LITTLEWOOD-PALEY TYPE ESTIMATES AND APPLICATIONS TO COMPOSITION OPERATORS

BOO RIM CHOE, HYUNGWOON KOO, ELODIE POZZI, AND WAYNE SMITH

Abstract. Let $\Omega \subset \mathbb{C}$ be a chord-arc domain, bounded or unbounded. We introduce a new Littlewood-Paley type estimate of the norm of a function in the Hardy-Smirnov space $E^2(\Omega)$ that is valid for all $f \in \mathrm{Hol}(\Omega)$ and includes a line integral of $|f|^2$ over a hyperbolic geodesic in Ω and an area integral of $|f'|^2$ over Ω. Given an analytic self-map φ of Ω, we introduce operators L_φ and A_φ that are associated with these integrals and show that the composition operator induced by φ is bounded on $E^2(\Omega)$ if and only if L_φ and A_φ are both bounded. We also study the Reproducing Kernel Thesis for these operators and give characterizations for when they are bounded. Finally, we show that a version of the new Littlewood-Paley type estimate is valid on the strip $\left\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{\pi}{2}\right\}$, even though that domain is not chord-arc, and study these operators in that setting.

1. Introduction

Throughout the paper Ω will always denote a simply connected domain properly contained in the complex plane $\mathbb{C}$. Also, throughout the paper, we use the term *conformal map* to mean a function that is *analytic and univalent* (not just locally univalent). We denote by $\mathrm{Hol}(\Omega)$ the space of all analytic functions on Ω. A function $\varphi \in \mathrm{Hol}(\Omega)$ with $\varphi(\Omega) \subset \Omega$ (i.e. φ is an analytic self-map of Ω) induces the composition operator C_φ defined by

$$C_\varphi f := f \circ \varphi, \quad f \in \mathrm{Hol}(\Omega).$$

We are interested in composition operators acting on generalizations of the classical Hardy spaces. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ and put $\mathbb{T} := \partial\mathbb{D}$. Let $\tau : \mathbb{D} \to \Omega$ be a Riemann map and for $0 < r < 1$ let J_r be the Jordan curve formed by the τ-image of $r\mathbb{T}$. For $0 < p < \infty$, the *Hardy-Smirnov*

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space $E^p(\Omega)$ is the set of all $f \in \mathrm{Hol}(\Omega)$ such that

$$\sup_{0<r<1} \left\{ \frac{1}{2\pi} \int_{J_r} |f(\zeta)|^p \, |d\zeta| \right\}^{1/p} =: \|f\|_{E^p(\Omega)} < \infty. \tag{1.1}$$

When $1 \leq p < \infty$, $\|\cdot\|_{E^p(\Omega)}$ is a norm for the space $E^p(\Omega)$, and in the case that $\Omega = \mathbb{D}$ and τ is the identity map of $\mathbb{D}$, this definition coincides with that of the classical Hardy space $H^p(\mathbb{D})$. More information is given below in Section 2.1, including that using a Riemann map other than τ to define $E^p(\Omega)$ will result in the same space of functions and an equivalent norm.

In the case that $\Omega = \mathbb{D}$, every composition operator is bounded on $H^p(\mathbb{D})$, $0 < p < \infty$, and understanding how properties of the inducing map φ affect operator theoretic properties of C_φ has attracted considerable attention; see [2] and [14]. The classical Littlewood-Paley Identity,

$$\|f\|^2_{H^2(\mathbb{D})} = |f(0)|^2 + \frac{2}{\pi} \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} \, dA(z), \qquad f \in \mathrm{Hol}(\mathbb{D}), \tag{1.2}$$

where dA is area measure on $\mathbb{C}$ and $\|f\|_{H^2(\mathbb{D})} = \infty$ means that $f \notin H^2(\mathbb{D})$, has been important in the study of composition operators in this setting. For example, it played an important role in J. Shapiro's seminal paper [13], in which he gave an expression for the essential norm of C_φ on $H^2(\mathbb{D})$ in terms of the classical Nevanlinna counting function of φ.

In the setting of the upper half-plane $\mathbb{H} := \{z \in \mathbb{D} : \mathrm{Im}\, z > 0\}$, $E^2(\mathbb{H})$ coincides with the classical Hardy space $H^2(\mathbb{H})$, and the Littlewood-Paley Identity takes the form

$$\|f\|^2_{E^2(\mathbb{H})} = 4 \int_{\mathbb{H}} |f'(z)|^2 \mathrm{Im}\, z \, dA(z), \qquad f \in E^2(\mathbb{H}). \tag{1.3}$$

Here it must be assumed that $f \in E^2(\mathbb{H})$ as, for example, if f is a non-zero constant function the right side of (1.3) is finite even though $f \notin E^2(\mathbb{H})$. Thus the Littlewood-Paley Identity has not been as useful in studying composition operators in this setting as, for example, the composition operator induced by a constant function φ is not bounded on $E^2(\mathbb{H})$ even though the right side of (1.3) is equal to zero for every function in the range of C_φ.

In the setting of composition operators acting on $E^p(\Omega)$, properties of the domain Ω play an important role in determining properties of a composition operator. For example, all composition operators are bounded on $E^p(\Omega)$ if and only if τ' and $1/\tau'$ are bounded, where $\tau : \mathbb{D} \to \Omega$ is a Riemann map; see[15, Theorem 6.1]. Thus, in general not all composition operators are bounded on $E^p(\Omega)$ and it is desirable to have a Littlewood-Paley type approximation of the norm that is valid for all functions in $Hol(\Omega)$.

When Ω is a chord-arc domain, defined below in Section 2.3, Littlewood-Paley type approximations are available from the work of Jerison and Kenig [8]. When Ω is a bounded chord-arc domain, this approximation was not explicitly stated in [8], and various forms of the approximation have appeared in the literature. In (3.3) below we give a form suitable for our applications. Significantly, it is valid for all $f \in \mathrm{Hol}(\Omega)$, just as in the case that $\Omega = \mathbb{D}$, and so is useful in the study of composition operators acting on $E^2(\Omega)$. Gallardo-Gutiérrez, González, and Nicolau [7] used a version of this approximation and a variation of the classical Nevanlinna counting function to characterize when C_φ is bounded or compact on $E^2(\Omega)$. On the other hand, when Ω is an unbounded chord-arc domain, the approximation given in [8] (see (3.6) below) is valid only for functions in $E^2(\Omega)$, and so it has limited utility in the study of composition operators on $E^2(\Omega)$, just as in the case that $\Omega = \mathbb{H}$.

Simple examples show that neither (3.3) nor (3.6) is valid for all chord-arc domains. In Theorem 3.6 we present a new Littlewood-Paley type estimate of $\|f\|_{E^2(\Omega)}$ that includes a line integral of $|f|^2$ over a certain curve in Ω and an area integral of $|f'|^2$ over Ω and is valid for all chord-arc domains Ω and for all $f \in \mathrm{Hol}(\Omega)$.

Next, we develop some applications of this new Littlewood-Paley type estimate to the study of composition operators. It is known that a composition operator is bounded (compact) on $E^p(\Omega)$ for some $p \in (0, \infty)$ if and only if it is bounded(compact) on $E^2(\Omega)$; see Section 4. Thus we may just consider the case $p = 2$ to which the new Littlewood-Paley type estimate applies. We will not be concerned with compactness of C_φ. In the case of bounded chord-arc domains, compactness of C_φ was characterized in [7]. On the other hand, the boundary of an unbounded chord-arc domain Ω is not rectifiable, and so there are no compact composition operators on $E^2(\Omega)$; see [15]. Given an analytic self-map φ of a chord-arc domain Ω, we introduce operators L_φ and A_φ that are associated with the line integral and area integral mentioned above in connection with the new Littlewood-Paley type estimate, and show that C_φ is bounded on $E^2(\Omega)$ if and only if both L_φ and A_φ are bounded on $E^2(\Omega)$; see Corollary 4.7.

It is often the case that some properties of an operator acting on various classical spaces can be determined by the operator's action on reproducing kernels of that space. When this happens, it is said that the *Reproducing Kernel Thesis* (RKT) holds for these properties of the operator. In Theorem 4.1 we show that the RKT holds for boundedness of composition operators acting on $E^2(\Omega)$ when Ω belongs to a class of domains more general than chord-arc domains. We also show that the RKT fails to hold in general for adjoints of composition operators acting on $E^2(\Omega)$. For L_φ acting on $E^2(\Omega)$, we give Carleson measure criteria that characterize boundedness and show that the RKT holds; see Proposition 4.5. For A_φ acting on $E^2(\Omega)$, where Ω is a chord-arc domain, we use counting function methods to characterize boundedness (see Theorem 4.6), but we have not been able to determine if the RKT holds in general.

Finally, we present some further developments in the case that Ω is the strip $S := \left\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{\pi}{2}\right\}$. We show that a version of the new Littlewood-Paley type estimate is valid in this setting even though S is not a chord-arc domain. Moreover, we prove that C_φ is bounded on $E^2(S)$ if and only if L_φ and A_φ are both bounded, and that the RKT holds for all three operators. Simple examples show that A_φ may be bounded on $E^2(S)$ while C_φ is unbounded, but the question of whether there is an analytic self-map φ of S such that L_φ is bounded on $E^2(S)$ but C_φ is unbounded remains open. However, we have been able to prove that if φ is a conformal self-map of S, then C_φ is bounded if and only if L_φ is bounded on $E^2(S)$; see Theorem 5.11.

The rest of the paper is organized as follows: Background material and some preliminary results are given in Section 2, and the new Littlewood-Paley type estimate on chord-arc domains is given in Section 3. The applications to composition operators are in Section 4, and the extensions to the strip S are in Section 5. The paper closes with a short section related to an attempt we made to resolve the question of whether L_φ being bounded on $E^2(S)$ implies that C_φ is bounded.

Constants. Throughout the paper we use the same letter C to denote positive constants which may vary at each occurrence but do not depend on the essential parameters. For nonnegative (possibly ∞) quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \le CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries

In this section we collect some preliminary results to be used in later sections. Throughout the section $\tau : \mathbb{D} \to \Omega$ is an arbitrary (but fixed) Riemann map, unless otherwise specified. Recall that Ω always denotes a general simply connected domain properly contained in $\mathbb{C}$.

2.1. **Hardy-Smirnov Spaces.**

We recall and summarize some basic facts about the Hardy-Smirnov spaces. For more detailed information, see [3, Chapter 10].

The definition of the Hardy-Smirnov spaces, $E^2(\Omega)$, $0 < p < \infty$, was given in the introduction. The change of variable $w = \tau(z)$ shows that $E^p(\Omega)$ is isometrically isomorphic to $H^p(\mathbb{D})$; see [3, Corollary, p. 169].

Proposition 2.1. *The operator V_p on $E^p(\Omega)$ by*

$$V_p f := (\tau')^{1/p}(f \circ \tau)$$

is an isometric isomorphism from $E^p(\Omega)$ onto $H^p(\mathbb{D})$.

One of consequences is that the supremum in the left side of (1.1) can be replaced by the limit as $r \to 1$, which is well-known for $H^p(\mathbb{D})$. Moreover, using a Riemann map other than τ to define $E^p(\Omega)$ will result in the same space of functions and an equivalent norm.

2.2. **Reproducing Kernels.** When $p = 2$, it is easily verified that $E^2(\Omega)$ is a Hilbert space with inner product defined by

$$\langle f, g\rangle := \langle V_2 f, V_2 g\rangle_{\mathbb{D}},$$

where V_2 is the isometry introduced in Proposition 2.1 and $\langle \cdot\,, \cdot\rangle_{\mathbb{D}}$ denotes the usual inner product on $H^2(\mathbb{D})$. Recalling that the reproducing kernel k_a at $a \in \mathbb{D}$ for $H^2(\mathbb{D})$ is given by $k_a(z) = (1 - \overline{a}z)^{-1}$, $z \in \mathbb{D}$, it is easy to check that the reproducing kernel K_b at $b \in \Omega$ for $E^2(\Omega)$ is given by

$$K_b(w) = \frac{\big(\overline{\lambda'(b)}\lambda'(w)\big)^{1/2}}{1 - \overline{\lambda(b)}\lambda(w)}, \qquad w \in \Omega, \tag{2.1}$$

where $\lambda : \Omega \to \mathbb{D}$ is the inverse map of τ. In particular, we obtain

$$\|K_b\|^2_{E^2(\Omega)} = \langle K_b, K_b\rangle = \frac{|\lambda'(b)|}{1 - |\lambda(b)|^2},$$

and so by a consequence of the Koebe Distortion Theorem (see [10, Corollary 1.4])

$$\|K_b\|^2_{E^2(\Omega)} \approx \frac{1}{\delta_\Omega(b)}, \qquad b \in \Omega, \tag{2.2}$$

with suppressed constants absolute. Here and in the rest of the paper

$$\delta_\Omega(b) := \inf_{\zeta\in\partial\Omega} |b - \zeta|$$

denotes the Euclidean distance between $b \in \Omega$ and $\partial\Omega$.

2.3. **Ahlfors-regular/Chord-arc Domains.** An *Ahlfors-regular domain* Ω is a domain for which there is a constant $\alpha > 0$ such that

$$\Lambda_1\big(\partial\Omega \cap D(z, r)\big) \le \alpha r, \qquad z \in \mathbb{C},\ r > 0.$$

Here, and in what follows, Λ_1 denotes one-dimensional Hausdorff measure and $D(z, r)$ denotes the open Euclidean disk with center z and radius r. The smallest such α is called the *Ahlfors-regular constant* for Ω.

A *bounded chord-arc domain* Ω, which is often called a *Lavrentiev domain* in the literature, is a Jordan domain with rectifiable boundary that satisfies a chord-arc condition. This means that there is a constant $\kappa > 0$ such that

$$\Lambda_1(J_a^b) \le \kappa|a - b|, \qquad a, b \in \partial\Omega, \tag{2.3}$$

where J_a^b is the shorter arc of $\partial\Omega$ between a and b. For a bounded chord-arc domain Ω, one has

$$\operatorname{diam}(\Omega) \le \Lambda_1(\partial\Omega) \le 2\kappa \operatorname{diam}(\Omega). \tag{2.4}$$

In fact, the first inequality is clear, and to see the second one, consider two points $a, b \in \partial\Omega$ with $\Lambda_1(J_a^b) = \frac{1}{2}\Lambda_1(\partial\Omega)$ and then apply (2.3). An *unbounded chord-arc domain* Ω is a domain on one side of an unbounded Jordan curve $J \subset \mathbb{C}$ that is locally rectifiable and satisfies the chord-arc condition (2.3) where J_a^b in this case is simply the arc of $\partial\Omega$ between a and b. In either case we call the smallest such κ the *chord-arc constant* for Ω.

Chord-arc domains are known to be Ahlfors-regular domains. See [10, Proposition 7.7] for bounded chord-arc domains. For unbounded chord-arc domains, one may easily deduce this from the definitions. However, an Ahlfors-regular Jordan domain may not be a chord-arc domain. For example, a Jordan domain with a cusp is not a chord-arc domain, but can be an Ahlfors-regular domain. Also, note that a strip is an Ahlfors-regular domain but not a chord-arc domain, simply because its boundary is not a Jordan curve.

2.4. **Carleson Domains.** A complex Borel measure μ on Ω is said to be a *Carleson measure* on Ω if the Carleson "norm"

$$\|\mu\| := \sup_{\substack{z\in\partial\Omega \\ r>0}} \frac{|\mu|\big(D(z,r)\big)}{r}$$

is finite. Zinsmeister [17] called Ω a *Carleson domain* if there exists a constant $C > 0$ such that

$$\int_\Omega |f| d|\mu| \le C\|\mu\| \|f\|_{E^1(\Omega)}, \qquad f \in E^1(\Omega),\ \mu \in \mathcal{M}(\Omega), \tag{2.5}$$

where $\mathcal{M}(\Omega)$ denotes the set of all Carleson measures on Ω. In case $\Omega = \mathbb{D}$, note that $\mathcal{M}(\mathbb{D})$ is the same as the set of all classical Carleson measures on $\mathbb{D}$. In the same paper Zinsmeister characterized Carleson domains as follows; see [17, Theorem 1].

Theorem 2.2 (Zinsmeister)**.** *Ω is a Carleson domain if and only if* $\log \tau' \in \mathrm{BMOA}(\mathbb{D})$.

Zinsmeister [16] has shown that $\log \tau' \in \mathrm{BMOA}(\mathbb{D})$ when Ω is an Ahlfors-regular domain. Thus Ahlfors-regular domains are Carleson domains, and hence chord-arc domains are as well. Also, as remarked in [17], as a consequence of Theorem 2.2, all domains that are starlike, spirallike, or close-to-convex are Carleson domains. Thus, for example, strips are Carleson domains.

2.5. **Carleson Norms.** Let μ be a complex Borel measure on Ω. Denote by $\mu\circ\tau$ the image measure on $\mathbb{D}$ defined by $(\mu\circ\tau)(E) = \mu\big(\tau(E)\big)$ for Borel sets $E \subset \mathbb{D}$. For Borel functions $h \ge 0$ on Ω, the change of variable $\tau(z) = w$, or $z = \tau^{-1}(w)$, gives

$$\int_\Omega h(w)\, d|\mu|(w) = \int_{\mathbb{D}} h\big(\tau(z)\big)|\tau'(z)|\, d|\nu|(z),$$

where $d\nu = |\tau'|^{-1}\, d(\mu\circ\tau)$. In particular, we have

$$\|K_b\|^{-2}_{E^2(\Omega)} \int_\Omega |K_b(w)|^2\, d|\mu|(w) = \int_{\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}z|^2}\, d|\nu|(z) \tag{2.6}$$

for $a \in \mathbb{D}$, $b \in \Omega$ with $\tau(a) = b$. We remark in passing that each side of (2.6) is sometimes called the Berezin transform of the measure under consideration. Put

$$\|\mu\|_* := \sup_{b\in\Omega} \|K_b\|^{-2}_{E^2(\Omega)} \int_\Omega |K_b|^2 \, d|\mu|,$$

which might be possibly ∞. In this notation the underlying domain, which must be obvious from the context, is suppressed for ease of notation. Using this notation, we have

$$\|\mu\|_* = \|\nu\|_* \approx \|\nu\|. \tag{2.7}$$

The first equality above holds by (2.6) and the second equivalence with suppressed constants absolute is well-known; see for example [5, Lemma VI.3.3]. Clearly, $\|\mu\|_* \lesssim \|\mu\|$ on Carleson domains by definition of Carleson domains. This estimate can be reversed on Ahlfors-regular domains as in the next proposition.

Proposition 2.3. *Let Ω be an Ahlfors-regular domain. For a complex Borel measure μ on Ω, the estimate*

$$\|\mu\| \approx \|\mu\|_* \approx \sup_{\|f\|_{E^2(\Omega)}=1} \int_\Omega |f|^2 \, d|\mu| \tag{2.8}$$

holds with suppressed constants depending only on the Ahlfors-regular constant for Ω.

Proof. By the aforementioned remark, the estimate $\|\mu\|_* \lesssim \|\mu\|$ holds whenever Ω is a Carleson domain, and hence holds here since Ahlfors-regular domains are Carleson domains. The estimate $\|\mu\| \lesssim \|\nu\|$ is implicit in the proof of [17, Proposition 4], where the Ahlfors-regularity of the domain plays an essential role. So, we obtain $\|\mu\| \lesssim \|\mu\|_*$ by (2.7). Denote by M the supremum in (2.8). It is immediate that $\|\mu\|_* \leq M$, and $M \lesssim \|\mu\|$ holds by the definition of Carleson domains. This establishes the estimates in (2.8).

Moreover, one may keep track of constants suppressed in the aforementioned estimates to find them depending only on the Ahlfors-regular constant for Ω, and so the proof is complete. □

3. Littlewood-Paley type estimates

In what follows, given $a \in \Omega$, Γ_a denotes an arbitrary (but fixed) hyperbolic geodesic of Ω that passes through a. For background on hyperbolic geodesics, we refer to [10, Section 4.6]. We will use the notation σ_{Γ_a} for the arclength measure on Γ_a.

The next proposition generalizes the classical Fejér-Riesz Inequality (see [3, Theorem 3.13]) which refers to the case that $\Omega = \mathbb{D}$, $a = 0$ and $\Gamma_0 = (-1, 1)$.

Proposition 3.1. *The inequality*
$$\int_{\Gamma_a} |f|^2 \, d\sigma_{\Gamma_a} \le \pi \|f\|^2_{E^2(\Omega)}$$
holds for $a \in \Omega$ and $f \in E^2(\Omega)$.

Proof. Let $a \in \Omega$ and let $\tau : \mathbb{D} \to \Omega$ be a Riemann map such that $\tau(0) = a$. Put $\lambda := \tau^{-1}$. Observe that $\lambda(\Gamma_a)$ is a hyperbolic geodesic in $\mathbb{D}$ passing through the origin and hence is a diameter in $\mathbb{D}$. So, it follows from the Fejér-Riesz Inequality that
$$\int_{\lambda(\Gamma_a)} |F|^2 \, d\sigma_{\lambda(\Gamma_a)} \le \pi \|F\|^2_{H^2(\mathbb{D})}$$
for all $F \in H^2(\mathbb{D})$. Now, given $f \in E^2(\Omega)$, apply Proposition 2.1 to pick F satisfying $f = (\lambda')^{1/2}(F \circ \lambda)$. For such F, note $F \in H^2(\mathbb{D})$ with $\|F\|_{H^2(\mathbb{D})} = \|f\|_{E^2(\Omega)}$ by Proposition 2.1. We thus obtain
$$\int_{\Gamma_a} |f|^2 \, d\sigma_{\Gamma_a} = \int_{\lambda(\Gamma_a)} |F|^2 \, d\sigma_{\lambda(\Gamma_a)} \le \pi \|F\|^2_{H^2(\mathbb{D})} = \pi \|f\|^2_{E^2(\Omega)},$$
which completes the proof. $\square$

The next lemma is the key to Littlewood-Paley type estimates for bounded chord-arc domains.

Lemma 3.2. *Let Ω be a bounded chord-arc domain and let $a \in \Omega$ be such that*
$$\delta_\Omega(a) > \frac{1}{2} \sup_{b \in \Omega} \delta_\Omega(b). \tag{3.1}$$
The estimates
$$\delta_\Omega(a) \approx \Lambda_1(\partial\Omega) \approx \Lambda_1(\Gamma_a) \tag{3.2}$$
hold with suppressed constants depending only on the chord-arc constant for Ω.

Proof. To begin with, consider an arbitrary $b \in \Omega$. Let $\tau : \mathbb{D} \to \Omega$ be the Riemann map with $\tau(0) = b$ and put $\Gamma_b := \tau\big((-1, 1)\big)$. Then, one has
$$\delta_\Omega(b) \le \Lambda_1(\Gamma_b) = \int_{-1}^{1} |\tau'(x)| \, dx \le \frac{1}{2} \int_{\mathbb{T}} |\tau'(\zeta)||d\zeta| = \frac{1}{2}\Lambda_1(\partial\Omega)$$
where the second inequality is the Fejér-Riesz Inequality. Next, we will find a point $b_0 \in \Omega$ such that $\Lambda_1(\partial\Omega) \lesssim \delta_\Omega(b_0)$, so that
$$\delta_\Omega(b_0) \approx \Lambda_1(\partial\Omega) \approx \Lambda_1(\Gamma_{b_0}).$$
It will be then clear that any point $a \in \Omega$ with (3.1) will work for (3.2).

Consider now $\zeta_1, \zeta_2 \in \partial\Omega$ such that $|\zeta_1 - \zeta_2| = \operatorname{diam}(\Omega)$ and let Γ be a hyperbolic geodesic with endpoints ζ_1 and ζ_2. Choose $b_0 \in \Omega \cap \Gamma$ such that $|b_0 - \zeta_1| = |b_0 - \zeta_2| \ge \frac{1}{2}\operatorname{diam}(\Omega)$. Let $\partial^+\Omega$ and $\partial^-\Omega$ be the two components of $\partial\Omega$ with endpoints ζ_1 and ζ_2. We may assume that there is $\zeta_3 \in \partial^+\Omega$

such that $|b_0-\zeta_3| = \delta_\Omega(b_0)$. By [11, Exercise 10.3.1], there is $\zeta_4 \in \partial^-\Omega$ such that $|b_0-\zeta_4| \lesssim \delta_\Omega(b_0)$. Let $\gamma := J_{\zeta_3}^{\zeta_4}$ be the shorter arc of $\partial\Omega$ between ζ_3 and ζ_4, and assume without loss of generality that $\zeta_2 \in \gamma$. One has

$$\begin{aligned}\Lambda_1(\gamma) &\geq |\zeta_3-\zeta_2| \\ &\geq |b_0-\zeta_2| - |b_0-\zeta_3| \\ &\geq \frac{1}{2}\mathrm{diam}(\Omega) - \delta_\Omega(b_0).\end{aligned}$$

It follows that $\mathrm{diam}(\Omega) \leq 2\delta_\Omega(b_0) + 2\Lambda_1(\gamma)$. As Ω is a chord-arc domain, using the chord-arc constant κ for Ω, we get $\Lambda_1(\gamma) \leq \kappa|\zeta_3-\zeta_4| \lesssim \delta_\Omega(b_0)$ which implies that $\mathrm{diam}(\Omega) \lesssim \delta_\Omega(b_0)$. Hence, $\Lambda_1(\partial\Omega) \lesssim \delta_\Omega(b_0)$ by (2.4) and the proof is complete. $\square$

Recall that A denotes the area measure on $\mathbb{C}$. For $f \in \mathrm{Hol}(\Omega)$, we use the notation $\|f\|_{E^2(\Omega)} = \infty$ to mean that $f \notin E^2(\Omega)$.

Theorem 3.3. *Let Ω be a bounded chord-arc domain and let $a \in \Omega$ be chosen as in* (3.1)*. For $f \in Hol(\Omega)$, the estimate*

$$\|f\|^2_{E^2(\Omega)} \approx \delta_\Omega(a)|f(a)|^2 + \int_\Omega |f'|^2\delta_\Omega\, dA \tag{3.3}$$

holds with suppressed constants depending only on the chord-arc constant for Ω.

Proof. Let $f \in \mathrm{Hol}(\Omega)$. Let $r > 0$ and $F(w) := f\left(\frac{w}{r}\right)$, $w \in r\Omega$. It is easy to check that $\|F\|^2_{E^2(r\Omega)} = r\|f\|^2_{E^2(\Omega)}$ and similarly, with $w_0 = ra$, each term on the right side of (3.3), when expressed with F and $r\Omega$, changes by a factor of r. Thus, we may assume that $\delta_\Omega(a) = 1$.

First, assume that $f \in E^2(\Omega)$. In this case (3.3) is obtained immediately by combining [1, Lemma 3.6] with Lemma 3.2. For the case that $f \notin E^2(\Omega)$, let J_r, $0 < r < 1$, be the Jordan curves introduced in Section 2.1 and let Ω_r be domain interior to J_r. Note that the Ω_r all satisfy the chord-arc condition (2.3) with a constant κ that depends only on the chord-arc constant for Ω. This was established for unbounded chord-arc domains in [8, pp. 225–6], and with routine modifications the proof works for bounded chord-arc domains; see [8, Remark 5.1]. Then $f \in E^2(\Omega_r)$ and from the first part of the proof we have that

$$\frac{1}{2\pi}\int_{J_r} |f(\zeta)|^2\,|d\zeta| \approx \delta_{\Omega_r}(a)|f(a)|^2 + \int_{\Omega_r} |f'|^2\delta_{\Omega_r}\, dA,$$

for $r_0 < r < 1$, where $r_0 \in (0,1)$ is chosen so that $a \in \Omega_{r_0}$ and condition (3.1) is satisfied by the point a and the domain Ω_r for $r \in [r_0, 1)$. Note from the afore-mentioned remark that the constants suppressed above are independent of r. Now, (3.3) is obtained by taking the limit as $r \to 1$ (see the remark right after Proposition 2.1) and using the Monotone Convergence Theorem. This completes the proof. $\square$

Remark 3.4. (1) We remark that (3.3) can fail if Ω is not a chord-arc domain. For example, the domain

$$\Omega := \{z \in \mathbb{D} : |\operatorname{Im} z| < |\operatorname{Re} z|^2,\ \operatorname{Re} z > 0\}$$

has an outward pointing cusp at 0 and is an Ahlfors-regular domain, but not a chord-arc domain. It is easy to check that

$$\|z^{\varepsilon-1/2}\|^2_{E^2(\Omega)} \approx \int_0^1 r^{2\varepsilon-1} dr \approx \varepsilon^{-1}, \qquad 0 < \varepsilon < 1.$$

On the other hand, noting that $\delta_\Omega(z) \le |z|^2$, we see that the right side of (3.3) (for any choice of $a \in \Omega$ as in (3.1)) remains comparable to 1 as $\varepsilon \to 0$.

(2) We also note that (3.3) can hold for a domain that is not a chord-arc domain, such as $\mathbb{D} \setminus \overline{\Omega}$, which has an inward pointing cusp at 0. The observation that $\mathbb{D} \setminus \overline{\Omega}$ can be split into two chord-arc domains leads to an easy proof that (3.3) holds for this domain.

In addition to the Littlewood-Paley estimate (3.3), valid for bounded chord-arc domains, there is a similar estimate due to Jerison and Kenig [8] that is valid for unbounded chord-arc domains; see (3.6) below for the statement. Simple examples show that neither (3.3) nor (3.6) is valid for all chord-arc domains. For example, when Ω is the upper half-plane and f is a non-zero constant function they both fail. We now proceed to the development of a new version of these estimates that is valid for both bounded and unbounded chord-arc domains. Later, in Section 5, we will show that a version of it is valid in some non-chord-arc domains such as a strip.

Next, we briefly review some background on the hyperbolic distance; for more see, for example, [10, Section 1.2]. The hyperbolic distance $\rho_{\mathbb{D}}(a, z)$ between $a, z \in \mathbb{D}$ is given by

$$\rho_{\mathbb{D}}(a, z) = \frac{1}{2} \log \frac{1 + |a - z|/|1 - \bar{a}z|}{1 - |a - z|/|1 - \bar{a}z|}.$$

This distance is invariant under conformal automorphisms of $\mathbb{D}$, which leads to a natural extension to any Ω, given by

$$\rho_\Omega(b, w) = \rho_{\mathbb{D}}\big(\tau^{-1}(b), \tau^{-1}(w)\big), \quad b, w \in \Omega,$$

where $\tau : \mathbb{D} \to \Omega$ is any Riemann map. It follows that hyperbolic distance is conformally invariant: If Ω_1 and Ω_2 are simply connected domains, not the whole $\mathbb{C}$, and $\psi : \Omega_1 \to \Omega_2$ is any Riemann map, then

$$\rho_{\Omega_1}(b, w) = \rho_{\Omega_2}\big(\psi(b), \psi(w)\big), \quad b, w \in \Omega_1.$$

If Ω is an unbounded chord-arc domain and $a \in \Omega$, there is a unique hyperbolic geodesic of Ω through a that is unbounded. These are the "vertical lines" that appear in [8]. To be precise, let $\Psi : \mathbb{H} \to \Omega$ (recall that $\mathbb{H}$ denotes the upper half-plane) be any Riemann map that takes the prime end of $\mathbb{H}$ at ∞ to the prime end of Ω at ∞. The unbounded hyperbolic geodesics of Ω are the images under Ψ of the vertical lines $\gamma_x := \{x + yi : y > 0\}$, $x \in \mathbb{R}$, in $\mathbb{H}$. For later use, we observe that

Lemma 3.5. *Let Ω be an unbounded chord-arc domain and let Γ_1 and Γ_2 be unbounded hyperbolic geodesics of Ω. Then*

$$\lim_{\Gamma_1 \ni z \to \infty} \rho_\Omega(z, \Gamma_2) = 0.$$

Proof. Choose $x_j \in \mathbb{R}$ such that $\Gamma_j = \Psi(\gamma_{x_j})$ for $j = 1, 2$. Then, for $t > 0$,

$$\begin{aligned}\rho_\Omega\big(\Psi(x_1 + it), \Psi(\gamma_{x_2})\big) &\leq \rho_\Omega\big(\Psi(x_1 + it), \Psi(x_2 + it)\big)\\ &= \rho_{\mathbb{H}}(x_1 + it, x_2 + it)\\ &= \rho_{\mathbb{H}}\left(\frac{x_1}{t} + i, \frac{x_2}{t} + i\right),\end{aligned}$$

where conformal invariance of hyperbolic distance was used for the two equalities. Letting $t \to \infty$ completes the proof. □

Again, $\|f\|_{E^2(\Omega)} = \infty$ means that $f \notin E^2(\Omega)$.

Theorem 3.6. *Let Ω be a chord-arc domain. If Ω is bounded, let $a \in \Omega$ be chosen as in* (3.1) *and let Γ_a be any hyperbolic geodesic of Ω through a; if Ω is unbounded, let $a \in \Omega$ be any point and let Γ_a be the unbounded hyperbolic geodesic of Ω through a. For $f \in Hol(\Omega)$, the estimate*

$$\|f\|^2_{E^2(\Omega)} \approx \int_{\Gamma_a} |f|^2 \, d\sigma_{\Gamma_a} + \int_\Omega |f'|^2 \delta_\Omega \, dA \tag{3.4}$$

holds with suppressed constants depending only on the chord-arc constant for Ω.

Proof. Before proceeding, we make a preliminary observation. Let $g \in \mathrm{Hol}(\Omega)$ and put

$$\widetilde{D}(z) := D\left(z, \frac{\delta_\Omega(z)}{2}\right), \qquad z \in \Omega.$$

Noting that $\delta_\Omega \approx \delta_\Omega(z)$ on $\widetilde{D}(z)$, and using subharmonicity of $|g'|^2$ we obtain the estimate that

$$|g'(z)|^2 \lesssim \frac{1}{\delta_\Omega^2(z)} \int_{\widetilde{D}(z)} |g'|^2 \, dA \lesssim \frac{1}{\delta_\Omega^3(z)} \int_\Omega |g'|^2 \delta_\Omega \, dA, \qquad z \in \Omega.$$

Thus, integrating g' along the line segment from z to $w \in \widetilde{D}(z)$ and again using that $\delta_\Omega \approx \delta_\Omega(z)$ on $\widetilde{D}(z)$ shows

$$|g(w) - g(z)| \leq \frac{\delta_\Omega(z)}{2} \sup_{\widetilde{D}(z)} |g'| \lesssim \frac{1}{\delta_\Omega(z)^{1/2}} \left\{\int_\Omega |g'|^2 \delta_\Omega \, dA\right\}^{1/2} \tag{3.5}$$

for $w \in \widetilde{D}(z)$.

Now we proceed to the proof of the theorem. First assume that Ω is bounded. If $f \notin E^2(\Omega)$, then (3.4) follows from Theorem 3.3 since the first term on the right side of (3.3) is finite. So we may assume that $f \in E^2(\Omega)$.

In this case, the estimate $\gtrsim$ in (3.4) holds by Theorem 3.3 and Proposition 3.1. To see the estimate $\lesssim$, we begin by noting from (3.5) that

$$|f(a)|^2 \lesssim |f(w)|^2 + |f(w) - f(a)|^2 \lesssim |f(w)|^2 + \frac{1}{\delta_\Omega(a)} \int_\Omega |f'|^2 \delta_\Omega \, dA.$$

for $w \in \widetilde{D}(a)$. So, integrating over $\Gamma_a \cap \widetilde{D}(a)$ with respect to the measure $d\sigma_{\Gamma_a}(w)$ and using that $\Lambda_1(\Gamma_a) \approx \delta_\Omega(a) \le \Lambda_1\big(\Gamma_a \cap \widetilde{D}(a)\big)$ by Lemma 3.2, one gets

$$\begin{aligned}\delta_\Omega(a)|f(a)|^2 &\lesssim \int_{\Gamma_a} |f|^2 \, d\sigma_{\Gamma_a} + \frac{\Lambda_1(\Gamma_a)}{\delta_\Omega(a)} \int_\Omega |f'|^2 \delta_\Omega \, dA \\ &\approx \int_{\Gamma_a} |f|^2 \, d\sigma_{\Gamma_a} + \int_\Omega |f'|^2 \delta_\Omega \, dA,\end{aligned}$$

which, together with Theorem 3.3, yields the estimate $\lesssim$, as desired.

Assume now that Ω is unbounded, and first consider the case that $f \in E^2(\Omega)$. In this case, Jerison and Kenig proved in [8, Corollary 2.5] that

$$\|f\|^2_{E^2(\Omega)} \approx \int_\Omega |f'|^2 \delta_\Omega \, dA, \tag{3.6}$$

and the asserted estimate holds by Proposition 3.1.

Finally, we consider the case that $f \notin E^2(\Omega)$. To prove that (3.4) holds when $f \notin E^2(\Omega)$, we must show that

$$\int_{\Gamma_a} |f|^2 \, d\sigma_{\Gamma_a} + \int_\Omega |f'|^2 \delta_\Omega \, dA =: M_1 + M_2 = \infty. \tag{3.7}$$

For this, we assume that $M_2 < \infty$ and prove that this implies $M_1 = \infty$. Since $f \notin E^2(\Omega)$, from [8, Corollary 2.5], $M_2 < \infty$ means that there is an unbounded hyperbolic geodesic Γ_* of Ω such that $f(z) \not\to 0$ as $z \to \infty$ with $z \in \Gamma_*$. After multiplying f by a constant, we may assume there is a sequence $\{z_n\}$ of points in Γ_* such that

$$\lim_{n\to\infty} |z_n| = \infty \quad \text{and} \quad \inf_n |f(z_n)| \ge 1. \tag{3.8}$$

Next, recalling that Γ_* is a vertical line in Ω, we see from [8, Proposition 1.1] that Γ_* is contained in a nontangential approach region $\{z \in \Omega : |z - z_*| \lesssim \delta_\Omega(z)\}$ to a point $z_* \in \partial\Omega$. Thus $\delta_\Omega(z_n) \to \infty$ as $n \to \infty$, and hence $\delta_\Omega(z_n) \ge (2C_1)^2 M_2$ for all n sufficiently large where $C_1 > 0$ is an absolute constant suppressed in the estimate (3.5). Hence, using (3.5) and (3.8), we see that

$$|f(z)| \ge |f(z_n)| - |f(z) - f(z_n)| \ge \frac{1}{2}, \qquad z \in \widetilde{D}(z_n), \tag{3.9}$$

for all n sufficiently large. We also note that

$$\rho_\Omega(z, z_n) \gtrsim 1, \qquad z \in \Omega \setminus D\left(z_n, \frac{\delta_\Omega(z_n)}{4}\right), \tag{3.10}$$

for all n. This, for example, is immediate from the Distance Lemma in [14, p.157].

Now let Γ_a be the unbounded hyperbolic geodesic of Ω from (3.7). From Lemma 3.5 (with $\Gamma_1 = \Gamma_*$ and $\Gamma_2 = \Gamma_a$) and (3.10),

$$\Gamma_a \cap D\left(z_n, \frac{\delta_\Omega(z_n)}{4}\right) \neq \emptyset$$

for all n sufficiently large. Thus, using (3.9), we have

$$\int_{\Gamma_a \cap \widetilde{D}(z_n)} |f|^2 \, d\sigma_{\Gamma_a} \geq \frac{1}{2}\Lambda_1\big(\Gamma_a \cap \widetilde{D}(z_n)\big) \geq \frac{\delta_\Omega(z_n)}{4},$$

for all n sufficiently large. Since $\delta_\Omega(z_n) \to \infty$ (in fact $\delta_\Omega(z_n) \geq 1$ is enough), this implies that $M_1 = \infty$, showing that (3.7) holds and completing the proof. □

4. Applications to composition operators

In this section we observe some consequences which are related to composition operators and the Littlewood-Paley type estimates given in Theorem 3.3 and Theorem 3.6. So, recall that each analytic self-map φ of Ω induces the composition operator C_φ defined on $\mathrm{Hol}(\Omega)$ by

$$C_\varphi f := f \circ \varphi, \qquad f \in \mathrm{Hol}(\Omega).$$

It is known that C_φ is bounded on $E^p(\Omega)$ for some $p \in (0,\infty)$ if and only if it is bounded on $E^p(\Omega)$ for all $p \in (0,\infty)$; see [15, Proposition 2.4]. In what follows we will just consider the case that $p = 2$, and without further mention note that the results on when C_φ is bounded extend to all $p \in (0,\infty)$. As discussed in the Introduction, we will not be concerned with compactness of C_φ.

We will use the notation

$$\|C_\varphi\| := \sup_{\|f\|_{E^2(\Omega)}=1} \|C_\varphi f\|_{E^2(\Omega)},$$

which, when finite, is the operator norm of $C_\varphi : E^2(\Omega) \to E^2(\Omega)$. Of course, unlike the well-known case of unit disk, $\|C_\varphi\| = \infty$ is quite often possible in general as mentioned in the Introduction.

Recall from the Introduction that when a property of an operator acting on a reproducing kernel space can be determined by the operator's action on the reproducing kernels of that space, it is said that the RKT holds for this property of the operator. First, we show that the RKT holds for boundedness of composition operators acting on $E^2(\Omega)$ when Ω is an Ahlfors-regular domain. To this end we introduce the notation

$$\|C_\varphi\|_* := \sup_{b\in\Omega} \|K_b\|^{-1}_{E^2(\Omega)} \|C_\varphi K_b\|_{E^2(\Omega)},$$

which again is possibly ∞. Clearly, $\|C_\varphi\|_* \leq \|C_\varphi\|$.

Theorem 4.1. *Let Ω be an Ahlfors-regular domain. For an analytic self-map φ of Ω, the estimate*

$$\|C_\varphi\| \approx \|C_\varphi\|_*$$

holds with suppressed constants depending only on the Ahlfors-regular constant for Ω*.*

Proof. It suffices to show $\|C_\varphi\| \lesssim \|C_\varphi\|_*$. Pick a Riemann map $\tau : \mathbb{D} \to \Omega$ and put $J_r := \tau(r\mathbb{T})$ for $0 < r < 1$. Let σ_r be the arclength measure on J_r. Note

$$\|\sigma_r \circ \varphi^{-1}\|_* = \sup_{b\in\Omega} \|K_b\|^{-2} \int_{J_r} |C_\varphi K_b|^2 d\sigma_r \leq 2\pi \|C_\varphi\|_*^2, \tag{4.1}$$

for each $r \in (0,1)$. Here, $\sigma_r \circ \varphi^{-1}$ denotes the pullback measure defined by $(\sigma_r \circ \varphi)^{-1}(E) = \sigma_r\big[\varphi^{-1}(E)\big]$ for Borel sets $E \subset \Omega$. The change of variable implicit in the equality in (4.1) can be easily verified by standard approximation starting with characteristic functions.

Now, we see from Proposition 2.3 that

$$\sup_{0<r<1} \|\sigma_r \circ \varphi^{-1}\| \lesssim \|C_\varphi\|_*^2 \tag{4.2}$$

with suppressed constant depending only on the Ahlfors-regular constant for Ω. Being an Ahlfors-regular domain, Ω is a Carleson domain; see Section 2.4. Thus, for $F \in E^2(\Omega)$, from (2.5) and (4.2) we get that

$$\int_{J_r} |C_\varphi F|^2 d\sigma_r = \int_\Omega |F|^2 \, d(\sigma_r \circ \varphi^{-1}) \lesssim \|C_\varphi\|_*^2 \|F\|_{E^2(\Omega)}^2 \tag{4.3}$$

for all $r \in (0,1)$. Taking the supremum over $0 < r < 1$, we get that

$$\|C_\varphi F\|_{E^2(\Omega)}^2 \lesssim \|C_\varphi\|_*^2 \|F\|_{E^2(\Omega)}^2,$$

and thus conclude $\|C_\varphi\| \lesssim \|C_\varphi\|_*$, as desired. The proof is complete. $\square$

Next, we show that the RKT fails, in general, to hold for adjoints of composition operators acting on $E^2(\Omega)$. For example, it fails to hold when Ω is a strip, which is an Ahlfors-regular domain.

Proposition 4.2. *Assume* $\Lambda_1(\partial\Omega) = \infty$ *and* $\|\delta_\Omega\|_\infty := \sup_{b\in\Omega} \delta_\Omega(b) < \infty$*. Let* $a \in \Omega$ *and let* $\varphi \equiv a$ *be the constant self-map of* Ω*. Then,* C_φ^* *is not bounded on* $E^2(\Omega)$ *but*

$$\sup_{b\in\Omega} \|K_b\|_{E^2(\Omega)}^{-2} \|C_\varphi^* K_b\|_{E^2(\Omega)}^2 < \infty. \tag{4.4}$$

Proof. For any $b \in \Omega$, using the well-known fact that $C_\varphi^* K_b = K_{\varphi(b)} = K_a$,

$$\begin{aligned}\|C_\varphi^* K_b\|_{E^2(\Omega)}^2 = \|K_a\|_{E^2(\Omega)}^2 &\approx \frac{1}{\delta_\Omega(a)} \quad \text{by (2.2)} \\ &\leq \frac{1}{\delta_\Omega(a)} \cdot \frac{\|\delta_\Omega\|_\infty}{\delta_\Omega(b)} \simeq \frac{\|\delta_\Omega\|_\infty}{\delta_\Omega(a)} \|K_b\|_{E^2(\Omega)}^2,\end{aligned}$$

which proves (4.4).

Now choose $f \in E^2(\Omega)$ with $f(a) \neq 0$, so that $C_\varphi f$ is a non-zero constant function. From the hypothesis that $\Lambda_1(\partial\Omega) = \infty$ it follows that $\tau' \notin H^1(\mathbb{D})$, where $\tau : \mathbb{D} \to \Omega$ is a Riemann map; see [11, Theorem 10.11]. Thus, by

Proposition 2.1, $C_\varphi f \notin E^2(\Omega)$, and so C_φ is not bounded on $E^2(\Omega)$. Hence C_φ^* is not bounded on $E^2(\Omega)$, and this completes the proof. $\square$

Next, we review some additional background on the hyperbolic distance ρ_Ω introduced in Section 3. A consequence of the Schwarz-Pick Lemma is that an analytic self-map φ of Ω is a ρ_Ω-contraction:

$$\rho_\Omega\big(\varphi(b), \varphi(w)\big) \leq \rho_\Omega(b, w), \qquad b, w \in \Omega. \tag{4.5}$$

For $w \in \Omega$ and $r > 0$, denote by

$$\Delta_\Omega(w, r) := \{z \in \Omega \,:\, \rho_\Omega(z, w) < r\}$$

the open hyperbolic disk in Ω with center w and radius r.

In the next lemma we formulate a version of [10, Corollary 1.5], which is a consequence of the Koebe Distortion Theorem, that will be convenient for our application. Recall that $D(w, r)$ denotes the open Euclidean disk with center w and radius r.

Lemma 4.3. *There exists an absolute constant $\gamma > 1$ such that*

$$D\big(w, \gamma^{-1}\delta_\Omega(w)\big) \subset \Delta_\Omega(w, 1) \subset D\big(w, \gamma\delta_\Omega(w)\big)$$

for $w \in \Omega$. Also, for $w \in \Omega$ and a Riemann map $\tau : \mathbb{D} \to \Omega$, the estimates

$$\delta_\Omega(z) \approx \delta_\Omega(w) \quad \text{and} \quad |\tau'\big(\tau^{-1}(z)\big)| \approx |\tau'\big(\tau^{-1}(w)\big)|, \qquad z \in \Delta_\Omega(w, 1),$$

hold with suppressed constants absolute.

Proposition 4.4. *For $b, z \in \Omega$ with $\rho_\Omega(b, z) < 1$, the estimate*

$$\delta_\Omega(b)|K_b(z)| \approx 1$$

holds with suppressed constants absolute.

Proof. Since the representation (2.1) for K_b is independent of the choice of Riemann map $\tau : \mathbb{D} \to \Omega$, we may assume $\tau(0) = b$. Put $\lambda := \tau^{-1}$ so that $\lambda(b) = 0$. Then, using (2.2), we see that

$$\delta_\Omega(b)|K_b(z)| \approx \left|\frac{K_b(z)}{K_b(b)}\right| = \left|\frac{\lambda'(z)}{\lambda'(b)}\right|^{1/2} = \left|\frac{\tau'(0)}{\tau'\big(\lambda(z)\big)}\right|^{1/2}.$$

The result now follows from Lemma 4.3. $\square$

Given $a \in \Omega$, as in Section 3, we denote by Γ_a an arbitrary (but fixed) hyperbolic geodesic of Ω that passes through a. Recall that the arclength measure of Γ_a is denoted by σ_{Γ_a}. We now define some operators induced by an analytic self-map φ of Ω. Their dependence on a, Γ_a and Ω, which are suppressed for ease of notation, will be clear from the context. First, the operator P_φ is defined by

$$P_\varphi f := (f \circ \varphi)(a), \qquad f \in \mathrm{Hol}(\Omega).$$

Note that P_φ is the point evaluation at $\varphi(a)$. So, we see from the reproducing formula that $\|P_\varphi\|$, the operator norm of $P_\varphi : E^2(\Omega) \to \mathbb{C}$, is precisely equal to $\|K_{\varphi(a)}\|_{E^2(\Omega)} \approx \delta_\Omega(\varphi(a))^{-1/2}$. Next, we define operators L_φ and A_φ by

$$L_\varphi f = \chi_{\Gamma_a}(f \circ \varphi) \quad \text{and} \quad A_\varphi f := (f \circ \varphi)', \qquad f \in \mathrm{Hol}(\Omega). \tag{4.6}$$

Here χ represents the characteristic function of the set in the subscript. Associated with these operators are the (possibly ∞) quantities

$$\|L_\varphi\|^2 := \sup \int_{\Gamma_a} |L_\varphi f|^2 \, d\sigma_{\Gamma_a} \quad \text{and} \quad \|A_\varphi\|^2 := \sup \int_\Omega |A_\varphi f|^2 \delta_\Omega \, dA$$

where the suprema are taken over all $f \in E^2(\Omega)$ with $\|f\|_{E^2(\Omega)} = 1$. Note that $\|L_\varphi\|$, when finite, is the operator norm of $L_\varphi : E^2(\Omega) \to L^2(d\sigma_{\Gamma_a})$. Similarly, $\|A_\varphi\|$, when finite, is the operator norm of $A_\varphi : E^2(\Omega) \to L^2(\delta_\Omega dA)$.

We put $\varphi_{\Gamma_a} := \varphi|_{\Gamma_a}$, the restriction of φ to Γ_a. Also, we denote by $\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1}$ the pullback measure defined by

$$(\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1})(E) := \sigma_{\Gamma_a}[\varphi_{\Gamma_a}^{-1}(E)]$$

for Borel sets $E \subset \Omega$. In terms of this pullback measure we have

$$\|L_\varphi f\|^2_{L^2(\sigma_{\Gamma_a})} = \int_\Omega |f|^2 \, d(\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1}) \tag{4.7}$$

for $f \in \mathrm{Hol}(\Omega)$.

Similarly, $\|A_\varphi f\|_{L^2(\delta_\Omega dA)}$ can be represented as an L^2-norm of f relative to an appropriate measure. To be more precise, we recall a Nevanlinna-type counting function that was introduced in [7]. Given an analytic self-map φ of Ω, define

$$N_\varphi(w) = \begin{cases} \displaystyle\sum_{z \in \varphi^{-1}(\{w\})} \delta_\Omega(z) & \text{if} \quad w \in \varphi(\Omega) \\ \qquad 0 & \text{if} \quad w \notin \varphi(\Omega). \end{cases}$$

Following the proof of the Area Formula [2, Theorem 2.32], one may verify the non-univalent change-of-variable formula

$$\int_\Omega (h \circ \varphi)|\varphi'|^2 \delta_\Omega \, dA = \int_\Omega h N_\varphi \, dA$$

for Borel functions $h \geq 0$ on Ω. In particular,

$$\|A_\varphi f\|^2_{L^2(\delta_\Omega dA)} = \int_\Omega |f'|^2 N_\varphi \, dA, \tag{4.8}$$

for $f \in \mathrm{Hol}(\Omega)$.

We also introduce the (possibly ∞) quantities

$$\|L_\varphi\|_* := \sup_{b \in \Omega} \frac{\|L_\varphi K_b\|_{L^2(\sigma_\Gamma)}}{\|K_b\|_{E^2(\Omega)}} \quad \text{and} \quad \|A_\varphi\|_* := \sup_{b \in \Omega} \frac{\|A_\varphi K_b\|_{L^2(\delta_\Omega dA)}}{\|K_b\|_{E^2(\Omega)}}.$$

In view of (4.7) and (4.8), one may expect a relation between $\|L_\varphi\|$, $\|L_\varphi\|_*$($\|A_\varphi\|$, $\|A_\varphi\|_*$, resp.) and the measure $\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1}$ ($N_\varphi dA$, resp.). Indeed, the next proposition is immediate from (4.7) and Proposition 2.3.

Proposition 4.5. *Let φ be an analytic self-map of an Ahlfors-regular domain Ω and let $a \in \Omega$ and Γ_a be as in* (4.6)*. Then the estimates*

$$\|L_\varphi\|^2 \approx \|\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1}\| \approx \|\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1}\|_* \approx \|L_\varphi\|_*^2$$

hold with suppressed constants depending only on the Ahlfors-regular constant for Ω.

To obtain estimates for $\|A_\varphi\|$, we need some notation. For a Borel measure $\mu \geq 0$ on Ω and $b \in \Omega$, put

$$\widehat{\mu}(b) := \frac{\mu\big(\Delta(b)\big)}{A\big(\Delta(b)\big)} \quad \text{where} \quad \Delta(b) := \Delta_\Omega(b, 1).$$

Note, from Lemma 4.3, that $A\big(\Delta(b)\big) \approx \delta_\Omega(b)^2$ with the suppressed constants absolute. We put $\widehat{N}_\varphi := \widehat{N_\varphi dA}$.

In [7] the counting function N_φ was used to study composition operators in the setting of Hardy-Smirnov spaces on bounded chord-arc domains. The next theorem should be compared to [7, Theorem 5.1].

Theorem 4.6. *Let φ be an analytic self-map of a chord-arc domain Ω. Then the estimate*

$$\|A_\varphi\|^2 \approx \sup_{b \in \Omega} \frac{\widehat{N}_\varphi(b)}{\delta_\Omega(b)}$$

holds with suppressed constants depending only on the chord-arc constant for Ω.

The proof of the lower bound for $\|A_\varphi\|^2$ follows the approach to the proof of [7, Theorem 5.1], which gave an estimate of $\|C_\varphi\|^2$ on a bounded chord-arc domain. However, we'd like to take this opportunity to point out what seems to be an error in part of the proof of the upper bound in [7, Theorem 5.1] where it relied on "$f \circ \varphi \in E^2(\Omega)$ when $f \in E^2(\Omega)$", which does not hold in general. Our approach to the proof of the upper bound for $\|A_\varphi\|^2$ is different and works on both bounded and unbounded chord-arc domains.

Proof. Let $b_0 \in \Omega$. Since $\partial\Omega$ is a chord-arc curve, we can choose $b_0^* \in \mathbb{C} \setminus \overline{\Omega}$ such that $|b_0 - b_0^*| \approx \delta_{\mathbb{C}\setminus\overline{\Omega}}(b_0^*) \approx \delta_\Omega(b_0)$; see [7] and [8]. Define the function

$$k_{b_0}(z) := \frac{\delta_\Omega(b_0)^{1/2}}{z - b_0^*}, \qquad z \in \Omega.$$

Since $\partial\Omega$ is a chord-arc curve, it follows that $\|k_{b_0}\|_{E^2(\Omega)} \approx 1$; see [8, (1.12)]. From the second containment in Lemma 4.3, $|z - b_0^*| \lesssim \delta_\Omega(b_0)$ for $z \in \Delta(b_0)$

and hence $|k'_{b_0}|^2 \gtrsim \delta_\Omega(b_0)^{-3}$ on $\Delta(b_0)$. Thus, using (4.8),

$$\begin{aligned}\|A_\varphi\|^2 &\gtrsim \int_\Omega |A_\varphi k_{b_0}|^2 \delta_\Omega \, dA = \int_\Omega |k'_{b_0}|^2 N_\varphi \, dA \\ &\gtrsim \frac{1}{\delta^3_\Omega(b_0)} \int_{\Delta(b_0)} N_\varphi \, dA \approx \frac{\widehat{N}_\varphi(b_0)}{\delta_\Omega(b_0)}.\end{aligned}$$

Since $b_0 \in \Omega$ was arbitrary, this proves the lower bound for $\|A_\varphi\|^2$.

For the reverse estimate, put $\mu_\varphi := N_\varphi \, dA$ and let

$$M_\varphi := \sup_{b\in\Omega} \frac{\widehat{\mu}_\varphi(b)}{\delta_\Omega(b)}. \tag{4.9}$$

If $M_\varphi = \infty$, the estimate is trivial. So assume $M_\varphi < \infty$ and let $f \in E^2(\Omega)$. Using (4.8) and then subharmonicity (relying on the first containment in Lemma 4.3) of $|f'|^2$, we have that

$$\begin{aligned}\|A_\varphi f\|^2_{L^2(\delta_\Omega dA)} &= \int_\Omega |f'(b)|^2 d\mu_\varphi(b) \\ &\lesssim \int_\Omega \left\{ \frac{1}{\delta_\Omega(b)^2} \int_{\Delta(b)} |f'(z)|^2 \, dA(z) \right\} d\mu_\varphi(b).\end{aligned}$$

From Lemma 4.3, $\delta_\Omega(z) \approx \delta_\Omega(b)$ for $z \in \Delta(b)$, so using Fubini's Theorem while noting that $\chi_{\Delta(b)}(z) = \chi_{\Delta(z)}(b)$ shows that

$$\|A_\varphi f\|^2_{L^2(\delta_\Omega dA)} \lesssim \int_\Omega \frac{\mu_\varphi\big(\Delta(z)\big)}{\delta_\Omega(z)^2} |f'(z)|^2 \, dA(z),$$

which in turn yields

$$\|A_\varphi f\|^2_{L^2(\delta_\Omega dA)} \lesssim M_\varphi \int_\Omega |f'|^2 \delta_\Omega \, dA. \tag{4.10}$$

Note that this estimate is valid on general Ω; this will be used later in the proof of Theorem 5.5. Now, using the assumption that Ω is a chord-arc domain, we obtain $\|A_\varphi\|^2 \lesssim M_\varphi$ by Theorem 3.6, completing the proof. □

Corollary 4.7. *Let Ω be a domain for which the Littlewood-Paley type estimate* (3.4) *holds with suppressed constants depending only on Ω. For an analytic self-map φ of Ω, the estimate*

$$\|C_\varphi\| \approx \|L_\varphi\| + \|A_\varphi\| \tag{4.11}$$

holds. If, in addition, Ω is an Ahlfors-regular domain, then the estimate

$$\|C_\varphi\| \approx \|L_\varphi\|_* + \|A_\varphi\|_* \tag{4.12}$$

holds. The suppressed constants in these estimates depend only on Ω. In particular, C_φ is bounded on $E^2(\Omega)$ if and only if both L_φ and A_φ are bounded on $E^2(\Omega)$.

Proof. The first part of the corollary is an easy consequence of (3.4). Also, as consequence of (3.4), we note $\|C_\varphi\|_* \approx \|L_\varphi\|_* + \|A_\varphi\|_*$. Thus the second part holds by Theorem 4.1. □

For example, we see from Theorem 3.6 that (4.11) and (4.12) hold for chord-arc domains; recall that chord-arc domains are Ahlfors-regular domains. The next corollary is thus immediate from Proposition 4.5, Theorem 4.6 and Corollary 4.7.

Corollary 4.8. *Let φ be an analytic self-map of a chord-arc domain Ω and let $a \in \Omega$ be chosen as in Theorem 3.6. The estimate*

$$||C_\varphi\|^2 \approx \|\sigma_{\Gamma_a} \circ \varphi_{\Gamma_a}^{-1}\| + \sup_{b\in\Omega} \frac{\widehat{N}_\varphi(b)}{\delta_\Omega(b)}$$

holds with suppressed constants depending only on the chord-arc constant for Ω.

Recall that $\|P_\varphi\|^2 \approx 1/\delta_\Omega(\varphi(a))$. So, for bounded chord-arc domains, using Theorem 3.3 we can say a bit more as follows.

Corollary 4.9. *Let Ω be a bounded chord-arc domain and let $a \in \Omega$ be chosen as in* (3.1)*. For an analytic self-map φ of Ω, the estimate*

$$\|C_\varphi\|^2 \approx \frac{\delta_\Omega(a)}{\delta_\Omega\big(\varphi(a)\big)} + \sup_{b\in\Omega} \frac{\widehat{N}_\varphi(b)}{\delta_\Omega(b)}$$

holds with suppressed constants depending only on the chord-arc constant for Ω.

In general we have the following result.

Theorem 4.10. *Let $a \in \Omega$ and let φ be an analytic self-map of Ω. The following assertions hold:*

(a) *There is an absolute constant $C > 1$ such that*

$$C^{-1} \le \delta_\Omega(a)\|P_\varphi\|^2 \le C\|L_\varphi\|_*^2;$$

(b) *It holds that*

$$\|L_\varphi\| \le \sqrt{\pi}\|C_\varphi\|;$$

(c) *If Ω is a Carleson domain, then there is a constant $C(\Omega) > 0$, depending on Ω, such that*

$$\|A_\varphi\| \le C(\Omega)\|C_\varphi\|.$$

Proof. For the proof of (a), using (2.2) we have that

$$\|L_\varphi\|_*^2 \ge \int_{\Gamma_a} \left| \frac{K_{\varphi(a)} \circ \varphi}{\|K_{\varphi(a)}\|_{E^2(\Omega)}} \right|^2 d\sigma_{\Gamma_a} \approx \delta_\Omega(\varphi(a)) \int_{\Gamma_a} |K_{\varphi(a)} \circ \varphi|^2 \, d\sigma_{\Gamma_a}.$$

Let $\Delta := \Delta_\Omega(a,1)$ and note that if $z \in \Delta$, then $\rho_\Omega\big(\varphi(a), \varphi(z)\big) < 1$ by (4.5). Hence, using Proposition 4.4 and (2.2), we get that

$$|K_{\varphi(a)}\big(\varphi(z)\big)| \approx |K_{\varphi(a)}\big(\varphi(a)\big)| \approx \frac{1}{\delta_\Omega\big(\varphi(a)\big)}, \qquad z \in \Delta.$$

Thus, using Lemma 4.3 for the last inequality, we have

$$\|L_\varphi\|_*^2 \gtrsim \frac{\sigma_{\Gamma_a}(\Gamma_a \cap \Delta)}{\delta_\Omega(\varphi(a))} \gtrsim \frac{\delta_\Omega(a)}{\delta_\Omega(\varphi(a))}.$$

Since $\|P_\varphi\|^2 = \|K_{\varphi(a)}\|^2_{E^2(\Omega)} \approx \frac{1}{\delta_\Omega(\varphi(a))}$, this completes the proof of (a).

To prove (b), we note $L_\varphi = L_{id} C_\varphi$ where id denotes the identity map of Ω. Moreover, we have $\|L_{id}\| \le \sqrt{\pi}$ by Proposition 3.1. Thus (b) holds with $C = \sqrt{\pi}$.

To prove (c), assume that Ω is a Carleson domain. It is enough to consider the case $\varphi = id$ as in the proof of (b), and show that $\|A_{id}\| \lesssim 1$. Pick a Riemann map $\tau : \mathbb{D} \to \Omega$. Making the change of variable $z = \tau(w)$ and recalling (see [10, Corollary 1.4]) that $\delta_\Omega(z) \approx |\tau'(w)|\delta_\mathbb{D}(w)$, we obtain

$$\|A_{id} f\|_{L^2(\delta_\Omega dA)} \approx \int_\mathbb{D} |(f\circ\tau)'|^2 |\tau'| \delta_\mathbb{D}\, dA \lesssim I + II, \tag{4.13}$$

for all $f \in E^2(\Omega)$ where

$$\begin{aligned} I :&= \int_\mathbb{D} \left| (f\circ\tau)' + (f\circ\tau)\cdot \frac{\tau''}{2\tau'} \right|^2 |\tau'| \delta_\mathbb{D}\, dA \\ &= \int_\mathbb{D} \left| \big[(\tau')^{1/2}(f\circ\tau)\big]' \right|^2 \delta_\mathbb{D}\, dA \end{aligned}$$

and

$$II := \frac{1}{4}\int_\mathbb{D} |f\circ\tau|^2 |\tau'| \left|\frac{\tau''}{\tau'}\right|^2 \delta_\mathbb{D}\, dA.$$

Now, using the Littlewood-Paley Identity for the $H^2(\mathbb{D})$-norm, we see that

$$I \lesssim \|(\tau')^{1/2}(f\circ\tau)\|^2_{H^2(\mathbb{D})} = \|f\|^2_{E^2(\Omega)}$$

where the equality holds by Proposition 2.1. Next, from Theorem 2.2 and the hypothesis that Ω is a Carleson domain, we have that $\log\tau' \in \mathrm{BMOA}(\mathbb{D})$, and hence the measure $\left|\frac{\tau''}{\tau'}\right|^2 \delta_\mathbb{D}\, dA$ is a Carleson measure on $\mathbb{D}$. Thus

$$II \le C(\Omega)\|(\tau')^{1/2}(f\circ\tau)\|^2_{H^2(\mathbb{D})} = \|f\|^2_{E^2(\Omega)};$$

this is the place where a constant depending on Ω comes into play. Putting these estimates for I and II in (4.13), we obtain

$$\|A_{id} f\|_{L^2(\delta_\Omega dA)} \lesssim \|f\|^2_{E^2(\Omega)}$$

for $f \in E^2(\Omega)$. So, we conclude $\|A_{id}\| \lesssim 1$, which completes the proof of (c). The proof is complete. $\square$

Example 4.11. *Let $\Omega = \mathbb{D}$, $a = 0$, and $\Gamma_0 = (-1, 1)$. For an analytic self-map φ of $\mathbb{D}$, the estimate*

$$\|C_\varphi\| \approx \|L_\varphi\| \approx \|P_\varphi\| = \frac{1}{(1 - |\varphi(0)|^2)^{1/2}}$$

holds with suppressed constants absolute.

Proof. Recall

$$\|P_\varphi\|^2 = \|K_{\varphi(0)}\|^2_{H^2(\mathbb{D})} = \frac{1}{1 - |\varphi(0)|^2}.$$

Also well-known is that

$$\|C_\varphi\|^2 \approx \frac{1}{1 - |\varphi(0)|^2};$$

see [2, Corollary 3.7]. Thus the example holds by Theorem 4.10. □

Our next example, with Ω the square

$$\mathbb{Q} := \{z \in \mathbb{C} : |\operatorname{Re} z| < 1 \text{ and } |\operatorname{Im} z| < 1\},$$

shows that $\|L_\varphi\| < \infty$ but $\|A_\varphi\| = \infty$ is possible.

Example 4.12. *Let $\Omega = \mathbb{Q}$, $a = 0$, and $\Gamma_0 = (-1, 1)$. Let $\tau : \mathbb{D} \to \mathbb{Q}$ be the Riemann map with $\tau(0) = 0$ and $\tau'(0) > 0$. For $\omega \in \mathbb{T}$, define $\varphi_\omega := \tau(\omega\tau^{-1})$, the self-map of $\mathbb{Q}$ that is conformally conjugate to the rotation of $\mathbb{D}$ by angle* $\arg \omega$. *There exists $\omega_0 \in \mathbb{T}$ such that $\|L_{\varphi_{\omega_0}}\| < \infty$, $\|A_{\varphi_{\omega_0}}\| = \infty$, and $C_{\varphi_{\omega_0}}$ is not bounded on $E^2(\mathbb{Q})$.*

Proof. By the Schwarz-Christoffel Formula and symmetry,

$$\tau'(z) = \tau'(0) \prod_{k=0}^{3} (1 - \overline{z}_k z)^{-1/2}, \qquad z \in \mathbb{D},$$

where $z_k = e^{\pi(2k+1)i/4}$, $0 \le k \le 3$, are the points mapped by τ to the corners of $\mathbb{Q}$. Thus τ' is unbounded on $\mathbb{D}$, and it follows from [15, Theorem 6.1] that, for a.e. $\omega \in \mathbb{T}$, C_{φ_ω} is not bounded on $E^2(\mathbb{Q})$.

Next, let $f \in E^2(\mathbb{Q})$. From the formula for τ' there is a positive constant M such that

$$|\tau'(t)| \le M|\tau'(\omega t)|, \quad \text{if} \quad -1 < t < 1 \quad \text{and} \quad |1 - \omega| < \frac{1}{2}.$$

Hence, for such ω, using the change of variable $t = \tau^{-1}(x)$, where $t \in \mathbb{D}$ and $x \in \Gamma_0$, we obtain

$$\begin{aligned}
\|L_{\varphi_\omega} f\|^2_{L^2(d\sigma_{\Gamma_0})} &= \int_{-1}^{1} |f(\varphi_\omega(x))|^2 \, dx \\
&= \int_{-1}^{1} |f(\tau(\omega t))|^2 |\tau'(t)| \, dt \\
&\leq M \int_{-1}^{1} |f(\tau(\omega t))|^2 |\tau'(\omega t)| \, dt \\
&\leq M\pi \|(\tau')^{1/2}(f \circ \tau)\|^2_{H^2(\mathbb{D})} \quad \text{by Proposition 3.1} \\
&= M\pi \|f\|^2_{E^2(\mathbb{Q})} \quad \text{by Proposition 2.1.}
\end{aligned}$$

This yields $\|L_{\varphi_\omega}\| \leq \sqrt{M\pi}$, for $|1-\omega| < \frac{1}{2}$. Since $\mathbb{Q}$ is a chord-arc domain, from Theorem 3.6 and Corollary 4.7, we obtain

$$\|L_{\varphi_\omega}\|^2 + \|A_{\varphi_\omega}\|^2 \approx \|C_{\varphi_\omega}\|^2.$$

Hence, taking $\omega_0 \in \mathbb{T}$ with $|\omega_0 - 1| < \frac{1}{2}$ and $\|C_{\varphi_{\omega_0}}\| = \infty$ completes the proof. $\square$

The next example, with Ω a strip, shows that $\|L_\varphi\| = \infty$ and $\|P_\varphi\| < \infty$ is possible. Recall

$$S = \left\{ z \in \mathbb{C} : |\operatorname{Im} z| < \frac{\pi}{2} \right\}. \tag{4.14}$$

Example 4.13. *Let $\Omega = S$, $a = 0$, and $\Gamma_0 = \mathbb{R}$. If φ is an analytic self-map of S and $\limsup_{x\to\infty} |\varphi(x)| < \infty$, then $\|L_\varphi\| = \infty$ and $\|P_\varphi\| < \infty$.*

Proof. Let $f(z) = (z+2i)^{-1}$, so that $f \in E^2(S)$, and let φ be an analytic self-map of S with $\limsup_{x\to\infty} |\varphi(x)| < \infty$. Then $|f|$ is bounded away from 0 on $\varphi[0,\infty)$, and hence $\|L_\varphi f\|_{E^2(S)} = \|L_\varphi\| = \infty$, while $\|P_\varphi\|^2 \approx \frac{1}{\delta_S(\varphi(0))} < \infty$. $\square$

5. Extensions when Ω is a strip

This section is focused on further developments in the case that Ω is the strip S from (4.14). Note that S is an Ahlfors-regular domain and hence is a Carleson domain by the remark in Section 2.3. However, S is not a chord-arc domain.

Before proceeding, we introduce the notation $T := \partial S$ and σ for the arclength measure on T. We also remark that, given $f \in E^2(S)$, its boundary function, still denoted by f, is uniquely defined σ-almost everywhere on T by means of nontangential limits and that

$$\|f\|^2_{E^2(S)} = \int_T |f|^2 \, d\sigma, \tag{5.1}$$

which can be easily verified using Proposition 2.1.

We begin with an easy observation that the Littlewood-Paley type estimates for chord-arc domains, (3.3) and (3.6), both fail for $E^2(S)$. For $t > 0$, consider the test function

$$f_t(z) := \frac{\sqrt{t}}{z - i\left(t + \frac{\pi}{2}\right)}, \qquad z \in S. \tag{5.2}$$

A straightforward estimate of the integral of $|f_t|^2$ over T shows that

$$\|f_t\|_{E^2(S)} \approx 1, \quad \text{for} \quad t > 0,$$

and another easy estimate shows that

$$\int_S |f_t'|^2 \delta_S \, dA \approx \frac{1}{t^2}, \quad \text{when} \quad t > 1.$$

Also, for any $a \in S$, $f_t(a) \to 0$ as $t \to \infty$. Hence, as $t \to \infty$, we see that both (3.3) and (3.6) fail for $E^2(S)$.

Nonetheless, we observe that a version of (3.4) still holds for S. To this end, we note that a typical Riemann map $\tau : \mathbb{D} \to S$ is given by

$$\tau(z) = \log\left(\frac{1+z}{1-z}\right). \tag{5.3}$$

So, $\mathbb{R} = \tau\big((-1,1)\big)$ is the unbounded hyperbolic geodesic of S through 0.

Theorem 5.1. *For $f \in Hol(S)$, the estimate*

$$\|f\|^2_{E^2(S)} \approx \int_{\mathbb{R}} |f|^2 \, d\sigma_{\mathbb{R}} + \int_S |f'|^2 \delta_S \, dA \tag{5.4}$$

holds with suppressed constants absolute.

Note that both terms on the right side of (5.4) are needed. The test functions f_t from (5.2) can be used to show this. Letting $t \to \infty$ shows that the integral over $\mathbb{R}$ is needed, while letting $t \to 0$ shows that the area integral is needed.

Proof. Let $f \in \mathrm{Hol}(S)$. We first consider the estimate $\gtrsim$. If $f \notin E^2(S)$, this estimate holds trivially, so consider the case that $f \in E^2(S)$. We have by Theorem 4.10(b)

$$\int_{\mathbb{R}} |f|^2 \, d\sigma_{\mathbb{R}} \le \sqrt{\pi} \|f\|^2_{E^2(S)}.$$

Since S is a Carleson domain, we also have by Theorem 4.10(c)

$$\int_S |f'|^2 \delta_S \, dA \le C \|f\|^2_{E^2(S)}$$

for some absolute constant $C > 0$. Adding these two estimates, we conclude the estimate $\gtrsim$.

We now turn to the estimate $\lesssim$. Let Q be the square given by

$$Q := \left\{ z \in S : -\frac{\pi}{2} < \mathrm{Re}\, z < \frac{\pi}{2} \right\}.$$

Clearly, Q is a chord-arc domain. Also, note $\delta_Q \le \delta_S$ on Q. Moreover, using the Riemann map $\tau : \mathbb{D} \to Q$ with $\tau(0) = 0$ and $\tau'(0) > 0$, one may verify by using the uniqueness property of the Riemann map that $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) = \tau\big((-1,1)\big)$ is a hyperbolic geodesic in Q. So, by Theorem 3.6, we obtain

$$\begin{aligned}\int_{\partial Q} |f(\zeta)|^2\,|d\zeta| &\approx \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(x)|^2\,dx + \int_Q |f'|^2\delta_Q\,dA\\ &\le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(x)|^2\,dx + \int_Q |f'|^2\delta_S\,dA.\end{aligned}$$

Clearly, the same estimate remains valid when Q is replaced by each of the squares

$$Q_j := \pi j + Q \qquad (j = 0, \pm 1, \pm 2, \dots).$$

More explicitly, we have

$$\int_{\partial Q_j} |f|^2\,d\sigma_j \lesssim \int_{-\frac{\pi}{2}+\pi j}^{\frac{\pi}{2}+\pi j} |f(x)|^2\,dx + \int_{Q_j} |f'|^2\delta_S\,dA \tag{5.5}$$

for all j where σ_j denotes the arclength measure on ∂Q_j. In conjunction with this, we note from (5.1)

$$\|f\|^2_{E^2(S)} = \int_T |f|^2\,d\sigma \le \sum_j \int_{\partial Q_j} |f|^2\,d\sigma_j.$$

Accordingly, summing both sides of (5.5) over all j, we obtain

$$\|f\|^2_{E^2(S)} \lesssim \int_{\mathbb{R}} |f|^2\,d\sigma_{\mathbb{R}} + \int_S |f'|^2\delta_S\,dA,$$

as required. The proof is complete. □

Remark 5.2. We remark that the proof of Theorem 5.1 can easily be modified to show that versions of (3.4) hold for domains such as a half-strip or the region above a parabola. We leave the details to the readers.

In what follows C_φ refers to the composition operator acting on $\mathrm{Hol}(S)$. Also, L_φ and A_φ refer to the operators introduced in (4.6) associated with $\Omega = S$, $a = 0$ and $\Gamma_0 = \mathbb{R}$. So, the next two corollaries are consequences of Proposition 4.5, Corollary 4.7, and Theorem 5.1.

Corollary 5.3. *For an analytic self-map φ of S, the estimates*

$$\|L_\varphi\|^2 \approx \|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| \approx \|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\|_* \approx \|L_\varphi\|_*^2$$

hold with suppressed constants absolute.

Corollary 5.4. *For an analytic self-map φ of S, the estimates*

$$\|C_\varphi\| \approx \|L_\varphi\|_* + \|A_\varphi\|_* \approx \|L_\varphi\| + \|A_\varphi\|$$

hold with suppressed constants absolute. In particular, C_φ is bounded on $E^2(S)$ if and only if both L_φ and A_φ are bounded on $E^2(S)$.

While $\|A_\varphi\|_* \leq \|A_\varphi\|$ always, in view of Corollary 5.3 and Corollary 5.4 one may conjecture that the estimate $\|A_\varphi\| \approx \|A_\varphi\|_*$ holds. This turns out to be the case in the setting of S.

Theorem 5.5. *For an analytic self-map φ of S, the estimates*

$$\|A_\varphi\|^2 \approx \sup_{b\in S} \frac{\widehat{N}_\varphi(b)}{\delta_S(b)} \approx \|A_\varphi\|_*^2$$

hold with suppressed constants absolute.

We defer a proof until after we establish optimal pointwise growth estimate for the derivatives of reproducing kernels.

Put $\lambda := \tau^{-1}$ where τ is the Riemann map as in (5.3). The explicit formula for λ is given by

$$\lambda(z) = \frac{e^z - 1}{e^z + 1} = 1 - \frac{2}{e^z + 1}.$$

Thus a straightforward calculation via (2.1) yields

$$K_b(z) = \frac{1}{e^{(z-\bar{b})/2} + e^{(\bar{b}-z)/2}} \tag{5.6}$$

and

$$K_b'(z) = \frac{e^{(\bar{b}-z)/2} - e^{(z-\bar{b})/2}}{2\left[e^{(z-\bar{b})/2} + e^{(\bar{b}-z)/2}\right]^2} \tag{5.7}$$

for $b, z \in S$. So, we are naturally led to the next lemma.

Lemma 5.6. *For $z \in S$, the estimates*

$$|e^z + e^{-z}| \approx e^{|z|} \min\left\{\left|z - \frac{\pi i}{2}\right|, \left|z + \frac{\pi i}{2}\right|, 1\right\}$$

and

$$|e^z - e^{-z}| \approx e^{|z|} \min\{|z|, 1\}$$

hold with suppressed constants absolute.

Proof. We prove the first estimate; the proof for the second estimate is similar and simpler.

By symmetry it suffices to consider $z \in S$ with $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z \geq 0$. Fix such a z and note that $e^{\operatorname{Re} z} \approx e^{|z|}$. We thus have

$$e^{-|z|}|e^z + e^{-z}| = e^{\operatorname{Re} z - |z|}|1 - e^{-2z+\pi i}| \approx |1 - e^{-2w}| \tag{5.8}$$

where $w := z - \frac{\pi i}{2}$. Note that w belongs to the set

$$F_1 := \left\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \geq 0,\ -\frac{\pi}{2} \leq \operatorname{Im} \zeta \leq 0\right\}.$$

Put

$$F_2 := \{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \leq 1 \leq |\zeta|\}.$$

Note $k\pi i \notin F_1 \cap F_2$ for any integer k. So, the function $\zeta \mapsto |1 - e^{-2\zeta}|$ on the compact set $F_1 \cap F_2$ is non-vanishing and thus attains positive minimum and maximum. Accordingly, when $w \in F_2$, we have

$$|1 - e^{-2w}| \approx 1 = \min\{|w|, 1\}. \tag{5.9}$$

Now, assume $w \notin F_2$. So, either $\operatorname{Re} w > 1$ or $|w| < 1$. When $\operatorname{Re} w > 1$, we note $|e^{-2w}| = e^{-2\operatorname{Re} w} < e^{-2}$, and so $|1 - e^{-2w}| \approx 1$. Also, when $|w| < 1$, we note $|1 - e^{-2w}| \approx |w|$. Hence we see that the estimate (5.9) remains valid for $w \notin F_2$. Now, since $|w| \le |z + \frac{\pi i}{2}|$, the proof is complete. □

We introduce the notation

$$b^* = \begin{cases} \overline{b} + \pi i & \text{if} \quad \operatorname{Im} b \ge 0 \\ \overline{b} - \pi i & \text{if} \quad \operatorname{Im} b < 0 \end{cases}, \qquad b \in S.$$

Thus, when $b \in S \setminus \mathbb{R}$, b^* can be thought of as the reflection of b across the nearest boundary of S. We are now ready to prove optimal pointwise growth estimates for the reproducing kernels and their derivatives. Note that only derivatives are involved in the operator A_φ, so only the estimate for the derivatives of the reproducing kernels will be used in the proof of Theorem 5.5. That said, the estimate for the reproducing kernels themselves, which might be of independent interest, is also included, as it comes easily from the proof.

Proposition 5.7. *For $b, z \in S$, the estimates*

$$|K_b(z)| \approx \frac{e^{-|z-b|/2}}{\min\{|z - b^*|, 1\}}$$

and

$$|K_b'(z)| \approx \frac{\min\{|z - \overline{b}|, 1\}}{\min\{|z - b^*|^2, 1\}} e^{-|z-b|/2}$$

hold with suppressed constants absolute.

Proof. Let $b, z \in S$. Since $\big||z - b^*| - |z - \overline{b}|\big| \le \pi$ and $\big||z - b| - |z - \overline{b}|\big| \le \pi$, we have

$$e^{|z-b^*|} \approx e^{|z-\overline{b}|} \approx e^{|z-b|}.$$

In addition, if $(\operatorname{Im} z)(\operatorname{Im} b) > 0$, then

$$\min\left\{\left|z - \left(\overline{b} + \pi i\right)\right|, \left|z - \left(\overline{b} - \pi i\right)\right|\right\} = |z - b^*|.$$

On the other hand, if $(\operatorname{Im} z)(\operatorname{Im} b) \le 0$, then

$$\min\left\{\left|z - \left(\overline{b} + \pi i\right)\right|, \left|z - \left(\overline{b} - \pi i\right)\right|\right\} \ge \frac{\pi}{2}.$$

Accordingly, we see that

$$\min\left\{\left|z - \left(\overline{b} + \pi i\right)\right|, \left|z - \left(\overline{b} - \pi i\right)\right|, 1\right\} \approx \min\left\{|z - b^*|, 1\right\}.$$

Thus the proposition holds by (5.6), (5.7) and Lemma 5.6. □

We are now ready to prove Theorem 5.5.

Proof of Theorem 5.5. Put $\mu_\varphi := N_\varphi dA$ for simplicity. Note from (2.2), (4.8) and Proposition 5.7

$$\frac{\|A_\varphi K_b\|^2_{L^2(\delta_S dA)}}{\|K_b\|^2_{E^2(S)}} \approx \delta_S(b) \int_S \frac{e^{-|z-b|}\min\{|z-\bar{b}|^2, 1\}}{\min\{|z-b^*|^4, 1\}}\, d\mu_\varphi(z) =: I_\varphi(b)$$

for all $b \in S$. This yields

$$\|A_\varphi\|^2_* \approx \sup_{b\in S} I_\varphi(b) =: \|I_\varphi\|_\infty.$$

Also, using the quantity M_φ specified in (4.9) (with $\Omega = S$), we have $\|A_\varphi\|^2 \lesssim M_\varphi$ by (4.10) with the following remark and Theorem 5.1. Thus, we will complete the proof by establishing

$$M_\varphi \lesssim \|I_\varphi\|_\infty \tag{5.10}$$

with suppressed constants absolute.

Let $b \in S$. If $|\operatorname{Im} b| \geq \frac{\pi}{4}$, then $|w-\bar{b}| \approx 1$ and $|w-b^*| \approx \delta_S(b)$ for $w \in \Delta(b)$ by the second containment in Lemma 4.3 . So, we obtain

$$\|I_\varphi\|_\infty \geq I_\varphi(b) \gtrsim \delta_S(b) \int_{\Delta(b)} \frac{d\mu_\varphi}{\delta_S^4(b)} \approx \frac{\widehat{\mu}_\varphi(b)}{\delta_S(b)}.$$

On the other hand, if $|\operatorname{Im} b| < \frac{\pi}{4}$, then $|w-(\bar{b}+1)| \approx 1 \approx |w-(b+1)^*|$ and $\delta_S(b) = \delta_S(b+1) \approx 1$ for $w \in \Delta(b)$. So, we obtain

$$\|I_\varphi\|_\infty \geq I_\varphi(b+1) \gtrsim \mu_\varphi\big(\Delta(b)\big) \approx \frac{\widehat{\mu}_\varphi(b)}{\delta_S(b)}.$$

Since these estimates hold for arbitrary $b \in S$, we conclude (5.10), completing the proof. ☐

In view of Corollary 5.4 and Example 4.11, we are naturally led to the study of similar properties on S. Theorem 5.11 below is our result in that direction. To state it we observe in Proposition 5.9 a certain property, which has no disk-analogue, of conformal self-maps of S.

Lemma 5.8. *The equality*

$$\rho_S(s,t) = \frac{|s-t|}{2}$$

holds for all $s, t \in \mathbb{R}$

Proof. Let $s, t \in \mathbb{R}$ and assume without loss of generality that $s < t$. Note (see, for example, [14, Example, p. 151] and also note that the definition of hyperbolic distance in [14] differs from our definition by a factor of 2)

$$2\rho_{\mathbb{D}}(0,r) = \log\frac{1+r}{1-r} = \tau(r), \quad 0 < r < 1,$$

where $\tau : \mathbb{D} \to S$ is the Riemann map specified in (5.3). Then two uses of conformal invariance of the hyperbolic metric gives

$$\rho_S(s,t) = \rho_S(0,t-s) = \rho_{\mathbb{D}}\big(0, \tau^{-1}(t-s)\big) = \frac{t-s}{2},$$

which completes the proof. $\square$

Given an analytic self-map φ of S, put

$$\varepsilon_\varphi := \inf_{x\in\mathbb{R}} \delta_{\varphi(S)}(\varphi(x)),$$

which is precisely the Euclidean distance between $\varphi(\mathbb{R})$ and $\partial\varphi(S)$.

Proposition 5.9. *Let φ be a conformal self-map of S such that $\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| < \infty$. Then $\varepsilon_\varphi > 0$.*

Proof. Suppose to the contrary that φ satisfies the hypotheses but $\varepsilon_\varphi = 0$. Then there is a sequence $\{x_n\} \subset \mathbb{R}$ such that $\delta_{\varphi(S)}(\varphi(x_n)) \le e^{-n}$. Put

$$D_n := D(\zeta_n, 5) \quad \text{where} \quad \zeta_n := \operatorname{Re}\varphi(x_n) - \frac{\pi i}{2}.$$

If $\partial D_{n_0} \cap \varphi[x_{n_0}, \infty) = \emptyset$ for some n_0, then $\varphi[x_{n_0}, \infty) \subset D_{n_0}$ and from Example 4.13 we see that $\|L_\varphi\| = \infty$. By Corollary 5.3, this is contrary to the hypothesis that $\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| < \infty$. So, for each n, assume that $\partial D_n \cap \varphi[x_{n_0}, \infty) \neq \emptyset$ and choose a real $y_n > x_n$ with $\varphi(y_n) \in \partial D_n$ and $\varphi(x_n, y_n) \subset D_n$. Then

$$\begin{aligned}|\varphi(x_n) - \varphi(y_n)| &\ge |\varphi(y_n) - \zeta_n| - |\varphi(x_n) - \zeta_n| \\ &= 5 - \left|\operatorname{Im}\varphi(x_n) + \frac{\pi}{2}\right| \\ &\ge 1.\end{aligned}$$

Thus, using conformal invariance of the hyperbolic distance and then the Distance Lemma [14, p. 157] (also recall, as mentioned before, that the definition of hyperbolic distance in [14] differs from our definition by a factor of 2), we have the estimate

$$\begin{aligned}\rho_S(x_n, y_n) &= \rho_{\varphi(S)}\big(\varphi(x_n), \varphi(y_n)\big) \\ &\ge \frac{1}{4}\log\big[1 + e^n|\varphi(x_n) - \varphi(y_n)|\big] \\ &\ge \frac{n}{4}.\end{aligned} \tag{5.11}$$

It follows that

$$(\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1})(D_n) \ge (\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1})\big(\varphi(x_n, y_n)\big) = |x_n - y_n| \ge \frac{n}{2},$$

where Lemma 5.8 and (5.11) were used to get the last estimate. Since n can be arbitrarily large, this means that $\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| = \infty$. This is contrary to the hypothesis, and so completes the proof by contradiction. $\square$

The following is an immediate consequence of Proposition 5.9.

Corollary 5.10. *Let φ be a conformal self-map of S such that $\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| < \infty$. Then $\lim_{|x|\to\infty} |\varphi(x)| = \infty$.*

Theorem 5.11. *There are absolute constants $c > 0$ and $C > 0$ such that, for a conformal self-map φ of S,*

$$\|C_\varphi\| \le C \frac{e^{c\|L_\varphi\|_*}}{\varepsilon_\varphi}.$$

In particular, $C_\varphi : E^2(S) \to E^2(S)$ is bounded if and only if $L_\varphi : E^2(S) \to L^2(\sigma_{\mathbb{R}})$ is bounded.

The second part of the theorem above is clear from the first part. For the first part, we defer a proof until after establishing/recalling some preliminary results.

Our proof of the first part of Theorem 5.11 will be deeply based on the notion of harmonic measures. So, we pause to review the properties of harmonic measures which will be used in our proof.

Brief Review of Harmonic measures: Let G be any domain in $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ such that its boundary relative to $\mathbb{C}_\infty$ is a set of positive (logarithmic) capacity. Given $z \in G$, it is known that there is a unique Borel probability measure $\omega(z, \cdot, G)$ with the property that the Perron (or Wiener) solution u_f to the Dirichlet Problem for boundary data $f \in C(\partial G)$ is given by

$$u_f(z) = \int_{\partial G} f(\zeta)\, d\omega(z, \zeta, G).$$

Moreover, $\omega(z, E, G)$ is a harmonic function of $z \in G$ for each Borel set $E \subset \partial G$. Such a measure $\omega(z, \cdot, G)$ is called the *harmonic measure for z relative to G*; see [12, Section 4.3] or [6, p.90].

We recall two properties of the harmonic measures which are useful for our purposes. One is the property that if G_1 is a sub-domain of G and E is a Borel subset of $\partial G_1 \cap \partial G$, then

$$\omega(z, E, G_1) \le \omega(z, E, G). \tag{5.12}$$

This is a consequence of the Subordination Principle; see [12, Theorem 4.3.8] and its corollary. The other is the conformal invariance of the harmonic measures. To be explicit, let $G \subsetneq \mathbb{C}$ be a simply connected domain and consider a Riemann map $\tau : \mathbb{D} \to G$. As is well-known (see for example [6, Lemma VI.3.1]), the boundary function $\tau_{\mathbb{T}}$ is defined by nontangential limits on $\mathbb{T}$ except for a Borel set of capacity zero. We have

$$\omega(z, E, G) = \omega\big(\tau^{-1}(z), \tau_{\mathbb{T}}^{-1}(E), \mathbb{D}\big) \tag{5.13}$$

for Borel sets E contained in the range of $\tau_{\mathbb{T}}$; see [6, Theorem VI.3.2]. Since a boundary Borel set of capacity zero has harmonic measure zero, we may regard $\tau_{\mathbb{T}}$ as a Borel function defined everywhere on $\mathbb{T}$. □

Now we proceed to the proof of Theorem 5.11. Given an analytic self-map φ of S, set

$$F_\varphi := \{\zeta \in T : \varphi \text{ fails to have a nontangential limit at } \zeta\}.$$

Denote by φ_T the boundary function of φ defined on $T \setminus F_\varphi$ by nontangential limits. Considering the analytic self-map $\tau^{-1} \circ \varphi \circ \tau$ of $\mathbb{D}$, where $\tau : \mathbb{D} \to S$ is the Riemann map given by (5.3), and noting that τ and τ^{-1} preserve boundary null sets, one may check that F_φ is a σ-null set. Moreover, if φ is conformal in addition, then F_φ is of capacity zero. To see it, note that $\tau^{-1}(F_\varphi)$ is precisely the set of all points in $\mathbb{T} \setminus \{\pm 1\}$ where $\varphi \circ \tau$ fails to have nontangential limits. So, $\tau^{-1}(F_\varphi)$ is of capacity zero by the aforementioned remark and hence so is F_φ by [12, Corollary 3.6.6]. Thus, when the boundary function φ_T is considered, we may regard φ_T as a Borel function defined everywhere on T by trivially extending it to F_φ. When φ is conformal, we get a version of (5.13) for S:

$$\omega\big(z, E, \varphi(S)\big) = \omega\big(\varphi^{-1}(z), {\varphi_T}^{-1}(E), S\big), \tag{5.14}$$

where E is a Borel set in the range of φ_T.

In what follows $\sigma \circ {\varphi_T}^{-1}$ denotes the pullback measure on $\overline{S}$ define by

$$(\sigma \circ {\varphi_T}^{-1})(E) := \sigma[{\varphi_T}^{-1}(E)]$$

for Borel sets $E \subset \overline{S}$; recall that σ denotes the arclength measure on T.

Lemma 5.12. *For an analytic self-map φ of S, the estimate*

$$\|C_\varphi\|^2 \approx \sup_{\substack{\zeta \in T \\ r > 0}} \frac{(\sigma \circ {\varphi_T}^{-1})\big(D(\zeta, r)\big)}{r}$$

holds with suppressed constants absolute.

Proof. One may easily modify the proof for the ball analogue, which is implicit in the proof of [2, Theorem 3.35]. □

Given $x \in \mathbb{R}$, we associate boundary intervals defined by

$$I_1^x := x - \frac{\pi i}{2} + [-1, 1] \quad \text{and} \quad I_2^x := x + \frac{\pi i}{2} + [-1, 1].$$

Recall that Λ_1 denotes one-dimensional Hausdorff measure.

Lemma 5.13. *There is an absolute constant K such that given any conformal self-map φ of S and $x \in \mathbb{R}$, for $k = 1, 2$ there is a hyperbolic geodesic J_k^x of S connecting x to I_k^x with $\Lambda_1\big(\varphi(J_k^x)\big) \le K$.*

Proof. Using the conformal invariance of hyperbolic geodesics and harmonic measures, the result is immediate from [10, Corollary 4.18] applied to the function $f = \varphi \circ \tau$ where $\tau : \mathbb{D} \to S$ is a Riemann map. □

For the proof below we introduce the notation

$$Q(a, b) := \{z \in S \,:\, a < \operatorname{Re} z < b\}$$

for $a, b \in \mathbb{R}$ with $a < b$.

Proof of Theorem 5.11: We assume $\|L_\varphi\|_* < \infty$, or equivalently by Corollary 5.3, $\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| < \infty$. Based on Lemma 5.12, we will complete the proof by finding absolute constants $c > 0$ and $C > 0$ such that

$$\frac{(\sigma \circ {\varphi_T}^{-1})\big(D(\zeta, r)\big)}{r} \le C \frac{e^{c\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\|}}{\varepsilon_\varphi^2} \tag{5.15}$$

for any $\zeta \in T$ and $r > 0$.

Given $x \in \mathbb{R}$, consider $\zeta := x \pm \frac{\pi i}{2} \in T$. Let $r > 0$. By symmetry we may assume that $\operatorname{Im}\zeta = -\frac{\pi}{2}$, and for the rest of the proof we will use the notations $\Omega := \varphi(S)$ and $D := D(\zeta, r)$ for simplicity. Using the constant K provided by Lemma 5.13, let U be any component of the set

$$\Omega \cap Q(x - r - 2K, x + r + 2K)$$

such that $U \cap D \neq \emptyset$. Using Corollary 5.10, there is a maximal interval $[a_1, a_2] \subset \mathbb{R}$ such that $\varphi(a_m) \in \partial U$, $m = 1, 2$. Also, note that

$$|\operatorname{Re}\varphi(a_m) - x| = r + 2K, \quad m = 1, 2. \tag{5.16}$$

From Proposition 5.9, we have that $D(a_m, \varepsilon_\varphi) \subset U$, $m = 1, 2$, and these disks are disjoint. Hence

$$\Lambda_1\big[U \cap \{z : |\operatorname{Re} z - x| = r + 2K\}\big] \ge 4\varepsilon_\varphi$$

and so there can be at most $\pi(2\varepsilon_\varphi)^{-1}$ such components. Thus to establish (5.15) it suffices to show that

$$(\sigma \circ {\varphi_T}^{-1})(D \cap \partial U) \le C \frac{e^{c\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\|}}{\varepsilon_\varphi} r \tag{5.17}$$

for some absolute constants $c > 0$ and $C > 0$.

To begin with, we first note that

$$\begin{aligned} a_2 - a_1 &= (\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1})(U) \\ &\le (\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1})\big(D(\zeta, r + 2K + \pi)\big) \\ &\le C_1(r + 1) \end{aligned} \tag{5.18}$$

where $C_1 := (2K + \pi)\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\|$.

Let $J_k^{a_m}$, $1 \le k, m \le 2$, be the hyperbolic geodesics from Lemma 5.13, and denote by V the portion of S bounded between the crosscuts $\eta_m = J_1^{a_m} \cup J_2^{a_m}$, $m = 1, 2$, of S. Then $\Lambda_1\big(\varphi(J_k^{a_m})\big) \le K$, and using (5.16) we note

$$\varphi(\eta_m) \cap D = \emptyset, \quad m = 1, 2.$$

It follows that

$$\varphi^{-1}(D \cap U) \subset V \ \text{ and hence } \ {\varphi_T}^{-1}(D \cap \partial U) \subset T \cap \partial V.$$

Also, note $\overline{\eta_m} \cap \ T \subset I_1^{a_m} \cup I_2^{a_m}$ by the definition of $J_k^{a_m}$ in Lemma 5.13. Hence, we get the estimate

$$(\sigma \circ {\varphi_T}^{-1})(D \cap \partial U) \le \sigma(T \cap \partial V) \le 2(a_2 - a_1 + 2) \le C_2(r + 1) \tag{5.19}$$

where $C_2 = 2C_1 + 4$ and the last inequality holds by (5.18).

We now split the proof into two cases, depending on how the size of r compares to that of ε_φ. We first consider the case that $r \geq \frac{\varepsilon_\varphi}{2}$. In this case, we see from (5.19) that

$$(\sigma \circ {\varphi_T}^{-1})(D \cap \partial U) \leq C_2 \left(1 + \frac{2}{\varepsilon_\varphi}\right) r \lesssim \frac{\|\sigma_{\mathbb{R}} \circ \varphi_{\mathbb{R}}^{-1}\| + 1}{\varepsilon_\varphi} r,$$

which implies (5.17), and so the proof is complete in this case.

We now turn to the remaining case $0 < r < \frac{\varepsilon_\varphi}{2}$. Consider an arbitrary $w_0 \in \varphi(\mathbb{R})$, so that $w_0 \in \Omega \setminus \overline{D}$ since $\delta_S(w_0) \geq \delta_\Omega(w_0) \geq \varepsilon_\varphi > 2r$. Denoting by Π the half-plane $\{z : \operatorname{Im} z > -\frac{\pi}{2}\}$, we have by (5.12)

$$\omega(w_0, \overline{D} \cap \partial\Omega, \Omega) \leq \omega(w_0, \overline{D} \cap \partial\Omega, \Pi \setminus (\overline{D} \cap \partial\Omega)). \tag{5.20}$$

In connection with this, we consider the function

$$\omega(z, \Pi \cap \partial\overline{D}, \Pi \setminus \overline{D}) - \omega(z, \overline{D} \cap \partial\Omega, \Pi \setminus (\overline{D} \cap \partial\Omega)) =: \omega_1(z) - \omega_2(z), \tag{5.21}$$

which is harmonic on $\Pi \setminus \overline{D}$. As for the boundary values, we have $\omega_1 = 1 \geq \omega_2$ on $\partial D \cap \Pi$ and $\omega_1 = \omega_2 = 0$ on $\partial\Pi \setminus \overline{D}$. Thus, by the Maximum Principle (see [6, Corollary 8.3]), we see that the function in (5.21) is non-negative at each point in $\Pi \setminus \overline{D}$. In particular it is non-negative at $z = w_0$, and this gives the inequality

$$\omega(w_0, \overline{D} \cap \partial\Omega, \Pi \setminus (\overline{D} \cap \partial\Omega)) \leq \omega(w_0, \Pi \cap \partial\overline{D}, \Pi \setminus \overline{D}),$$

which, together with (5.20), yields

$$\omega(w_0, \overline{D} \cap \partial\Omega, \Omega) \leq \omega(w_0, \Pi \cap \partial\overline{D}, \Pi \setminus \overline{D}). \tag{5.22}$$

As is probably well-known, this last harmonic measure can be computed exactly. We include details for completeness. The harmonic measure of an interval $J \subset \partial\Pi$ at w_0 is

$$\omega(w_0, J, \Pi) = \frac{1}{\pi} \theta(w_0, J),$$

where $\theta(w_0, J)$ is the angle formed at w_0 by J; see for example [5, p. 13]. Thus, with the interval $I := x - \frac{\pi i}{2} + [-r, r]$, we have

$$\omega(z, I, \Pi) = \frac{1}{2}, \quad z \in \Pi \cap \partial\overline{D},$$

and so

$$\omega(w_0, \Pi \cap \partial\overline{D}, \Pi \setminus \overline{D}) = 2\omega(w_0, I, \Pi) = 2\theta(w_0, I).$$

In the above the first equality holds by the Maximum Principle; note that two harmonic functions under consideration on $\Pi \setminus \overline{D}$ have the same boundary values (except possibly at ∞ and two corner points). Moreover, a simple geometric argument shows that

$$\theta(w_0, I) \leq \theta(x + i\operatorname{Im} w_0, I) \leq \theta(x + i\varepsilon_\varphi, I) = 2\tan^{-1}\left(\frac{r}{\varepsilon_\varphi}\right) \leq \frac{2r}{\varepsilon_\varphi}.$$

Using this and (5.22), we see that

$$\omega(w_0, \overline{D} \cap \partial\Omega, \Omega) \le \frac{4r}{\varepsilon_\varphi} \tag{5.23}$$

for all $w_0 \in \varphi(\mathbb{R})$.

Now choose $w_0 \in \varphi(\mathbb{R})$ such that $x_0 := \varphi^{-1}(w_0) \in V$. Let $\tau_0(z) := x_0 + \log \frac{1+z}{1-z}$ for $z \in \mathbb{D}$ so that $\tau_0 : \mathbb{D} \to S$ is the Riemann map with $\tau_0(0) = x_0$ and $\tau_0'(0) > 0$. Since $\sigma(T \cap \partial V) \le C_2(1 + \varepsilon_\varphi) < 3C_2$ by (5.19), we see from Lemma 5.13 that $\overline{V}$ is contained in $Q\big(x_0 - C_3, x_0 + C_3\big)$ where $C_3 := 3C_2 + 2K$. It follows that

$$\left|\log\left|\frac{1+z}{1-z}\right|\right| = |\operatorname{Re}\tau_0(z) - x_0| \le C_3, \qquad z \in \tau_0^{-1}(\overline{V}).$$

This yields

$$|\tau_0'(z)| = \frac{2}{|1-z^2|} \le 2e^{C_3} \lesssim e^{6C_1}, \qquad z \in \tau_0^{-1}(\overline{V}).$$

Hence, given a Borel set $E \subset T \cap \partial V$, denoting by $\widetilde{\sigma}$ the normalized arclength measure on $\mathbb{T}$, we obtain by the conformal invariance of harmonic measure

$$\begin{aligned} \omega(x_0, E, S) &= \omega\big(0, \tau_0^{-1}(E), \mathbb{D}\big) = \widetilde{\sigma}[\tau_0^{-1}(E)] \\ &\gtrsim e^{-6C_1} \int_{\tau_0^{-1}(E)} |\tau_0'|\, d\widetilde{\sigma} = e^{-6C_1}\sigma(E). \end{aligned}$$

Applying this to $E = {\varphi_T}^{-1}(D \cap \partial U)$, we obtain

$$\begin{aligned} (\sigma \circ {\varphi_T}^{-1})(D \cap \partial U) &\lesssim e^{6C_1}\omega\big(x_0, {\varphi_T}^{-1}(D \cap \partial U), S\big) \\ &= e^{6C_1}\omega(w_0, \overline{D} \cap \partial U, \Omega) \quad \text{by} \quad (5.14) \\ &\lesssim \frac{e^{6C_1}}{\varepsilon_\varphi} r \quad \text{by} \quad (5.23). \end{aligned}$$

This establishes (5.17) in the case that $r \le \frac{\varepsilon_\varphi}{2}$, and so completes the proof. □

6. Remark

One may ask whether Theorem 5.11 extends to a general analytic self-map φ of S. The answer is likely *no*, but we do not have any explicit example. We tried to construct an example φ for which $\|L_\varphi\|_* < \infty$ but $\|C_\varphi\| = \infty$, but failed. In fact we considered analytic maps $\Phi : \mathbb{D} \to \mathbb{D}$ with $\Phi(\pm 1) = \pm 1$ and angular derivatives at ± 1, in which case a change of variable argument shows that its conformal conjugate $\varphi : S \to S$ satisfies $\|L_\varphi\|_* < \infty$, and tried to find such Φ with φ satisfying $\|C_\varphi\| = \infty$. We were not able to find such an example, and instead ended up proving that no such example is possible; see Proposition 6.1 below.

Given an analytic function β on $\mathbb{D}$, let $H^2_\beta(\mathbb{D})$ be the space defined by

$$H^2_\beta(\mathbb{D}) := \{\beta h \,:\, h \in H^2(\mathbb{D})\}$$

with norm

$$\|\beta h\|_{H^2_\beta(\mathbb{D})} = \|h\|_{H^2(\mathbb{D})}.$$

From Proposition 2.1 it is easy to see that

$$C_\varphi \textit{ is bounded on } E^2(\Omega) \iff C_{\Phi_\eta} \textit{ is bounded on } H^2_{(\eta')^{-1/2}}(\mathbb{D}); \tag{6.1}$$

see for example [9, Theorem 14]. Here, φ is an analytic self-map of Ω, $\eta : \mathbb{D} \to \Omega$ is a Riemann map, and $\Phi_\eta := \eta^{-1} \circ \varphi \circ \eta$ is the conformal conjugate of φ associated with η.

Proposition 6.1. *Suppose that $\Phi : \mathbb{D} \to \mathbb{D}$, $\Phi(\pm 1) = \pm 1$, and Φ has angular derivatives at ± 1. Put $\varphi = \tau \circ \Phi \circ \tau^{-1} : S \to S$ where $\tau : \mathbb{D} \to S$ denotes the Riemann map given in* (5.3)*. Then C_φ is bounded on $E^2(S)$.*

Proof. By (6.1), it suffices to show that C_Φ is bounded on $H^2_{(\tau')^{-1/2}}(\mathbb{D})$, so let $h \in H^2(\mathbb{D})$ and put $f := (\tau')^{-1/2} h \in H^2_{(\tau')^{-1/2}}(\mathbb{D})$. Note $\tau'(z) = 2(1-z^2)^{-1}$. Put $\beta_-(z) := (1-z)^{-1}$ and $\beta_+(z) := (1+z)^{-1}$ so that $\tau' = 2\beta_-\beta_+$. Using the Cauchy-Schwarz Inequality, we have

$$\begin{aligned}
\|C_\Phi f\|^2_{H^2_{(\tau')^{-1/2}}(\mathbb{D})} &= \left\|(1-\Phi^2)(h \circ \Phi)^2 \beta_-\beta_+\right\|_{H^1(\mathbb{D})} \\
&\le \|(1-\Phi)(h\circ\Phi)\beta_-\|_{H^2(\mathbb{D})} \|(1+\Phi)(h\circ\Phi)\beta_+\|_{H^2(\mathbb{D})} \\
&= \left\|C_\Phi\left(\frac{h}{\beta_-}\right)\right\|_{H^2_{\beta_-^{-1}}(\mathbb{D})} \left\|C_\Phi\left(\frac{h}{\beta_+}\right)\right\|_{H^2_{\beta_+^{-1}}(\mathbb{D})}.
\end{aligned} \tag{6.2}$$

From the proof of the sufficiency in [9, Theorem 15], and the hypotheses on Φ, we have that C_Φ is bounded on $H^2_{\beta_\pm^{-1}}(\mathbb{D})$. Hence

$$\left\|C_\Phi\left(\frac{h}{\beta_\pm}\right)\right\|_{H^2_{\beta_\pm^{-1}}(\mathbb{D})} \le \|C_\Phi\|_\pm \|h\|_{H^2(\mathbb{D})} = \|C_\Phi\|_\pm \|f\|_{H^2_{(\tau')^{-1/2}}(\mathbb{D})}$$

where $\|C_\Phi\|_\pm$ denote the operator norms of C_Φ acting on $H^2_{\beta_\pm^{-1}}(\mathbb{D})$. Combined with (6.2), we get that

$$\|C_\Phi f\|^2_{H^2_{(\tau')^{-1/2}}(\mathbb{D})} \le \|C_\Phi\|_- \|C_\Phi\|_+ \|f\|^2_{H^2_{(\tau')^{-1/2}}(\mathbb{D})},$$

which completes the proof. □

Remark 6.2. Matache has proven [9, Theorem 15] that an analytic self-map ψ of the upper half-plane $\mathbb{H}$ induces a bounded composition operator C_ψ on $E^2(\mathbb{H})$ if and only if $\eta^{-1} \circ \psi \circ \eta$, where $\eta : \mathbb{D} \to \mathbb{H}$ is a Riemann map, fixes the boundary point η maps to infinity and has an angular derivative at that point; see also [4]. Thus, it is reasonable to ask whether the converse to Proposition 6.1 is valid. The simple self-map $\varphi(z) = \frac{z}{2}$ of S shows that it is not. It is easy to see that $\|C_\varphi\| < \infty$, for example by using Corollary 5.4 or Theorem 5.11. But the self-map of $\mathbb{D}$ conformally conjugate to φ is a "lens map" that does not have an angular derivative at 1 or -1; see [14, p. 27].

The question *whether there is an analytic self-map φ of S such that $\|L_\varphi\| < \infty$ (or, equivalently by Corollary 5.3, $\|L_\varphi\|_* < \infty$) but $\|C_\varphi\| = \infty$* remains open.

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(Choe) Department of Mathematics, Korea University, Seoul 02845, Korea
Email address: cbr@korea.ac.kr

(Koo) Department of Mathematics, Korea University, Seoul 02845, Korea
Email address: koohw@korea.ac.kr

(Pozzi) Department of Mathematics and Statistics, Saint Louis University, St Louis, MO 63103, USA
Email address: elodie.pozzi@slu.edu

(Smith) Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822, USA
Email address: wayne@math.hawaii.edu