero initial value problem

$$\partial_t^\alpha u = \Delta u, \quad t>0\ ; \quad u(0)=u_0,\ 1_{\alpha>1}\partial_t u(0)=0.$$

The solution is given in the form of $p(t,\cdot)*u_0$, and we study the regularity of this convolution.

Define

$$\begin{aligned} p_j(t,x) &= (\Psi_j(\cdot)*p(t,\cdot))(x) = \mathcal{F}^{-1}(\hat{\Psi}(2^{-j}\cdot)\hat{p}(t,\cdot))(x) \\ &= 2^{jd}\mathcal{F}^{-1}(\hat{\Psi}(\cdot)\hat{p}(t,2^j\cdot))(2^jx) := 2^{jd}\bar{p}_j(t,2^jx). \end{aligned} \tag{3.29}$$

Lemma 3.8. *Let $p>1$, $0<\alpha<2$ and $\alpha\neq 1$. Then there exists a constant N depending only on α, d such that*

$$\|p_j(t,\cdot)\|_{L_1} \le N(2^{-\frac{2j}{\alpha}}t^{-1}\wedge 1), \quad t>0. \tag{3.30}$$

Proof. Let $R(t,x):=|x|^2t^{-\alpha}$. Then by (3.4), and (3.5),

$$\begin{aligned} \int_{\mathbb{R}^d}|p(t,x)|dx &= \int_{R\ge 1}|p(t,x)|dx + \int_{R<1}|p(t,x)|dx \\ &\le N\int_{R\ge1} t^{-\frac{\alpha d}{2}}\exp\{-c|x|^{\frac{2}{2-\alpha}}t^{-\frac{\alpha}{2-\alpha}}\}dx \\ &\quad + N\int_{R<1}|x|^{-d}(R+R|\log R|1_{d=2}+R^{1/2}1_{d=1})dx. \end{aligned}$$

By using change of variables and the relation

$$r^\nu|\log r| \le N(\nu) \quad 0<r\le 1, \quad \nu>0$$

we have $\|p(t,\cdot)\|_{L_1}\le N$. Due to this and the relation $\|p_j(t,\cdot)\|_{L_1} = \|\bar{p}_j(t,\cdot)\|_{L_1}$, it only remains to show

$$\|\bar{p}_j(t,\cdot)\|_{L_1} \le N2^{-\frac{2j}{\alpha}}t^{-1}.$$

By definition (see (3.29))

$$\mathcal{F}(\bar{p}_j)(t,\xi) = \hat{\Psi}(\cdot)\hat{p}(t,2^j\xi). \tag{3.31}$$

Since $q_{\alpha,\alpha} := D_t^{\alpha-\alpha}p = p$, by (3.6) and (3.12), we have

$$\begin{aligned} |\mathcal{F}\bar{p}_j(t,\xi)| \quad &\le \quad N1_{\frac{1}{2}\le|\xi|\le2}\int_0^1 r^{\alpha-1}\exp(-2^{\frac{2j}{\alpha}}|\xi|^{\frac{2}{\alpha}}tr)rdr \\ &\quad + N1_{\frac{1}{2}\le|\xi|\le2}\int_1^\infty r^{-\alpha-1}\exp(-2^{\frac{2j}{\alpha}}|\xi|^{\frac{2}{\alpha}}tr)rdr. \end{aligned} \tag{3.32}$$

Note that for any polynomial Q of degree m and constant $c>0$, we have

$$Q(r)e^{-cr} \le N(c,m)r^{-1}.$$

This and (3.32) easily yield

$$|\mathcal{F}\bar{p}_j(t,\xi)| \le N2^{-\frac{2j}{\alpha}}t^{-1}1_{\frac{1}{2}\le|\xi|\le2}.$$

Similarly, using (3.31) and following above computations, for any multi-index γ we get

$$|D_\xi^\gamma\mathcal{F}\bar{p}_j(t,\xi)| \le N(\alpha,\gamma,d)2^{-\frac{2j}{\alpha}}t^{-1}1_{\frac{1}{2}\le|\xi|\le2}.$$

Therefore, we finally have

$$\begin{aligned}\|\bar{p}_j(t,\cdot)\|_{L_1} &= \int_{\mathbb{R}^d}(1+|x|^{2d})^{-1}(1+|x|^{2d})|\bar{p}_j(t,x)|dx\\ &\leq N\int_{\mathbb{R}^d}(1+|x|^{2d})^{-1}\sup_{\xi}|(1+\Delta_{\xi}^d)\mathcal{F}(\bar{p}_j)(t,\xi)|dx\\ &\leq N2^{-\frac{2j}{\alpha}j}t^{-1}.\end{aligned}$$

The lemma is proved. □

Theorem 3.9. *Let, $p>1$, $0<\alpha<2$ and $f\in C_c^\infty(\mathbb{R}^d)$. Then we have*

$$\int_0^T\int_{\mathbb{R}^d}|p(t,\cdot)*f|^p dxdt \leq N\|f\|^p_{B_p^{-\frac{2}{\alpha p}}}, \tag{3.33}$$

where the constant N depends only on $\alpha, d, p,$ and T.

Proof. Since the case $\alpha=1$ is a classical result, we assume $\alpha\neq 1$. By (3.24), and the relation $\mathcal{F}(f_1*f_2)=\mathcal{F}(f_1)\mathcal{F}(f_2)$,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d}|p(t,\cdot)*f|^p dxdt &\leq N\int_0^T(\|p_0(t,\cdot)\|_{L_1}+\|p_1(t,\cdot)\|_{L_1})^p\|f_0\|^p_{L_p}dt\\ &\quad + N\int_0^T(\sum_{j=1}^\infty\sum_{i=j-1}^{j+1}\|p_i(t,\cdot)\|_{L_1}\|f_j\|_{L_p})^p dt.\end{aligned}$$

By (3.30),

$$\int_0^T(\|p_0(t,\cdot)\|_{L_1}+\|p_1(t,\cdot)\|_{L_1})^p\|f_0\|^p_{L_p}dt \leq N(T)\|f_0\|^p_{L_p}, \tag{3.34}$$

and

$$\int_0^T(\sum_{j=1}^\infty\sum_{i=j-1}^{j+1}\|p_i(t,\cdot)\|_{L_1}\|f_j\|_{L_p})^p dt \leq N\int_0^T(\sum_{j=1}^\infty(2^{-\frac{2j}{\alpha}}t^{-1}\wedge 1)\|f_j\|_{L_p})^p dt.$$

Observe that

$$\begin{aligned}&\int_0^T(\sum_{j=1}^\infty(2^{-\frac{2j}{\alpha}}t^{-1}\wedge 1)\|f_j\|_{L_p})^p dt\\ &\leq \int_0^T(\sum_{j=1}^\infty 1_J(t,j)\|f_j\|_{L_p})^p dt + \int_0^T(\sum_{j=1}^\infty 1_{J^c}(t,j)2^{-\frac{2j}{\alpha}}t^{-1}\|f_j\|_{L_p})^p dt,\end{aligned}$$

where $J=\{(t,j)|2^{\frac{2j}{\alpha}}t\leq 1\}$. By Hölder's inequality,

$$\begin{aligned}\int_0^T(\sum_{j=1}^\infty 1_J\|f_j\|_{L_p})^p dt &= \int_0^T(\sum_{j\in J(t)}2^{-\frac{2j}{\alpha}a}2^{\frac{2j}{\alpha}a}\|f_j\|_{L_p})^p dt\\ &\leq \int_0^T(\sum_{j\in J(t)}2^{-\frac{2j}{\alpha}aq})^{p/q}(\sum_{j\in J(t)}2^{\frac{2j}{\alpha}ap}\|f_j\|^p_{L_p})dt,\end{aligned}$$

where $a\in(-\frac{1}{p},0)$, $q=\frac{p}{p-1}$, and $J(t)=\{j=1,2,\dots|(t,j)\in J\}$. Since

$$\sum_{j\in J(t)}2^{-\frac{2j}{\alpha}aq}\leq N(q,a)t^{aq},$$

we have

$$\begin{aligned}\int_0^T \big(\sum_{j=1}^{\infty} 1_J \|f_j\|_{L_p}\big)^p dt &\leq N \sum_{j=1}^{\infty} \int_0^{2^{-\frac{2j}{\alpha}}} t^{ap} 2^{\frac{2j}{\alpha}ap} \|f_j\|_{L_p}^p dt \\ &\leq N \sum_{j=1}^{\infty} 2^{-\frac{2j}{\alpha}} \|f_j\|_{L_p}^p.\end{aligned} \tag{3.35}$$

By Hölder's inequality again, for $b \in (-1, -\frac{1}{p})$ and $q = \frac{p}{p-1}$,

$$\begin{aligned}&\int_0^T \big(\sum_{j=1}^{\infty} 1_{J^c} 2^{-\frac{2j}{\alpha}} t^{-1} \|f_j\|_{L_p}\big)^p dt \\ &= \int_0^T \big(\sum_{j \notin J(t)} 2^{-\frac{2j}{\alpha}b} 2^{\frac{2j}{\alpha}b} 2^{-\frac{2j}{\alpha}} t^{-1} \|f_j\|_{L_p}\big)^p dt \\ &\leq \int_0^T t^{-p} \big(\sum_{j \notin J(t)} 2^{-\frac{2j}{\alpha}bq} 2^{-\frac{2j}{\alpha}q}\big)^{p/q} \big(\sum_{j \notin J(t)} 2^{\frac{2j}{\alpha}bp} \|f_j\|_{L_p}^p\big) dt.\end{aligned}$$

Since

$$\sum_{j \notin J(t)} 2^{-\frac{2j}{\alpha}(b+1)q} \leq N(q,b) t^{(b+1)q},$$

we have

$$\begin{aligned}\int_0^T \big(\sum_{j=1}^{\infty} 1_{J^c} 2^{-\frac{2j}{\alpha}} t^{-1} \|f_j\|_{L_p}\big)^p dt &\leq N \sum_{j=1}^{\infty} \int_{2^{-\frac{2j}{\alpha}}}^{\infty} t^{-p} t^{(b+1)p} 2^{\frac{2j}{\alpha}bp} \|f_j\|_{L_p}^p dt \\ &= N \sum_{j=1}^{\infty} 2^{-\frac{2j}{\alpha}} \|f_j\|_{L_p}^p.\end{aligned} \tag{3.36}$$

Combining (3.34), (3.35) and (3.36), we have (3.33). The theorem is proved. □

The last part of this section is related to the non-zero initial date problem of the type

$$\partial_t^\alpha u = \Delta u, \quad t > 0\ ; \quad u(0,x) = 0,\ 1_{\alpha>1} \partial_t u(0,x) = 1_{\alpha>1} v_0(x). \tag{3.37}$$

Let $\alpha > 1$. Then using Lemma 3.2, each $x \neq 0$, one can check that

$$P(t,x) := q_{\alpha,\alpha-1} = \int_0^t p(s,x) ds$$

is well defined and becomes a fundamental solution to (3.37).

For $j = 0, 1, 2, \dots$ define

$$\begin{aligned}P_j(t,x) &= (\Psi_j(\cdot) * (-\Delta)^{\varepsilon/2} P(t,\cdot))(x) \\ &= \mathcal{F}^{-1}(\hat{\Psi}(2^{-j}\cdot) \mathcal{F}((-\Delta)^{\varepsilon/2} P)(t,\cdot))(x) \\ &= 2^{jd} \mathcal{F}^{-1}(\hat{\Psi}(\cdot) \mathcal{F}((-\Delta)^{\varepsilon/2} P)(t, 2^j\cdot))(2^j x) \\ &:= 2^{jd} \bar{P}_j(t, 2^j x).\end{aligned} \tag{3.38}$$

Lemma 3.10. *Let $\alpha \in (1,2)$. Then, for any $\delta \in (0,\alpha)$, there exists a constant N depending only on $\alpha, d, \varepsilon, \delta$ such that for any $t > 0$,*

$$\|P_j(t,\cdot)\|_{L_1} \leq N(2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta} \wedge t). \tag{3.39}$$

Proof. By (3.7), (3.8) and (3.9), we easily get

$$\|P(t,\cdot)\|_{L_1} \le Nt.$$

Therefore, it suffices to show that

$$\|\bar{P}_j(t,\cdot)\|_{L_1} \le N 2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta}.$$

By definition,

$$\mathcal{F}(\bar{P}_j)(t,\xi) = \hat{\Psi}(\xi)\mathcal{F}(P)(t,2^j\xi). \tag{3.40}$$

Also, by Lemma 3.3 and Lemma 3.5 with $\mu = 0$, $\beta = \alpha - 1$, we have

$$\begin{aligned}
&|\mathcal{F}\{P(t,\cdot)\}(2^j\xi)| \\
&\le N|2^j\xi|^{-\frac{2}{\alpha}} \int_0^\infty \frac{\exp\left(-m_1 t|2^j\xi|^{\frac{2}{\alpha}} r\right)\left(|r^\alpha \sin(\psi+m_3)| + |\sin(\psi+m_4)|\right)}{r^{2\alpha} - 2r^\alpha m_5 + 1} r^{\alpha-2} dr,
\end{aligned}$$

where $\psi = m_2 t|2^j\xi|^{\frac{2}{\alpha}} r$. Note that for any polynomial Q of degree m, and $c > 0$, we have

$$Q(r)\exp(-cr) \le N(c,m) r^{1-\alpha+\delta}, \quad r > 0. \tag{3.41}$$

This with the condition $\delta \in (0,\alpha)$ gives

$$\begin{aligned}
&|\mathcal{F}(\bar{P}_j)(t,\xi)| \le 1_{1/2\le|\xi|\le 2}|\mathcal{F}(P)(t,2^j\xi)| \\
\le\ & N 1_{1/2\le|\xi|\le 2} 2^{-\frac{2j}{\alpha}} \Big(\int_0^1 (t|2^j\xi|^{\frac{2}{\alpha}} r)^{1-\alpha+\delta} r^{\alpha-2} dr + \int_1^\infty (t|2^j\xi|^{\frac{2}{\alpha}} r)^{1-\alpha+\delta} r^{-2} dr \Big) \\
\le\ & N 2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta} 1_{\frac{1}{2}\le|\xi|\le 2}.
\end{aligned}$$

Using (3.40) and similar computations above, we also get for any multi-index γ

$$|D^\gamma_\xi \mathcal{F}(\bar{P}_j)| \le N 2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta} 1_{\frac{1}{2}\le|\xi|\le 2}.$$

Therefore, we have

$$\begin{aligned}
\|\bar{P}_j(t,\cdot)\|_{L_1} &= \int_{\mathbb{R}^d} (1+|x|^{2d})^{-1}(1+|x|^{2d})|\bar{P}_j(t,x)| dx \\
&\le N \int_{\mathbb{R}^d} (1+|x|^{2d})^{-1} \sup_\xi |(1+\Delta^d_\xi)\mathcal{F}(\bar{P}_j)(t,\xi)| dx \\
&\le N 2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta}.
\end{aligned}$$

The lemma is proved. □

Theorem 3.11. *Let $\alpha \in (1,2)$ and $h \in C_c^\infty(\mathbb{R}^d)$. Then there exists a constant $N = N(\alpha,d,p,T)$ such that*

$$\int_0^T \int_{\mathbb{R}^d} |(P(t)*f)(x)|^p dx dt \le N\|h\|^p_{B_p^{-\frac{2}{\alpha p}-\frac{2}{\alpha}}}, \quad \text{if} \quad \alpha > 1 + \frac{1}{p} \tag{3.42}$$

and

$$\int_0^T \int_{\mathbb{R}^d} |(P(t)*f)(x)|^p dx dt \le N\|h\|^p_{B_p^{-\frac{2}{\alpha}}}, \quad \text{if} \quad 1 < \alpha \le 1 + 1/p. \tag{3.43}$$

Proof. **Case 1**. Let $\alpha > 1 + 1/p$. Then by assumption on α, we can take $\delta \in (0,\alpha)$ such that

$$\alpha - 1 - \delta - \frac{1}{p} > 0, \quad -2 + \frac{2\delta}{\alpha} < 0. \tag{3.44}$$

By (3.24) and the relation $\mathcal{F}(h_1 * h_2) = \mathcal{F}(h_1)\mathcal{F}(h_2)$,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} |(P(t) * h)(x)|^p dxdt \\ &\leq N \int_0^T (\|P_0(t,\cdot)\|_{L_1} + \|P_1(t,\cdot)\|_{L_1})^p \|h_0\|_{L_p}^p dt \\ &\quad + N \int_0^T \big(\sum_{j=1}^{\infty} \sum_{i=j-1}^{j+1} \|P_i(t,\cdot)\|_{L_1} \|h_j\|_{L_p}\big)^p dt.\end{aligned}$$

Note that (3.39) with (3.44) easily yields

$$\int_0^T (\|P_0(t,\cdot)\|_{L_1} + \|P_1(t,\cdot)\|_{L_1})^p \|h_0\|_{L_p}^p dt \leq N(T)\|h_0\|_{L_p}^p. \tag{3.45}$$

Also by (3.44),

$$\begin{aligned}&\int_0^T \big(\sum_{j=1}^{\infty} \sum_{i=j-1}^{j+1} \|P_i(t,\cdot)\|_{L_1} \|h_j\|_{L_p}\big)^p dt \\ &\leq N \int_0^T \big(\sum_{j=1}^{\infty} L_j(t) \|h_j\|_{L_p}\big)^p dt \\ &\leq N \int_0^T \big(\sum_{j=1}^{\infty} 1_J(t,j) L_j(t) \|h_j\|_{L_p}\big)^p dt + N \int_0^T \big(\sum_{j=1}^{\infty} 1_{J^c}(t,j) L_j(t) \|h_j\|_{L_p}\big)^p dt,\end{aligned}$$

where $J := \{(t,j) | 2^j t^{\frac{\alpha}{2}} \geq 1\}$, and

$$L_j(t) := (2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta} \wedge t) = \begin{cases} 2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha+\delta} & : (t,j) \in J \\ t & : (t,j) \notin J. \end{cases}$$

By Hölder's inequality,

$$\begin{aligned}&\int_0^T \big(\sum_{j=1}^{\infty} 1_J L_j(t) \|h_j\|_{L_p}\big)^p dt \\ &= \int_0^T \big(\sum_{j=1}^{\infty} 1_J 2^{-2j+\frac{2\delta}{\alpha}j} t^{1-\alpha-\delta} 2^{-\frac{2bj}{\alpha}} 2^{\frac{2bj}{\alpha}} \|h_j\|_{L_p}\big)^p dt \\ &\leq \int_0^T t^{(1-\alpha+\delta)p} \big(\sum_{j\in J(t)} 2^{-\frac{2bqj}{\alpha}}\big)^{p/q} \big(\sum_{j\in J(t)} 2^{-2pj+\frac{2\delta}{\alpha}pj} 2^{\frac{2pbj}{\alpha}} \|h_j\|_{L_p}^p\big) dt,\end{aligned} \tag{3.46}$$

where $b \in (0, \alpha - 1 - \frac{1}{p} - \delta)$, $q = \frac{p}{p-1}$, and $J(t) = \{j = 1, 2, \dots | (t,j) \in J\}$. Since

$$\big(\sum_{j\in J(t)} 2^{-\frac{2qb}{\alpha}j}\big)^{p/q} \leq N(\alpha,p) t^{bp},$$

we have

$$\begin{aligned}
&\int_0^T \big(\sum_{j=1}^{\infty} 1_J L_j(t)\|h_j\|_{L_p}\big)^p dt \\
&\quad\leq N \int_0^T t^{(1-\alpha+\delta)p} t^{bp} \big(\sum_{j\in J(t)} 2^{-2pj+\frac{2\delta}{\alpha}pj} 2^{\frac{2pbj}{\alpha}} \|h_j\|_{L_p}^p\big) dt \\
&\quad\leq N \sum_{j=1}^{\infty} \int_{2^{-\frac{2j}{\alpha}}}^{\infty} t^{(1-\alpha+\delta)p} t^{bp} 2^{-2pj+\frac{2\delta}{\alpha}pj} 2^{\frac{2pbj}{\alpha}} \|h_j\|_{L_p}^p \\
&\quad\leq N \sum_{j=1}^{\infty} 2^{-\frac{2pj}{\alpha}-\frac{2j}{\alpha}} \|h_j\|_{L_p}^p.
\end{aligned} \tag{3.47}$$

Again by Hölder's inequality,

$$\begin{aligned}
\int_0^T \big(\sum_{j=1}^{\infty} 1_{J^c} L_j(t)\|h_j\|_{L_p}\big)^p dt &= \int_0^T \big(\sum_{j\notin J(t)} t 2^{-\frac{2aj}{\alpha}} 2^{\frac{2aj}{\alpha}} \|h_j\|_{L_p}\big)^p dt \\
&\leq \int_0^T t^p \big(\sum_{j\notin J(t)} 2^{-\frac{2aqj}{\alpha}}\big)^{p/q} \big(\sum_{j\notin J(t)} 2^{\frac{2paj}{\alpha}} \|h_j\|_{L_p}^p\big) dt,
\end{aligned}$$

where $a \in (-1-\frac{1}{p}, 0)$, and $q = \frac{p}{p-1}$. Since

$$\big(\sum_{j\notin J(t)} 2^{-\frac{2aqj}{\alpha}}\big)^{p/q} \leq N(\alpha,p) t^{ap},$$

we have

$$\begin{aligned}
\int_0^T \big(\sum_{j=1}^{\infty} 1_{J^c} L_j(t)\|h_j\|_{L_p}\big)^p dt &\leq N \int_0^T t^{p+ap} \big(\sum_{j\notin J(t)} 2^{\frac{2paj}{\alpha}} \|h_j\|_{L_p}^p\big) dt \\
&\leq N \sum_{j=1}^{\infty} \int_0^{2^{-\frac{2j}{\alpha}}} t^{p+ap} 2^{\frac{2paj}{\alpha}} \|h_j\|_{L_p}^p dt \\
&\leq N \sum_{j=1}^{\infty} 2^{-\frac{2pj}{\alpha}-\frac{2j}{\alpha}} \|h_j\|_{L_p}^p.
\end{aligned} \tag{3.48}$$

Combining (3.45), (3.47), and (3.48), we get (3.42). The theorem is proved.

Case 2. Let $1 \leq \alpha < 1 + 1/p$. This time, we choose $\delta, b > 0$ such that

$$\alpha - 1 - \delta > 0, \quad b \in (0, \alpha - 1 - \delta), \tag{3.49}$$

and repeat the proof of Case 1. The only difference is we need to replace (3.47) by the following:

$$\begin{aligned}
&\int_0^T \Big(\sum_{j=1}^{\infty} 1_J L_j(t)\|h_j\|_{L_p}\Big)^p dt \\
&\leq N\int_0^T t^{(1-\alpha+\delta)p}t^{bp}\Big(\sum_{j\in J(t)} 2^{-2pj+\frac{2\delta}{\alpha}pj}2^{\frac{2pbj}{\alpha}}\|h_j\|^p_{L_p}\Big)dt \\
&\leq N\int_0^T t^{(1-\alpha+\delta-1/p)p}t^{bp}\Big(\sum_{j\in J(t)} 2^{-2pj+\frac{2\delta}{\alpha}pj}2^{\frac{2pbj}{\alpha}}\|h_j\|^p_{L_p}\Big)dt \\
&\leq N\sum_{j=1}^{\infty}\int_{2^{-\frac{2j}{\alpha}}}^{\infty} t^{(1-\alpha+\delta-1/p)p}t^{bp}2^{-2pj+\frac{2\delta}{\alpha}pj}2^{\frac{2pbj}{\alpha}}\|h_j\|^p_{L_p} \\
&\leq N\sum_{j=1}^{\infty} 2^{-\frac{2pj}{\alpha}}\|h_j\|^p_{L_p}.
\end{aligned}$$

On the other hand, (3.48) still holds without any changes, and this certainly implies

$$\int_0^T \Big(\sum_{j=1}^{\infty} 1_{J^c} L_j(t)\|h_j\|_{L_p}\Big)^p dt \leq N\sum_{j=1}^{\infty} 2^{-\frac{2pj}{\alpha}}\|h_j\|^p_{L_p}.$$

Hence, Case 2 is also proved. □

4. Proof of Theorem 2.15

We first prove a version of Theorem 2.15 for (deterministic) equation (4.1).

Lemma 4.1. *Let $0<\alpha<2$, $1<p<\infty$ and $\gamma\in\mathbb{R}$. Then for any $u_0\in U_p^{\gamma+2}$, $v_0\in V_p^{\gamma+2}$ and $f\in\mathbb{H}_p^{\gamma}(T)$ the equation*

$$\partial_t^{\alpha}u=\Delta u+f,\quad t>0, x\in\mathbb{R}^d;\quad u(0)=u_0,\ 1_{\alpha>1}\partial_t u(0)=1_{\alpha>1}v_0 \tag{4.1}$$

has a unique solution $u\in\mathcal{H}_p^{\gamma+2}(T)$, and moreover

$$\|u\|_{\mathcal{H}_p^{\gamma+2}(T)}\leq N\big(\|u_0\|_{U_p^{\gamma+2}}+1_{\alpha>1}\|v_0\|_{V_p^{\gamma+2}}+\|f\|_{\mathbb{H}_p^{\gamma}(T)}\big), \tag{4.2}$$

where the constant N depends only on α, d, p, γ, and T.

Proof. Due to Remark 3.1, it is enough to prove the lemma for a particular γ, and therefore we assume $\gamma=-2$.

The statements of the lemma hold if $u_0=v_0=0$ due to [10, Theorem 2.3] (or [11, Theorem 2.10]), from which the uniqueness result follows. Furthermore, considering $u-v$, where v is the solution to the equation with $u_0=v_0=0$ taken from [10, Theorem 2.3], we may assume that $f=0$.

Now, let $u_0,v_0\in\mathbb{L}_c$ and define

$$u(t,x):=(p(t,\cdot)*u_0(\cdot))(x)+1_{\alpha>1}(P(t,\cdot)*v_0(\cdot))(x).$$

Then by Lemma 3.2 (or see [11, Lemma 3.5] for more detail), u satisfies equation (4.1), and $u\in\mathbb{H}_p^n(T)$ for any $n\in\mathbb{R}$, since $u_0,v_0\in\mathbb{L}_c$. Moreover, for this solution we have

$$\|u\|_{\mathbb{H}_p^{\gamma+2}(T)}\leq N\big(\|u_0\|_{U_p^{\gamma+2}}+1_{\alpha>1}\|v_0\|_{V_p^{\gamma+2}}+\|f\|_{\mathbb{H}_p^{\gamma}(T)}\big)$$

with $\gamma = -2$ due to Theorem 3.9, Theorem 3.11, and Remark 3.1. This estimate and the definition of norm in $\mathcal{H}_p^0(T)$ certainly yield (4.2).

In general, take $u_0^n, v_0^n \in \mathbb{L}_c$ such that $u_0^n \to u_0$ in U_p^0 and $v_0^n \to v_0$ in V_p^0, and for each n let u_n denote the solution to the equation with initial data u_0^n and v_0^n. Then estimate (4.2) corresponding to $u_n - u_m$, where $n, m \in \mathbb{N}$, shows that u_n is a Cauchy sequence in $\mathcal{H}_p^0(T)$, which is a Banach space. Now it is easy to check that the limit of the Cauchy sequence becomes a solution to the equation with initial data u_0 and v_0, and the estimate also follows. The lemma is proved. □

Remark 4.2. The proof of Lemma 4.1 actually shows that the lemma holds for any $p \in (1, \infty)$ with appropriate Besov spaces. Precisely speaking, if $\alpha > 1 + 1/p$, then we can use $B_p^{\gamma+2-2/\alpha p}$ and $B_p^{\gamma+2-2/\alpha-2/\alpha p}$ instead of $U_p^{\gamma+2}$, and $V_p^{\gamma+2}$ respectively, and for $\alpha \le 1 + 1/p$, then we can use $B_p^{\gamma+2-2/\alpha p}$ and $B_p^{\gamma+2-2/\alpha}$ instead of $U_p^{\gamma+2}$, and $V_p^{\gamma+2}$ respectively.

Lemma 4.3. *Let $\alpha \in (0, 2), \beta_2 < \alpha + 1/p$ and $h \in \mathbb{H}_c^\infty(T)$. Denote*

$$u(t, x) := \sum_{k=1}^{\infty} \int_0^t \left(\int_{\mathbb{R}^d} q_{\alpha,\beta_2}(t - s, x - y) h^k(s, y) dy \right) \cdot dZ_s^k. \tag{4.3}$$

Then $u \in \mathcal{H}_p^2(T)$ and satisfies

$$\partial_t^\alpha u = \Delta u + \partial_t^{\beta_2} \int_0^t h^k(s, x) \cdot dZ_s^k,\ t > 0, x \in \mathbb{R}^d; \quad u(0) = \partial_t u(0) 1_{\alpha>1} = 0 \tag{4.4}$$

in the sense of Definition 2.7.

Proof. It is enough to repeat the proof of [4, Lemma 3.10], which deals with the equation driven by Brownian motions. □

Next we prove a version of Theorem 2.15 for the linear equation

$$\begin{aligned} &\partial_t^\alpha u = \Delta u + f + \partial_t^{\beta_1} \int_0^t g^k(s, x) dW_s^k + \partial_t^{\beta_2} \int_0^t h^k(s, x) \cdot dZ_s^k, \quad t > 0, x \in \mathbb{R}^d, \\ &u(0) = u_0, \quad \partial_t u(0) 1_{\alpha>1} = 1_{\alpha>1} v_0, \end{aligned} \tag{4.5}$$

Theorem 4.4. *Let $\gamma \in \mathbb{R}$, $p \ge 2$, $\beta_1 < \alpha + 1/2$ and $\beta_2 < \alpha + 1/p$. Then, for any $u_0 \in U_p^{\gamma+2}$, $v_0 \in V_p^{\gamma+2}$, $f \in \mathbb{H}_p^\gamma(T)$, $g \in \mathbb{H}_p^{\gamma+c_0}(T, l_2)$ and $h \in \mathbb{H}_p^{\gamma+\bar{c}_0}(T, l_2, d_1)$, equation (4.5) has a unique solution u in the class $\mathcal{H}_p^{\gamma+2}(T)$, and for this solution it holds that*

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\gamma+2}(T)} \le N\big(&\|u_0\|_{U_p^{\gamma+2}} + 1_{\alpha>1} \|v_0\|_{V_p^{\gamma+2}} + \|f\|_{\mathbb{H}_p^\gamma(T)} \\ &+ \|g\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)} + \|h\|_{\mathbb{H}_p^{\gamma+\bar{c}_0'}(T,l_2,d_1)} \big), \end{aligned} \tag{4.6}$$

where $N = N(\alpha, \beta_1, \beta_2, d, d_1, p, \gamma, T)$.

Proof. Due to Remark 3.1 it is enough to prove the lemma for $\gamma = 0$. The uniqueness follows from Lemma 4.1.

Recall that the lemma holds if $h = 0$ and $u_0 = v_0 = 0$ by [10, Theorem 2.3], and it holds if $f = 0, g = 0, h = 0$ by Lemma 4.1. By the linearity of the equation, if $h = 0$ then the existence and the desired estimate is easily obtained by combining [10, Theorem 2.3] and Lemma 4.1. The case $h = 0$ is proved.

Furthermore, by the result for the case $h = 0$ and the linearity of the equation, to finish the proof of the lemma, we only need to prove the existence result and estimate (4.6), provided that $u_0 = v_0 = 0, f = 0$ and $g = 0$. Also it suffices to prove (4.6) with $\|u\|_{\mathbb{H}_p^{\gamma+2}(T)}$ in place of $\|u\|_{\mathcal{H}_p^{\gamma+2}(T)}$ due to the definition of $\|u\|_{\mathcal{H}_p^{\gamma+2}(T)}$. We divide the proof of this into following three cases.

Case 1. Let $\beta_2 > 1/p$.

If $h \in \mathbb{H}_c^\infty(T, l_2, d_1)$, we define $u \in \mathcal{H}_p^2(T)$ as in (4.3) such that it becomes a solution to equation (4.4). Denote $c_1 := \frac{2(\alpha+1/p-\beta_2)}{\alpha}$ and take a small constant $\varepsilon \in (0, c_1)$ satisfying (3.16) with β_2 in place of β, and set

$$v := (-\Delta)^{(2-c_1-\varepsilon)/2} u, \quad \bar{h} := (-\Delta)^{(2-c_1-\varepsilon)/2} h.$$

By Burkholder-Davis-Gundy inequality, and (2.9)

$$\begin{aligned}\|\Delta u\|_{\mathbb{L}_p(T)}^p &= \|(-\Delta)^{(c_1+\varepsilon)/2} v\|_{\mathbb{L}_p(T)}^p \\ &\leq N\mathbb{E}\int_{\mathbb{R}^d}\int_0^T \left(\int_0^t \sum_{k=1}^\infty \left|(-\Delta)^{\frac{c_1+\varepsilon}{2}} q_{\alpha,\beta_2}(t-s,\cdot) * \bar{h}^k(s,\cdot)\right|^2 (x) ds\right)^{\frac{p}{2}} dtdx \\ &\quad + N\mathbb{E}\int_{\mathbb{R}^d}\int_0^T \int_0^t \sum_{k=1}^\infty \left|(-\Delta)^{\frac{c_1+\varepsilon}{2}} q_{\alpha,\beta_2}(t-s,\cdot) * \bar{h}^k(s,\cdot)\right|^p (x) dsdtdx.\end{aligned}$$

By [10, Theorem 3.1] we have

$$\mathbb{E}\int_{\mathbb{R}^d}\int_0^T \left(\int_0^t \sum_{k=1}^\infty \left|(-\Delta)^{\frac{c_1+\varepsilon}{2}} q_{\alpha,\beta_2}(t-s,\cdot) * \bar{h}^k(s,\cdot)\right|^2 (x) ds\right)^{\frac{p}{2}} dtdx \leq N\|\bar{h}\|_{\mathbb{L}_p(T,l_2,d_1)}^p,$$

where the constant N depends only on α, β_2, d, d_1, and p. Also by Theorem 3.7 and Remark 3.1

$$\begin{aligned}&\mathbb{E}\int_{\mathbb{R}^d}\int_0^T \int_0^t \sum_{k=1}^\infty \left|(-\Delta)^{\frac{c_1+\varepsilon}{2}} q_{\alpha,\beta_2}(t-s,\cdot) * \bar{h}^k(s,\cdot)\right|^p (x) dsdtdx \\ &\leq N\mathbb{E}\sum_{r=1}^{d_1}\sum_{k=1}^\infty \int_0^T \|\bar{h}^{rk}(t,\cdot)\|_{B_p^\varepsilon}^p dt \leq N\mathbb{E}\sum_{r=1}^{d_1}\sum_{k=1}^\infty \int_0^T \|h^{rk}(t,\cdot)\|_{H_p^{2-c_1}}^p dt,\end{aligned}$$

where the constant N depends only on α, β_2, d, d_1, and p. The above estimations and the inequality $\sum_{k=1}^\infty |a_k|^p \leq \left(\sum_{k=1}^\infty |a_k|^2\right)^{p/2}$ yield

$$\|\Delta u\|_{\mathbb{L}_p(T)}^p \leq N\|h\|_{\mathbb{H}_p^{2-c_1}(T,l_2,d_1)}^p = N\|h\|_{\mathbb{H}_p^{\bar{c}_0'}(T,l_2,d_1)}^p. \tag{4.7}$$

Also, due to (2.20) and the inequality $\|\cdot\|_{\mathbb{L}_p(s)} \leq \|\cdot\|_{\mathbb{L}_p(T)}$ for $s \leq T$, we have

$$\begin{aligned}\|u\|_{\mathbb{L}_p(T)}^p &\leq N\int_0^T (T-s)^{\theta-1}\left(\|\Delta u\|_{\mathbb{L}_p(T)}^p + \|h\|_{\mathbb{L}_p(T,l_2,d_1)}^p\right) ds \\ &\leq N(\|\Delta u\|_{\mathbb{L}_p(T)}^p + \|h\|_{\mathbb{L}_p(T,l_2,d_1)}^p) \leq N\|h\|_{\mathbb{H}_p^{\bar{c}_0'}(T,l_2,d_1)}^p.\end{aligned}$$

This, (4.7) and the inequality $\|u\|_{H_p^2} \leq \|u\|_{L_p} + \|\Delta u\|_{L_p}$ yield estimate (4.6).

For general $h \in \mathbb{H}_p^{\bar{c}_0'}(T, l_2, d_1)$, it is enough to repeat the approximation argument used in the proof of Lemma 4.1.

Case 2. Let $\beta_2 = 1/p$. The argument used in Case 1 shows that to prove the existence result and estimate (4.6) we may assume $h \in \mathbb{H}_c^\infty(T, l_2, d_1)$. In this case the existence is a consequence of Lemma 4.3.

Let $u \in \mathcal{H}_p^2(T)$ be the solution to the equation. Take $\kappa > 0$ from (2.12), and put $\kappa' = \kappa\alpha/2$ and $\beta_2' = 1/p + \kappa' > 1/p$. Then by Case 1 with $\bar{c}_0' = (2\beta_2' - 2/p)/\alpha = \kappa$, if we define v as in (4.3) with β_2' in place of β_2, then v satisfies (4.4) (with β_2'), and

$$\|v\|_{\mathcal{H}_p^2(T)} \le N\|h\|_{\mathbb{H}_p^\kappa(T,l_2,d_1)}. \tag{4.8}$$

Since $I_t^{\kappa'} v$ satisfies (4.4) (with β_2), by the uniqueness of solutions, we obtain $u(t,x) = I_t^{\kappa'} v(t,x)$, and (4.6) holds due to (2.1) and (4.8). Hence the case $\beta_2 = 1/p$ is also proved.

Case 3. Let $\beta_2 < 1/p$. As in Case 2, we only need to prove estimate (4.6), provide that $h \in \mathbb{H}_c^\infty(T, l_2, d_1)$ and the solution u already exists.

Put

$$\bar{f}(t,x) =: \frac{1}{\Gamma(1-\beta_2)} \int_0^t (t-s)^{-\beta_2} h^k(x) \cdot dZ_s^k.$$

Then by the Burkerholder-Davis-Gundy inequality and (2.10),

$$\|\bar{f}\|_{\mathbb{L}_p(T)}^p \le N\mathbb{E}\int_0^T \int_0^t (t-s)^{-\beta_2 p} |h(s,\cdot)|_{L_p(l_2,d_1)}^p ds dt \le N\|h\|_{\mathbb{L}_p(T,l_2,d_1)}^p. \tag{4.9}$$

Note that by Lemma 2.6 (iii), u satisfies

$$\partial_t^\alpha u = \Delta u + \bar{f},\ t > 0\,; \quad u(0) = 1_{\alpha>1}\partial_t u(0) = 0.$$

Therefore, estimate (4.6) follows from (4.9) and Lemma 4.1. The theorem is proved. □

Proof of Theorem 2.15.

1. Linear case. Due to the method of continuity (see e.g. [10, Lemma 5.1]) and Theorem 4.4 we only need to prove that a priori estimate (2.25) holds, provided that a solution $u \in \mathcal{H}_p^{\gamma+2}(T)$ to equation (1.1) already exists. Also note that due to the definition of the norm in $\mathcal{H}_p^{\gamma+2}(T)$, we only need to prove (2.25) with $\|u\|_{\mathbb{H}_p^{\gamma+2}(T)}$ in place of $\|u\|_{\mathcal{H}_p^{\gamma+2}(T)}$.

Step 1. Assume $u_0 = v_0 = 0$. Denote

$$\bar{f} := b^i u_{x^i} + cu + f, \quad \bar{g}^k := \mu^{ik} u_{x^i} + \nu^k u + g^k, \quad \bar{h}^k := \bar{\mu}^{ik} u_{x^i} + \bar{\nu}^k u + h^k.$$

Recall that $c_0, \bar{c}_0 < 2$. By Assumption 2.13, $\mu = 0$ if $c_0 \ge 1$, and $\bar{\mu} = 0$ if $\bar{c}_0 \ge 1$. Therefore, by (2.23),

$$\begin{aligned} \|\bar{g}\|_{\mathbb{H}_p^{\gamma+c_0}(t,l_2)} &\le N 1_{c_0<1}\|u_x\|_{\mathbb{H}_p^{\gamma+c_0}(T)} + N\|u\|_{\mathbb{H}_p^{\gamma+c_0}(T)} + \|g\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)} \\ &\le N 1_{c_0<1}\|u\|_{\mathbb{H}_p^{\gamma+c_0+1}(T)} + N\|u\|_{\mathbb{H}_p^{\gamma+c_0}(T)} + \|g\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)}. \end{aligned}$$

The similar estimate holds for $\bar{f}$ and $\bar{h}$. Using these and the embedding inequality

$$\|u\|_{H_p^{\gamma+\delta}} \le \varepsilon\|u\|_{H_p^{\gamma+2}} + N(\delta,\varepsilon)\|u\|_{H_p^\gamma}, \quad \delta \in (0,2),\ \varepsilon > 0, \tag{4.10}$$

we get, for any $\varepsilon > 0$ and $t \le T$,

$$\begin{aligned} &\|\bar{f}\|_{\mathbb{H}_p^\gamma(t)} + \|\bar{g}\|_{\mathbb{H}_p^{\gamma+c_0}(t,l_2)} + \|\bar{h}\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(t,l_2,d_1)} \\ &\quad \le \varepsilon\|u\|_{\mathbb{H}_p^{\gamma+2}(t)} + N\|u\|_{\mathbb{H}_p^\gamma(t)} \\ &\qquad + \|f\|_{\mathbb{H}_p^\gamma(t)} + \|g\|_{\mathbb{H}_p^{\gamma+c_0}(t,l_2)} + \|h\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(t,l_2,d_1)} < \infty. \end{aligned} \tag{4.11}$$

Recall $\frac{\partial}{\partial x^i}: H_p^\nu \to H_p^{\nu-1}$ is a bounded operator for any $\nu \in \mathbb{R}$. Using this, (2.23) and Assumption 2.13, we easily have

$$\begin{aligned} &\|\bar{f}\|_{\mathbb{H}_p^\gamma(T)} + \|\bar{g}\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)} + \|\bar{h}\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(T,l_2,d_1)} \\ \leq\ & N\|u\|_{\mathbb{H}_p^{\gamma+2}(T)} + \|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)} + \|h\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(T,l_2,d_1)}. \end{aligned} \tag{4.12}$$

Due to Theorem 4.4 and (4.11), we can define $v \in \mathcal{H}_p^{\gamma+2}(T)$ as the solution to equation (4.5) with $\bar{g}$ and $\bar{h}$ in place of g and h, respectively. Furthermore, for each $t \leq T$ we have

$$\|v\|_{\mathbb{H}_p^{\gamma+2}(t)} \leq N\|f\|_{\mathbb{H}_p^\gamma(t)} + N\|\bar{g}\|_{\mathbb{H}_p^{\gamma+c_0}(t,l_2)} + N\|\bar{h}\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(t,l_2,d_1)}.$$

Note that $\bar{u} := u - v \in \mathcal{H}_p^{\gamma+2}(T)$ satisfies

$$\partial_t^\alpha \bar{u} = a^{ij}\bar{u}_{x^ix^j} + \tilde{f},\ t > 0\,; \quad \bar{u}(0) = 1_{\alpha>1}\bar{u}_t(0) = 0,$$

where

$$\tilde{f} := (a^{ij} - \delta^{ij})v_{x^ix^j} + \bar{f} - f.$$

Therefore, by [11, Theorem 2.10] and (4.11), for each $t \leq T$

$$\begin{aligned} \|u\|_{\mathbb{H}_p^{\gamma+2}(t)} &\leq \|u-v\|_{\mathbb{H}_p^{\gamma+2}(t)} + \|v\|_{\mathbb{H}_p^{\gamma+2}(t)} \\ &\leq N\varepsilon\|u\|_{\mathbb{H}_p^{\gamma+2}(t)} + N\|u\|_{\mathbb{H}_p^\gamma(t)} + N\|f\|_{\mathbb{H}_p^\gamma(t)} \\ &\quad + N\|g\|_{\mathbb{H}_p^{\gamma+c_0}(t,l_2)} + N\|h\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(t,l_2,d_1)}. \end{aligned}$$

Hence,

$$\begin{aligned} \|u\|^p_{\mathbb{H}_p^{\gamma+2}(t)} &\leq N\|u\|^p_{\mathbb{H}_p^\gamma(t)} + N\|f\|^p_{\mathbb{H}_p^\gamma(t)} \\ &\quad + N\|g\|^p_{\mathbb{H}_p^{\gamma+c_0}(t,l_2)} + N\|h\|^p_{\mathbb{H}_p^{\gamma+\bar{c}_0}(t,l_2,d_1)}. \end{aligned} \tag{4.13}$$

Combining this, (4.12) and (2.20), we get for each $t \leq T$

$$\begin{aligned} \|u\|^p_{\mathbb{H}_p^\gamma(t)} &\leq N\int_0^t (t-s)^{\theta-1}\|u\|^p_{\mathbb{H}_p^\gamma(s)}ds + N\|f\|_{\mathbb{H}_p^\gamma(T)} \\ &\quad + N\|g\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)} + N\|h\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(T,l_2,d_1)}. \end{aligned} \tag{4.14}$$

We use (4.14) and Gronwall's inequality (see [22]) to estimate $\|u\|^p_{\mathbb{H}_p^\gamma(T)}$. Then, applying this estimate to (4.13) and using (4.12), we get a priori estimate (2.25) if $u_0 = v_0 = 0$.

Step 2. We consider non-zero initial condition. Let $v \in \mathcal{H}_p^{\gamma+2}(T)$ denote the solution to equation (4.5) taken from Theorem 4.4. Then $\bar{u} := u - v \in \mathcal{H}_p^{\gamma+2}(T)$ satisfies equation (1.1) with $u_0 = v_0 = 0$, $\tilde{f}, \tilde{g}$ and $\tilde{h}$, where

$$\tilde{f} := (a^{ij} - \delta^{ij})v_{x^ix^j} + b^iv_{x^i} + cv, \quad \tilde{g}^k := \mu^{ik}v_{x^i} + \nu^kv, \quad \tilde{h}^{rk} := \bar{\mu}^{rk}v_{x^i} + \bar{\nu}^{rk}v.$$

By the result of Step 1,

$$\begin{aligned} \|u-v\|_{\mathcal{H}_p^{\gamma+2}(T)} &\leq N\|\tilde{f}\|_{\mathbb{H}_p^\gamma(T)} + N\|\tilde{g}\|_{\mathbb{H}_p^{\gamma+c_0}(T,l_2)} + N\|\tilde{h}\|_{\mathbb{H}_p^{\gamma+\bar{c}_0}(T,l_2,d_1)} \\ &\leq N\|v\|_{\mathbb{H}_p^{\gamma+2}(T)}. \end{aligned} \tag{4.15}$$

For the second inequality above we used the calculations in Step 1 (see (4.12)). Combining (4.15) with the estimate for v, that is (4.6), we finally get a priori estimate (2.25) for u. Hence, the theorem for the linear case is proved.

2. Non-linear case. First, set

$$\bar{f} = b^i u_{x^i} + cu + f(u), \quad \bar{g}^k = \mu^{ik} u_{x^i} + \nu^k u + g^k(u), \quad \bar{h}^{rk} = \bar{\mu}^{irk} u_{x^i} + \bar{\nu}^{rk} u + h^{rk}(u).$$

Then by Assumption 2.13, (4.10), and Assumption 2.14 we have

$$\begin{aligned}
&\|\bar{f}(u) - \bar{f}(v)\|_{H_p^\gamma} + \|\bar{g}(u) - \bar{g}(v)\|_{H_p^{\gamma+c_0'}(l_2)} + \|\bar{h}(u) - \bar{h}(v)\|_{H_p^{\gamma+\bar{c}_0'}(l_2,d_1)} \\
&\leq N\left(\|u - v\|_{H_p^{\gamma+1}} + 1_{c_0'<1} \|\mu^i (u-v)_{x^i}\|_{H_p^{\gamma+c_0'}(l_2)} + \|u - v\|_{H_p^{\gamma+c_0'}(l_2)} \right) \\
&\quad + N\left(1_{\bar{c}_0'<1} \|\bar{\mu}^i (u-v)_{x^i}\|_{H_p^{\gamma+\bar{c}_0'}(l_2,d_1)} + \|u - v\|_{H_p^{\gamma+\bar{c}_0'}(l_2,d_1)} \right) \\
&\quad + \|f(u) - f(v)\|_{H_p^\gamma} + \|g(u) - g(v)\|_{H_p^{\gamma+c_0'}(l_2)} + \|h(u) - h(v)\|_{H_p^{\gamma+\bar{c}_0'}(l_2,d_1)} \\
&\leq \varepsilon \|u - v\|_{H_p^{\gamma+2}} + N\|u - v\|_{H_p^\gamma},
\end{aligned}$$

where $u, v \in H_p^{\gamma+2}$ and the constant N depends only on $\alpha, \beta_1, \beta_2, d, d_1, \gamma, p, \delta, \kappa$ and ε. Hence by considering $\bar{f}, \bar{g}^k$ and $\bar{h}^{rk}$ in place of f, g^k and h^{rk} respectively, we may assume that $b^i = c = \mu^{ik} = \nu^k = 0$, and $\bar{\mu}^{irk} = \bar{\nu}^{rk} = 0$.

By the result for the linear case, for each $u \in \mathcal{H}_p^{\gamma+2}(T)$, one can define $v = \mathcal{R}u \in \mathcal{H}_p^{\gamma+2}(T)$ as the solution to the equation

$$\begin{aligned}
&\partial_t^\alpha v = a^{ij} v_{x^i x^j} + f(u) + \partial_t^{\beta_1} \int_0^t g^k(u) dW_s^k + \partial_t^{\beta_2} \int_0^t h^{rk}(u) dZ_s^{rk}, \quad t > 0 \\
&\quad v(0) = u_0, \quad 1_{\alpha>1} \partial_t v(0) = 1_{\alpha>1} v_0,
\end{aligned}$$

and for this solution we have

$$\begin{aligned}
\|v\|_{\mathcal{H}_p^{\gamma+2}(T)} \leq N\big(&\|u_0\|_{U_p^{\gamma+2}} + 1_{\alpha>1} \|v_0\|_{V_p^{\gamma+2}} + \|f(u)\|_{\mathbb{H}_p^\gamma(T)} \\
&+ \|g(u)\|_{\mathbb{H}_p^{\gamma+c_0'}(T,l_2)} + \|h(u)\|_{\mathbb{H}_p^{\gamma+\bar{c}_0'}(T,l_2,d_1)} \big).
\end{aligned}$$

By (2.20) for any $\varepsilon > 0$, $t \leq T$, and $n = 1, 2, \ldots$ we have

$$\begin{aligned}
\|\mathcal{R}u - \mathcal{R}v\|_{\mathcal{H}_p^{\gamma+2}(t)}^p &\leq N\big(\|f(u) - f(v)\|_{\mathbb{H}_p^\gamma(t)}^p + \|g(u) - g(v)\|_{\mathbb{H}_p^{\gamma+c_0'}(t,l_2)}^p \\
&\qquad + \|h(u) - h(v)\|_{\mathbb{H}_p^{\gamma+\bar{c}_0'}(t,l_2,d_1)}^p \big) \\
&\leq \varepsilon^p \|u - v\|_{\mathbb{H}_p^{\gamma+2}(t)}^p + N_0 \|u - v\|_{\mathbb{H}_p^\gamma(t)}^p \\
&\leq \varepsilon^p \|u - v\|_{\mathcal{H}_p^{\gamma+2}(t)}^p + N_0 \int_0^t (t-s)^{\theta-1} \|u - v\|_{\mathcal{H}_p^{\gamma+2}(s)}^p ds,
\end{aligned}$$

where the constant N_0 depends also on ε. Therefore, by using the identity

$$\int_0^t (t-s_1)^{\theta-1} \int_0^{s_1} (s_1 - s_2)^{\theta-1} \cdots \int_0^{s_{n-1}} (s_{n-1} - s_n)^{\theta-1} ds_n \ldots ds_1 = \frac{\Gamma(\theta)^n}{\Gamma(n\theta+1)} t^{n\theta},$$

and repeating above inequality, we get

$$\begin{aligned}
\|\mathcal{R}^n u - \mathcal{R}^n v\|_{\mathcal{H}_p^{\gamma+2}(T)}^p &\leq \sum_{k=0}^n \binom{n}{k} \varepsilon^{(n-k)p} (T^\theta N_0)^k \frac{\Gamma(\theta)^k}{\Gamma(k\theta+1)} \|u - v\|_{\mathcal{H}_p^{\gamma+2}(T)}^p \\
&\leq 2^n \varepsilon^{np} \max_k \left(\frac{(\varepsilon^{-1} T^\theta N_0 \Gamma(\theta))^k}{\Gamma(k\theta+1)} \right) \|u - v\|_{\mathcal{H}_p^{\gamma+2}(T)}^p.
\end{aligned}$$

Now fix $\varepsilon < 1/8$, and note that the above maximum is finite. This implies that if n is large enough, then $\mathcal{R}^n$ is a contraction on $\mathcal{H}_p^{\gamma+2}(T)$. This proves the existence and uniqueness results, and estimate (2.25) also follows. The theorem is proved.

5. Application to Lévy space-time white noise

In this section, we assume that

$$\beta_2 < \frac{3}{4}\alpha + \frac{1}{p}, \tag{5.1}$$

and the spatial dimension d satisfies

$$d < 4 - \frac{2(2\beta_2 - 2/p)^+}{\alpha} =: d_0. \tag{5.2}$$

Note that $d_0 \in (1, 4]$, and if $\beta_2 < \alpha/4 + 1/p$, then one can take $d = 1, 2, 3$. Also if $\alpha = \beta_2 = 1$ (in this case $p < 4$), then $d < 4/p \leq 2$, and thus d must be 1.

Let $\{\eta_k : k = 1, 2, \dots\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$ and let Z_t^k be i.i.d. one-dimensional $\mathscr{F}_t$ adapted Lévy processes satisfying Assumption 2.1. Define a cylindrical Lévy process $\mathcal{Z}_t$ on $L_2(\mathbb{R}^d)$ as

$$\mathcal{Z}_t = \sum_{k=1}^{\infty} \eta^k(x) Z_t^k.$$

In this section, we consider the SPDE

$$\begin{aligned} \partial_t^\alpha u &= a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u) + \partial_t^{\beta_2} \int_0^t h(u) d\mathcal{Z}_t, \\ u(0, \cdot) &= u_0, \quad 1_{\alpha > 1} \partial_t u(0, \cdot) = 1_{\alpha > 1} v_0 \end{aligned} \tag{5.3}$$

where a^{ij}, b^i, c are functions of (ω, t, x), and f and h depend on (ω, t, x) and the unknown u. Using the expansion of $\mathcal{Z}_t$, we can rewrite (5.3) as

$$\begin{aligned} \partial_t^\alpha u &= a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f(u) + \partial_t^{\beta_2} \sum_{k=1}^{\infty} \int_0^t g^k(u) dZ_t^k, \\ u(0, \cdot) &= u_0, \quad 1_{\alpha > 1} \partial_t u(0, \cdot) = 1_{\alpha > 1} v_0 \end{aligned}$$

where $g^k(\omega, t, x, u) = h(\omega, t, x, u) \eta^k(x)$.

The following result is from [10, Lemma 7.1].

Lemma 5.1. *Assume*

$$\kappa_0 \in \left(\frac{d}{2}, d\right], \quad 2 \leq 2r \leq p, \quad 2r < \frac{d}{d - \kappa_0}.$$

Also assume that $h(x, u)$ is a function of (x, u), and there is a function $\xi = \xi(x)$ such that

$$|h(x, u) - h(x, v)| \leq \xi(x) |u - v|.$$

Then for $u, v \in L_p$, we have

$$\|g(u) - g(v)\|_{H_p^{-\kappa_0}(l_2)} \leq N \|\xi\|_{L_{2s}} \|u - v\|_{L_p},$$

where $s = r/r - 1$, and $N = N(r) < \infty$. In particular, if $r = 1$, and $\xi \in L_\infty$, then

$$\|g(u) - g(v)\|_{H_p^{-\kappa_0}(l_2)} \leq N \|u - v\|_{L_p}.$$

Assumption 5.2. (i) The coefficients a^{ij}, b^i, and c are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^d)$-measurable.
(ii) The functions $f(t,x,u)$ and $h(t,x,u)$ are $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}^{d+1})$-measurable.
(iii) For each ω, t, x, u and v,

$$|f(t,x,u)-f(t,x,v)|\leq K|u-v|,\quad |h(t,x,u)-h(t,x,v)|\leq K\xi(x)|u-v|,$$

where ξ is a function of (ω,t,x).

Theorem 5.3. *Suppose that Assumption 5.2 holds and*

$$\|f(0)\|_{\mathbb{H}_p^{-\kappa_0-\bar{c}_0'}(T)}+\|h(0)\|_{\mathbb{L}_p(T)}+\sup_{\omega,t}\|\xi\|_{L_{2s}}\leq K<\infty,$$

where κ_0 and s satisfy

$$\frac{d}{2}<\kappa_0<\left(2-\frac{(2\beta_2-2/p)^+}{\alpha}\right)\wedge d,\qquad \frac{d}{2\kappa_0-d}<s. \tag{5.4}$$

Also assume that the coefficients a^{ij}, b^i and c satisfy Assumption 2.13 with $\gamma=-\kappa_0-\bar{c}_0'$, $u_0\in U_p^{-\kappa_0-\bar{c}_0'+2}$, and $v_0\in V_p^{-\kappa_0-\bar{c}_0'+2}$. Then equation (5.3) has unique solution $u\in\mathcal{H}_p^{2-\kappa_0-\bar{c}_0'}(T)$, and for this solution we have

$$\begin{aligned}\|u\|_{\mathcal{H}_p^{2-\kappa_0-\bar{c}_0'}(T)}\leq N\big(&\|u_0\|_{U_p^{-\kappa_0-\bar{c}_0'+2}}+1_{\alpha>1}\|v_0\|_{V_p^{-\kappa_0-\bar{c}_0'+2}}\\&+\|f(0)\|_{\mathbb{H}_p^{-\kappa_0-\bar{c}_0'}(T)}+\|h(0)\|_{\mathbb{L}_p(T)}\big).\end{aligned} \tag{5.5}$$

Proof. It suffices to check the conditions for Theorem 2.15 holds for $\gamma=-\kappa_0-\bar{c}_0'$. Since $f(u)$ is Lipschitz continuous, we only need to check the conditions for $g^k(u)=\eta^k h(u)$. Let $r=s/(s-1)$. Then $2r<d/(d-\kappa_0)$ due to the assumption on s. Since $\gamma+\bar{c}_0'=-\kappa_0$, by Lemma 5.1 for any $\varepsilon>0$, we have

$$\|g(u)-g(v)\|_{H_p^{\gamma+\bar{c}_0'}(l_2)}\leq N\|\xi\|_{L_{2s}}\|u-v\|_{L_p}\leq\varepsilon\|u-v\|_{H_p^{\gamma+2}}+N(\varepsilon)\|u-v\|_{H_p^{\gamma}},$$

where the second inequality holds due to the assumption on κ_0. Therefore, the condition for g^k is also fulfilled. Hence, by Theorem 2.15 we prove the claims of the theorem with estimate (5.4) replaced by

$$\begin{aligned}\|u\|_{\mathcal{H}_p^{2-\kappa_0-\bar{c}_0'}(T)}\leq N\big(&\|u_0\|_{U_p^{-\kappa_0-\bar{c}_0'+2}}+1_{\alpha>1}\|v_0\|_{V_p^{-\kappa_0-\bar{c}_0'+2}}\\&+\|f(0)\|_{\mathbb{H}_p^{-\kappa_0-\bar{c}_0'}(T)}+\|g(0)\|_{\mathbb{H}_p^{-\kappa_0}(T,l_2)}\big).\end{aligned}$$

Furthermore, by inspecting the proof of Lemma 5.1, one can easily check

$$\|g(0)\|_{\mathbb{H}_p^{-\kappa_0}(T,l_2)}\leq N\|h(0)\|_{\mathbb{L}_p(T)}.$$

Hence, we have (5.4), and the theorem is proved. □

Remark 5.4. (i) By (5.2) one can always choose κ_0 satisfying (5.4).

(ii) Note that the constant $2-\kappa_0-\bar{c}_0'$ represents the regularity (or differentiability) of the solution with respect to the space variables. By using the definition of $\bar{c}_0'$ we have

$$0<2-\kappa_0-\bar{c}_0'<\begin{cases}2-\frac{d}{2}-\frac{2\beta_2-2/p}{\alpha} & \beta_2>1/p\\ 2-\frac{d}{2} & \beta_2\leq 1/p.\end{cases}$$

If ξ is bounded, then one can choose $r=1$. Thus by taking κ_0 sufficiently close to $d/2$, one can make $2-\kappa_0-\bar{c}_0'$ as close to the above upper bounds as one wishes.

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