DIFFERENCE OF WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACES

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ABSTRACT. We characterize boundedness and compactness of differences of weighted composition operators acting from a standard Fock space $\mathscr{F}^p$ to a Fock–Lebesgue space $\mathscr{L}^q$. Our characterizations are expressed in terms of the reproducing kernel thesis and appropriate Carleson measure conditions. By allowing the weights to be arbitrary measurable functions, we substantially extend earlier results of Tien and Khoi, which were obtained under the assumption that the weights are entire. Our approach is fundamentally different from the known theory for Bergman spaces and reveals several additional subtleties specific to the Fock setting. A key ingredient in our analysis is an improved Fock-weighted submean estimate for derivatives of Fock functions, which may be of independent interest. As applications when the weights are entire, we provide new proofs of the rigidity phenomenon due to Tien and Khoi, and Yang. We also provide some examples illustrating our results.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and A be the ordinary area measure on $\mathbb{C}$. Let $\mathcal{O}(\mathbb{C})$ be the space of all entire functions on $\mathbb{C}$.

For $0 < p < \infty$ and a positive Borel measure μ on $\mathbb{C}$ (simply write $\mu \geq 0$ later on), denote by $\mathscr{L}^p(\mu)$ the *Fock-Lebesgue* space of all measurable functions h on $\mathbb{C}$ such that

$$\|h\|_{\mathscr{L}^p(\mu)} := \left\{ \int_{\mathbb{C}} \left| h(z) e^{-\frac{|z|^2}{2}} \right|^p d\mu(z) \right\}^{1/p} < \infty.$$

When $\mu = \frac{p}{2\pi} A$ where the factor $\frac{p}{2\pi}$ is chosen only for normalization purpose, we write

$$\mathscr{L}^p := \mathscr{L}^p\left(\frac{p}{2\pi} A\right) \quad \text{and} \quad \|\!|\cdot|\!\|_p := \|\cdot\|_{\mathscr{L}^p}$$

for simplicity. So, putting

$$dG_p(z) := \frac{p}{2\pi} e^{-\frac{p}{2}|z|^2}\, dA(z),$$

we note

$$\mathscr{L}^p := L^p(\mathbb{C}, G_p).$$

Here, and elsewhere, L^p stands for the standard Lebesgue space.

The well-known Fock space $\mathscr{F}^p$ is the linear subspace of $\mathscr{L}^p$ consisting of all entire functions in $\mathscr{L}^p$, i.e.,

$$\mathscr{F}^p := \mathscr{L}^p \cap \mathcal{O}(\mathbb{C}).$$

Date: September 29, 2026.

2020 *Mathematics Subject Classification.* Primary 47B33; Secondary 30H20.

Key words and phrases. Difference; Weighted composition operator; Carleson measure; Boundedness; Compactness; Fock space.

B. Choe was supported by NRF(RS-2026-25476546) of Korea and H. Koo was supported by NRF(RS-2025-23324102) of Korea.

As is well known, $\mathscr{F}^p$ is a closed subspace of $\mathscr{L}^p$. So, $\mathscr{F}^p$ is a Banach space for $1 \le p < \infty$ and a complete metric space for $0 < p < 1$ with the metric $d(f,g) = \|f-g\|_p^p$. In particular, $\mathscr{F}^2$ is a Hilbert space. In fact, $\mathscr{F}^2$ is a reproducing kernel Hilbert space whose reproducing kernel at $a \in \mathbb{C}$ is given by

$$K_a(z) = e^{\overline{a}z}, \qquad z \in \mathbb{C}.$$

It is easy to verify that $\|K_a\|_p = e^{\frac{1}{2}|a|^2}$ for any $0 < p < \infty$. Hence, the $\mathscr{F}^p$-normalized reproducing kernel

$$k_a(z) := \frac{K_a(z)}{\|K_a\|_p} = e^{\overline{a}z - \frac{1}{2}|a|^2} \tag{1.1}$$

is independent of the parameter p and its modulus admits a neat formula

$$|k_a(z)| = e^{\frac{1}{2}(|z|^2 - |z-a|^2)}. \tag{1.2}$$

Given a function $\varphi \in \mathcal{O}(\mathbb{C})$ and a weight (=measurable function) u on $\mathbb{C}$, the weighted composition operator $C_{\varphi,u}$ with inducing pair (φ, u) is defined by

$$C_{\varphi,u} f := u \cdot (f \circ \varphi)$$

for $f \in \mathcal{O}(\mathbb{C})$. So, when $u \equiv 1$, $C_{\varphi,1}$ is the standard composition operator acting on $\mathcal{O}(\mathbb{C})$. For earlier studies on the Fock composition operators, we refer to [5, 6, 10, 12]. We also refer to classical references [7, 11] for various aspects on the theory of the composition operators on the Bergman space over the unit disk or the unit ball.

In [12], Tien and Khoi studied differences of weighted composition operators with entire weights. Their results extend the remarkable rigidity phenomenon first observed in [5] for non-weighted differences. More precisely, when $p \ge q$, they proved that such a difference is bounded (compact, resp.) from $\mathscr{F}^p$ into $\mathscr{F}^q$ if and only if each of the individual weighted composition operators is bounded (compact, resp.). When $q < p$, they established the same rigidity only for non-weighted differences, and their result has been recently extended to the weighted case by Yang [12].

The aim of the present paper is to investigate this question in the more general setting of measurable weights. We obtain complete characterizations of the boundedness and compactness of general weighted differences acting from $\mathscr{F}^p$ into $\mathscr{L}^q$. These characterizations are formulated in terms of the reproducing kernel thesis and Carleson measure conditions. For the notion of Carleson measures associated with Fock and Fock-Lebesgue spaces, we refer to Section 2.

To begin with, we set some notation/terminology to be used for the rest of the paper without any further references.

- Throughout the paper, we fix two inducing pairs (φ, u) and (ψ, v), and set

  $$T := C_{\varphi,u} - C_{\psi,v}.$$

  Note

  $$Tk_a = k_a(\varphi)\big[u - ve^{\overline{a}(\psi-\varphi)}\big], \qquad a \in \mathbb{C}. \tag{1.3}$$

- Let $0 < p, q < \infty$. We say that $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded if

  $$\|T\|_{\mathscr{F}^p \to \mathscr{L}^q} := \sup_{\|f\|_p = 1} \|Tf\|_q < \infty.$$

  We say that $T : \mathscr{F}^p \to \mathscr{L}^q$ is *compact* if $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded; and if $Tf_j \to 0$ in $\mathscr{L}^q$ for any bounded sequence $\{f_j\}$ in $\mathscr{F}^p$ that converges to 0 uniformly on compact subsets of $\mathbb{C}$.

Now, in order to state our results, we introduce some pullback measures associated with φ, ψ, u, v and q. Recall that, given $\mu \geq 0$, the pullback measure $\mu \circ \varphi^{-1}$ on $\mathbb{C}$ is defined by $(\mu \circ \varphi^{-1})(E) = \mu[\varphi^{-1}(E)]$ for Borel sets $E \subset \mathbb{C}$. For $s > 0$, setting

$$d(z) := |\varphi(z) - \psi(z)|, \quad A_s := \{z \in \mathbb{C} : d(z) \geq s\} \quad \text{and} \quad N_s := \mathbb{C} \setminus A_s,$$

we introduce pullback measures

$$\omega_{q,s}^{(\varphi,u)} := \left(\chi_{A_s} \left|ue^{\frac{1}{2}|\varphi|^2}\right|^q \, dG_q\right) \circ \varphi^{-1}$$

and

$$\sigma_{q,s}^{(\varphi,u;\psi,v)} := \left(\chi_{N_s} \left|\left[|u - ve^{\overline{\varphi}(\psi-\varphi)}| + \left|dve^{\overline{\varphi}(\psi-\varphi)}\right|\right] e^{\frac{1}{2}|\varphi|^2}\right|^q \, dG_q\right) \circ \varphi^{-1}.$$

Similarly, we define $\omega_{q,s}^{(\psi,v)}$ and $\sigma_{q,s}^{(\psi,v;\varphi,u)}$ by exchanging the roles of (φ, u) and (ψ, v). Here, and elsewhere, χ_E denotes the characteristic function of the set E. Finally, we put

$$\omega_{q,s} := \omega_{q,s}^{(\varphi,u)} + \omega_{q,s}^{(\psi,v)} \quad \text{and} \quad \sigma_{q,s} := \sigma_{q,s}^{(\varphi,u;\psi,v)} + \sigma_{q,s}^{(\psi,v;\varphi,u)}.$$

Our result for $p \leq q$ is the following theorem. The characterization (c) below is often referred to as the reproducing kernel thesis for the operators under consideration.

Theorem 1.1. *Let $0 < p \leq q < \infty$ and $s > 0$. Let $\varphi, \psi \in \mathcal{O}(\mathbb{C})$ and u, v be weights. Then the following assertions are equivalent:*

(a) *$T : \mathscr{F}^q \to \mathscr{L}^q$ is bounded (compact, resp.);*
(b) *$T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded (compact, resp.);*
(c) *$\sup_{a\in\mathbb{C}} \|Tk_a\|_q < \infty$ (and in addition $\lim_{|a|\to\infty} \|Tk_a\|_q = 0$, resp.);*
(d) *$\omega_{q,s} + \sigma_{q,s}$ is a (compact, resp.) 1-Carleson measure;*
(e) *$\omega_{q,s} + \sigma_{q,s}^{(\varphi,u;\psi,v)}$ or $\omega_{q,s} + \sigma_{q,s}^{(\psi,v;\varphi,u)}$ is a (compact, resp.) 1-Carleson measure.*

Our result for $q < p$ is the following theorem. In this theorem we cannot expect the reproducing kernel thesis even for a single operator. For example, take $u \equiv 1$, $v \equiv 0$ and $\varphi(z) = z$ so that T is the identity operator.

Theorem 1.2. *Let $0 < q < p < \infty$ and $s > 0$. Let $\varphi, \psi \in \mathcal{O}(\mathbb{C})$ and u, v be weights. Put $\lambda := \frac{p}{q}$. Then the following assertions are equivalent:*

(a) *$T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded;*
(b) *$T : \mathscr{F}^p \to \mathscr{L}^q$ is compact;*
(c) *$\omega_{q,s} + \sigma_{q,s}$ is a λ-Carleson measure;*
(d) *$\omega_{q,s} + \sigma_{q,s}^{(\varphi,u;\psi,v)}$ or $\omega_{q,s} + \sigma_{q,s}^{(\psi,v;\varphi,u)}$ is a λ-Carleson measure.*

As applications of our characterizations, we provide new proofs of the aforementioned rigidity results in [12, 13]; see Corollary 4.7 and Corollary 5.4.

In the setting of weighted Bergman spaces on the disk and the ball, analogous Carleson measure characterizations were obtained in [2, 3, 4]. Although the present formulation of the pullback measures was inspired by analogous constructions in the Bergman setting, it differs from its Bergman counterpart and turns out to be much more subtle to handle. In fact, we initially worked with pullback measures that arose naturally from the Bergman framework, but this approach led to significant technical obstacles. Only after considerable trial and error did we arrive at the present formulation, whose structure is fundamentally different from that of the Bergman analogue. The key ingredient in our approach is a sharp Fock-weighted submean estimate for the first-order derivatives of Fock functions and its consequences; see Theorems 3.1 and 3.6. Furthermore, we provide explicit examples

demonstrating that the Bergman-like condition is irrelevant to boundedness/compactness of T in the Fock setting; see Example 6.3 and Example 6.4.

The paper is organized as follows:

In Section 2 we recall the notion of Carleson measures associated with Fock and Fock-Lebesgue spaces. We also recall geometric characterizations for Carleson measures.

In Section 3 we establish the main technical estimates, including the sharp Fock-weighted submean estimate for first-order derivatives and its consequences. In fact we establish a more general higher-order version and observe its consequences, which might be of independent interest.

In Section 4 we prove Theorem 1.1 by applying the sharp Fock-weighted submean estimates obtained in Section 3. As an application when the weights are entire, we provide a new proof of the aforementioned rigidity phenomenon for $p \geq q$.

In Section 5 we prove Theorem 1.2. Khinchine's Inequality is the key to its proof. As an application when the weights are entire, we also provide a new proof of the rigidity phenomenon for $q < p$.

In Section 6 examples are constructed to illustrate compact differences via cancellation and the failure of the Bergman-like approach.

In Section 7 we close the paper by adding a short remark concerning extensions to the weighted setting with possibly different weights for the domain and target spaces.

Constants. Throughout the paper we use the same letter C to denote positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be often specified inside parentheses. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \leq CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries: Carleson measures

In this section we recall the notion of Carleson measures relative to Fock and Fock-Lebesgue spaces. We also recall the geometric characterizations for Carleson measures which were obtained in [9].

Let $0 < p, q < \infty$ and let $\mu \geq 0$. We call μ a (p, q)-Carleson measure if the embedding $\mathscr{F}^p \hookrightarrow \mathscr{L}^q(\mu)$ is bounded, i.e., if

$$\left\{ \int_{\mathbb{C}} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^q d\mu(z) \right\}^{1/q} \leq (\text{constant}) \, \|f\|_p \,, \qquad f \in \mathscr{F}^p.$$

If this is the case, we write $\|\mu\|_{\mathscr{F}^p \to \mathscr{L}^q(\mu)}$ for the operator norm of the embedding $\mathscr{F}^p \hookrightarrow \mathscr{L}^q(\mu)$. If, in addition, the embedding $\mathscr{F}^p \hookrightarrow \mathscr{L}^q(\mu)$ is compact, then μ is called a *compact* (p, q)-Carleson measure.

Here, we recall geometric characterizations of (p, q)-Carleson measures. Let $r > 0$. The average of μ over the disk $D_r(a)$ is $\frac{\mu[D_r(a)]}{A[D_r(a)]}$. Since $A[D_r(a)] = \pi r^2$ is independent of a, we simply define the *r-averaging* function $\widehat{\mu}_r$ of μ by

$$\widehat{\mu}_r(z) := \mu[D_r(z)].$$

We define the *r-Berezin transform* $\widetilde{\mu}_r$ of μ by

$$\widetilde{\mu}_r(z) := \int_C \left(\frac{|K_z(w)|}{\|K_z\|_2 \, \|K_w\|_2} \right)^r d\mu(w) = \int_{\mathbb{C}} e^{-\frac{r}{2}|z-w|^2} d\mu(w).$$

Also, following [9], we call a sequence $\{a_j\} = \{a_j\}_{j=1}^{\infty} \subset \mathbb{C}$ an $\frac{r}{2}$-*lattice* if

$$\mathbb{C} = \bigcup_{j=1}^{\infty} D_{\frac{r}{2}}(a_j)$$

and

$$D_{\frac{r}{4}}(a_j) \cap D_{\frac{r}{4}}(a_k) = \emptyset \quad \text{for} \quad j \neq k.$$

In what follows $\|\cdot\|_{L^p}$ refers to the norm for the ordinary Lebesgue space $L^p(\mathbb{C}, A)$ and $\|\cdot\|_{\ell^p}$ refers to the norm for the standard sequence space ℓ^p over the set of all positive integers. We recall the following characterizations; see [9, Section 3] for details.

Theorem 2.1 ([9]). *Let $0 < p \leq q < \infty$, $\delta > 0$, $r > 0$ and $t > 0$. Let $\{a_j\}$ be an $\frac{r}{2}$-lattice. Then the following assertions are equivalent for $\mu \geq 0$:*

(a) *μ is a (compact, resp.) (p, q)-Carleson measure;*
(b) *$\widehat{\mu}_\delta$ is bounded on $\mathbb{C}$ (and in addition vanishes at ∞, resp.);*
(c) *$\{\widehat{\mu}_r(a_j)\} \in \ell^\infty$ (c_0, resp.);*
(d) *$\widetilde{\mu}_t$ is bounded on $\mathbb{C}$ (and in addition vanishes at ∞, resp.).*

Moreover, the quantities

$$\|\mu\|^q_{\mathscr{F}^p \to \mathscr{L}^q(\mu)}, \quad \|\widetilde{\mu}_t\|_{L^\infty}, \quad \|\widehat{\mu}_\delta\|_{L^\infty}, \quad \|\{\widehat{\mu}_r(a_j)\}\|_{\ell^\infty}$$

are comparable to one another.

Theorem 2.2 ([9]). *Let $0 < q < p < \infty$, $\delta > 0$, $r > 0$ and $t > 0$. Let $\{a_j\}$ be an $\frac{r}{2}$-lattice. Then the following assertions are equivalent for $\mu \geq 0$:*

(a) *μ is a (p, q)-Carleson measure;*
(b) *μ is a compact (p, q)-Carleson measure;*
(c) *$\widehat{\mu}_\delta \in L^{\frac{p}{p-q}}(\mathbb{C}, A)$;*
(d) *$\{\widehat{\mu}_r(a_j)\} \in \ell^{\frac{p}{p-q}}$;*
(e) *$\widetilde{\mu}_t \in L^{\frac{p}{p-q}}(\mathbb{C}, A)$.*

Moreover, the quantities

$$\|\mu\|^q_{\mathscr{F}^p \to \mathscr{L}^q(\mu)}, \quad \|\widetilde{\mu}_t\|_{L^{p/p-q}}, \quad \|\widehat{\mu}_\delta\|_{L^{p/p-q}}, \quad \|\{\widehat{\mu}_r(a_j)\}\|_{\ell^{p/p-q}}$$

are comparable to one another.

Thus the notion of (compact) (p, q)-Carleson measures is independent of (p, q) when $p \leq q$, and depends only on the ratio $\frac{p}{q}$ when $q < p$. Thus, when $\lambda := \frac{p}{q} > 1$, we may simply say (compact) λ-*Carleson* measure instead of (compact) (p, q)-Carleson measure. In this case note that $\frac{p}{p-q}$ is the conjugate exponent of λ. So, when $p \leq q$, noting that ∞ is the conjugate exponent of 1, we may simply say (compact) 1-*Carleson* measure instead of (compact) (p, q)-Carleson measure.

3. Fock-weighted submean estimates

For $0 < p < \infty$, $\beta > 0$ and $r > 0$, we put

$$A_{p,\beta,r} f(a) := \left\{ \int_{D_r(a)} \left| f(z) e^{-\beta|z|^2} \right|^p \, dA(z) \right\}^{1/p}$$

for $f \in \mathcal{O}(\mathbb{C})$ and $a \in \mathbb{C}$. This can be viewed (modulo the constant fact $(\pi r^2)^{-1/p}$) as a Fock-weighted average at a of $|f|$ associated with (p, β, r).

For a proof of the following Fock-weighted submean estimate for $k = 0$, we refer to [1, Lemma 3.1] or [14, Lemma 2.32].

Lemma 3.1. *Given $0 < p < \infty$, $\beta > 0$, $r > 0$ and an integer $k \geq 0$, there is a constant $C = C(p, \beta, r, k) > 0$ such that*

$$|f^{(k)}(a)|e^{-\beta|a|^2} \leq C(1 + |a|)^k A_{p,\beta,r} f(a)$$

for $f \in \mathcal{O}(\mathbb{C})$ and $a \in \mathbb{C}$.

Proof. By the aforementioned remark we may assume $k \geq 1$. A slightly weaker version is proved in [1, Proposition 2.2] whose proof can be easily adapted for the current version. Here, we provide details for convenience of readers.

Let $f \in \mathcal{O}(\mathbb{C})$ and pick $a \in \mathbb{C}$. Fix a positive number $\epsilon < r$. Applying the Cauchy Estimates over $D_{\frac{\epsilon}{1+|a|}}(a)$, we have

$$|f^{(k)}(a)| \leq \frac{1}{k!} \left(\frac{1 + |a|}{\epsilon} \right)^k \left(\sup_{|z-a|=\frac{\epsilon}{1+|a|}} |f(z)| \right). \tag{3.1}$$

To estimate the above supremum, let $|z - a| = \frac{\epsilon}{1+|a|}$. Note $D_{r-\epsilon}(z) \subset D_r(a)$. Also, note $|z|^2 \leq |a|^2 + 2\epsilon + \epsilon^2$, because $|z| \leq |a| + \frac{\epsilon}{1+|a|}$. Thus, by the case $k = 0$, we obtain

$$|f(z)|e^{-\beta|a|^2} \lesssim |f(z)|e^{-\beta|z|^2} \lesssim A_{p,\beta,r-\epsilon} f(z) \leq A_{p,\beta,r} f(a)$$

with hidden constant depending only on allowed parameters. Combining this with (3.1), we conclude the asserted inequality, completing the proof. □

Corollary 3.2. *Given $0 < p < \infty$, $\beta > 0$, $0 < r < t$ and an integer $k \geq 0$, there is a constant $C = C(p, \beta, r, t, k) > 0$ such that*

$$A_{p,\beta,r} f^{(k)}(a) \leq C(1 + |a|)^k A_{p,\beta,t} f(a);$$

for $f \in \mathcal{O}(\mathbb{C})$ and $a \in \mathbb{C}$.

Proof. Let $f \in \mathcal{O}(\mathbb{C})$ and $a \in \mathbb{C}$. For $z \in D_r(a)$, we note $1 + |z| \leq (1 + r)(1 + |a|)$ and thus by Lemma 3.1

$$|f^{(k)}(z)|e^{-\beta|z|^2} \lesssim (1 + |a|)^k A_{p,\beta,t-r} f(z) \leq (1 + |a|)^k A_{p,\beta,t} f(a)$$

with hidden constant depending only on allowed parameters. This implies the asserted inequality. □

In our argument for the proof of Theorem 1.1 we will need a Fock-weighted submean estimate for the first-order derivative. However, the first-order estimate in Lemma 3.1 turns out to be too rough for our purpose and we need a more accurate version. To this end, we prove a more general higher-order version which might be of its own interest.

Theorem 3.3. *Given $0 < p < \infty$, $\beta > 0$, $r > 0$ and an integer $k \geq 0$, there is a constant $C = C(p, \beta, r, k) > 0$ such that*

$$\left| \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(-2\beta\overline{a}\right)^{k-j} f^{(j)}(a) \right| e^{-\beta|a|^2} \leq C A_{p,\beta,r} f(a)$$

for $f \in \mathcal{O}(\mathbb{C})$ and $a \in \mathbb{C}$.

Proof. Let $f \in \mathcal{O}(\mathbb{C})$ and $a \in \mathbb{C}$. Put

$$g_a(z) := e^{-\beta(2\overline{a}z+|a|^2)} f(a+z) \in \mathcal{O}(\mathbb{C}).$$

Note

$$g_a^{(k)}(0) = e^{-\beta|a|^2} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(-2\beta\overline{a}\right)^{k-j} f^{(j)}(a).$$

Since

$$|g_a(z)|e^{-\beta|z|^2} = |f(z+a)|e^{-\beta|z+a|^2},$$

we note by Lemma 3.1

$$\sup_{|z|=\frac{r}{2}} \left|g_a(z)\right| e^{-\frac{\beta r^2}{4}} \lesssim \sup_{|z|=\frac{r}{2}} A_{p,\beta,\frac{r}{2}} g_a(z) \leq A_{p,\beta,r} f(a).$$

Accordingly, we obtain by the Cauchy Estimates

$$|g_a^{(k)}(0)| \lesssim \left(\frac{2}{r}\right)^k A_{p,\beta,r} f(a)$$

with hidden constants depending only on allowed parameters. This completes the proof. □

The following corollary is stated for easier reference later.

Corollary 3.4. *Given $0 < p < \infty$, $\beta > 0$ and $r > 0$, there is a constant $C = C(p,\beta,r) > 0$ such that*

$$\int_{\mathbb{C}} \left[|f(z)|^p + |f'(z) - 2\beta\overline{z}f(z)|^p\right] e^{-p\beta|z|^2}\, d\mu(z) \leq C \int_{\mathbb{C}} \left|f(w)e^{-\beta|w|^2}\right|^p \widehat{\mu}_r(w)\, dA(w)$$

for $f \in \mathcal{O}(\mathbb{C})$ and $\mu \geq 0$.

Proof. Fix $f \in \mathcal{O}(\mathbb{C})$ and $\mu \geq 0$. We have by Lemma 3.1 and Theorem 3.3

$$\left[|f(z)|^p + |f'(z) - 2\beta\overline{z}f(z)|^p\right] e^{-p\beta|z|^2} \lesssim A^p_{p,\beta,r} f(z), \qquad z \in \mathbb{C},$$

with hidden constants depending only on allowed parameters. Now, integrating both sides of the above over $\mathbb{C}$ against the measure $d\mu(z)$, and then interchanging the order of integrations, we conclude the lemma. □

Remark 3.5. The significance of Theorem 3.3 lies in the fact that the growth of higher-order derivatives can be completely cancelled, up to the Fock-weighted averaging factor, by introducing suitable cancellation terms. Building on Theorem 3.3, one can go a step further and obtain an improved growth estimate for higher-order derivatives by means of appropriately adjusted cancellation terms. Here, we illustrate the simplest example, which is closely connected with our argument later. We continue using the notation as in Theorem 3.3. Pick $t > r$.

We claim that

$$\left|f^{(k)}(a) - (2\beta\overline{a})^k f(a)\right| e^{-\beta|a|^2} \lesssim (1+|a|)^{k-1} A_{p,\beta,t} f(a) \tag{3.2}$$

with hidden constants depending only on allowed parameters. This reveals that subtracting the term $(2\beta\overline{a})^k f(a)$ from $f^{(k)}(a)$ improves the growth order by 1. We prove this by induction on k. By the case $k=1$ of Theorem 3.3, we have

$$|f'(a) - 2\beta\overline{a}f(a)|e^{-\beta|a|^2} \lesssim A_{p,\beta,r}f(a) \tag{3.3}$$

and thus (3.2) holds for $k=1$. Now, assume (3.2) holds for some integer $k \geq 1$. Applying the above with $f^{(k)}$ in place of f, we note

$$\begin{aligned}\left|f^{(k+1)}(a) - 2\beta\overline{a}f^{(k)}(a)\right| e^{-\beta|a|^2} &\lesssim A_{p,\beta,r}f^{(k)}(a) \\ &\lesssim (1+|a|)^k A_{p,\beta,t}f(a);\end{aligned}$$

the second estimate holds by Corollary 3.2. This, together with induction hypothesis (3.2), implies that (3.2) also holds for $k+1$. So, (3.2) holds for all k.

Now, in conjunction with (3.2), we note

$$\sum_{k=1}^{\infty} \frac{(z-a)^k}{k!}\left[f^{(k)}(a) - (2\beta\overline{a})^k f(a)\right] = f(z) - e^{2\beta\overline{a}(z-a)}f(a).$$

In view of this, one may intuitively expect Fock-weighted averages to be involved in the error estimate when $f(z)$ is approximated near a by $e^{2\beta\overline{a}(z-a)}f(a)$. In Theorem 3.6 below, which is a key to the proof of Theorem 1.1, we will use the estimate (3.3) as a main tool to show that such intuition turns out to be the case. This reveals that, in our setting with $\beta = \frac{1}{2}$, the correct first-order derivative quantity in the Fock-weighted submean estimate near a appears to be $f'(a) - \overline{a}f(a)$, not $f'(a)$ alone.

As mentioned above, the case $\beta = \frac{1}{2}$ of the following inequality, which is a non-trivial consequence of (3.3), will play a key role in our proof of Theorem 1.1 in the next section. Note that the Euclidean analogue of the next theorem, which is obtained by simply taking $\beta = 0$, is also true and must have been well known.

Theorem 3.6. *Given $0 < p < \infty$, $\beta > 0$ and $0 < r < t$, there is a constant $C = C(p,\beta,r,t) > 0$ such that*

$$\left|f(a) - e^{2\beta\overline{a}(a-b)}f(b)\right| e^{-\beta|a|^2} \leq C|a-b|A_{p,\beta,t}f(a)$$

for $f \in \mathcal{O}(\mathbb{C})$ and $a, b \in \mathbb{C}$ with $|a-b| < r$.

Proof. Fix $f \in \mathcal{O}(\mathbb{C})$ and $a, b \in \mathbb{C}$ with $|a-b| < r$. Put

$$F(z) := e^{2\beta\overline{a}(a-z)}f(z)$$

so that

$$f(a) - e^{2\beta\overline{a}(a-b)}f(b) = F(a) - F(b) = \int_b^a F'(\zeta)\, d\zeta.$$

It follows that

$$\left|f(a) - e^{2\beta\overline{a}(a-b)}f(b)\right| \leq |a-b| \left(\sup_{\zeta\in[a,b]} |F'(\zeta)|\right)$$

where $[a,b]$ denotes the line segment connecting a and b. We will complete the proof by establishing the estimate

$$\sup_{\zeta\in[a,b]} |F'(\zeta)| \lesssim e^{\beta|a|^2} A_{p,\beta,t}f(a) \tag{3.4}$$

with hidden constant depending only on allowed parameters.

To begin with, we note for $\zeta \in [a,b]$

$$\begin{aligned} F'(\zeta) &= e^{2\beta\overline{a}(a-\zeta)}[f'(\zeta) - 2\beta\overline{a}f(\zeta)] \\ &= e^{2\beta\overline{a}(a-\zeta)}[f'(\zeta) - 2\beta\overline{\zeta}f(\zeta) + 2\beta(\overline{\zeta}-\overline{a})f(\zeta)] \end{aligned}$$

and hence

$$|F'(\zeta)| \le e^{2\beta[|a|^2 - \mathrm{Re}\,(\overline{a}\zeta)]}\big[\big|f'(\zeta) - 2\beta\overline{\zeta}f(\zeta)\big| + 2\beta r|f(\zeta)|\big]. \tag{3.5}$$

In conjunction with the exponent in the right hand side of the above, we note

$$2|a|^2 - 2\mathrm{Re}\,(\overline{a}\zeta) = |a|^2 + |a-\zeta|^2 - |\zeta|^2 \le |a|^2 - |\zeta|^2 + r^2.$$

Combined with (3.5), this yields

$$|F'(\zeta)| \lesssim e^{\beta(|a|^2 - |\zeta|^2)}\big[|f'(\zeta) - 2\beta\overline{\zeta}f(\zeta)| + |f(\zeta)|\big] \tag{3.6}$$

with hidden constant depending only on β, r and t. In addition, taking $\epsilon := t - r > 0$ so that $D_\epsilon(\zeta) \subset D_t(a)$ for all $\zeta \in [a,b]$, we note from Lemma 3.1 and (3.3)

$$\big[|f'(\zeta) - 2\beta\overline{\zeta}f(\zeta)| + |f(\zeta)|\big]e^{-\beta|\zeta|^2} \lesssim A_{p,\beta,t}f(a)$$

with hidden constant depending only on allowed parameters. Since this estimate is uniform in $\zeta \in [a,b]$, we conclude (3.4) by (3.6), completing the proof. □

The estimate in Theorem 3.6 is sharp in the sense that it can be reversed for some f. For example, taking $f(z) = e^{(\beta+\epsilon)z^2}$ with $\epsilon > 0$, one may check that the estimate can be reversed near $a := \epsilon^{-1}$.

4. The Case $p \le q$

In this section we prove Theorem 1.1 and, as an application, provide a new proof of the rigidity phenomenon for the range $p \le q$.

Our proof of Theorem 1.1 will rely on two key observations. Namely, we will observe in Lemma 4.2 and Lemma 4.3 below that the averaging functions of the pullback measures under consideration are controlled by the norms of T-images of normalized reproducing kernels and vice versa.

Lemma 4.1. *Let $0 < p < \infty$ and $r > 0$. Let $\varphi, \psi \in \mathcal{O}(\mathbb{C})$ and u, v be weights. Then there is a constant $C = C(p,r) > 0$ such that*

$$|Tk_a(b)|^p \le C\int_{D_r(a)} |Tk_z(b)|^p\, dA(z)$$

for $a, b \in \mathbb{C}$.

Proof. Fix $a, b \in \mathbb{C}$. Note

$$Tk_a(b) = e^{-\frac{1}{2}|a|^2}F_b(a) \quad \text{where} \quad F_b(a) = u(b)e^{\overline{a}\varphi(b)} - v(b)e^{\overline{a}\psi(b)}.$$

Note that $\overline{F}_b(a)$ is entire as a function of a. Hence, we have by Lemma 3.1

$$\begin{aligned} |Tk_a(b)|^p &= \left|F_b(a)e^{-\frac{1}{2}|a|^2}\right|^p \\ &\lesssim \int_{D_r(a)} \left|F_b(z)e^{-\frac{1}{2}|z|^2}\right|^p dA(z) \\ &= \int_{D_r(a)} |Tk_z(b)|^p\, dA(z) \end{aligned}$$

with hidden constant depending only on allowed parameters. The proof is complete. □

Lemma 4.2. *Let $0 < q < \infty$, $s > 0$ and $0 < r < t$. Let $\varphi, \psi \in \mathcal{O}(\mathbb{C})$ and u, v be weights. Put $\mu := \omega_{q,s}^{(\varphi,u)} + \sigma_{q,s}^{(\varphi,u;\psi,v)}$. Then there is a constant $C = C(q,r,t,s) > 0$ such that*

$$\widehat{\mu}_r(a) \le C \left(\sup_{\xi \in D_t(a)} \int_{\varphi^{-1}(D_r(a))} |Tk_\xi|^q \, dG_q \right)$$

for all $a \in \mathbb{C}$. In particular,

$$\widehat{\mu}_r(a) \le C \left(\sup_{\xi \in D_t(a)} \|Tk_\xi\|_q^q \right)$$

for all $a \in \mathbb{C}$.

Proof. The second part of the lemma being immediate from the first part, we only need to prove the first part. Before proceeding, we set some notation and make some preliminary observations.

Fix $z \in \mathbb{C}$. Throughout the proof we set

$$\epsilon := \frac{t-r}{2} > 0, \;\; b := \varphi(z) \;\text{ and }\; \delta := \psi(z) - b$$

for ease of notation. Note from (1.3)

$$Tk_{b+\eta}(z) = k_{b+\eta}(b) \left(u(z) - v(z) e^{(\overline{b}+\overline{\eta})\delta} \right).$$

In connection with this, we put

$$X_\eta := \frac{Tk_{b+\eta}(z)}{k_{b+\eta}(b)} = u(z) - v(z) e^{(\overline{b}+\overline{\eta})\delta}.$$

By (1.2) we note

$$|X_\eta| = |Tk_{b+\eta}(z)| e^{\frac{1}{2}(|\eta|^2 - |b|^2)}. \tag{4.1}$$

When $|\eta| < \epsilon$ and $b \in D_r(a)$, note $D_\epsilon(b+\eta) \subset D_{r+2\epsilon}(a) = D_t(a)$. Consequently, we note from Lemma 4.1

$$|Tk_{b+\eta}(z)|^q \lesssim \int_{D_t(a)} |Tk_\xi(z)|^q \, dA(\xi) \tag{4.2}$$

for $\eta \in D_\epsilon(0)$ and $b \in D_r(a)$, The constant hidden above depends only on q, r and t.

Estimate on A_s. Let $z \in A_s$. Pick $c \in (0, \epsilon s)$ and put

$$\eta_0 := \frac{c\delta}{|\delta|^2}.$$

Since $|\delta| \ge s$, we note

$$|\eta_0| = \frac{c}{|\delta|} \le \frac{c}{s} < \epsilon. \tag{4.3}$$

Since $\overline{\eta}_0 \delta = c$, we also note

$$\frac{X_0 - X_{\eta_0}}{e^c - 1} = v(z) e^{\overline{b}\delta}.$$

Consequently,

$$|u(z)| \le |X_0| + |v(z) e^{\overline{b}\delta}| \le \frac{e^c}{e^c - 1} \left(|X_0| + |X_{\eta_0}| \right). \tag{4.4}$$

Moreover, we have by (4.1) and (4.3)

$$(|X_0| + |X_{\eta_0}|) e^{\frac{1}{2}|b|^2} \le |Tk_b(z)| + e^{\frac{1}{2}\epsilon^2} |Tk_{b+\eta_0}(z)|.$$

Combined with (4.4), this yields

$$|u(z)|e^{\frac{1}{2}|b|^2} \lesssim |Tk_b(z)| + |Tk_{b+\eta_0}(z)|.$$

So, given $a \in \mathbb{C}$, this, together with (4.2), yields (recall $b = \varphi(z)$)

$$\left|u(z)e^{\frac{1}{2}|\varphi(z)|^2}\right|^q \lesssim \int_{D_t(a)} |Tk_\xi(z)|^q \, dA(\xi) \quad \text{for } z \in \varphi^{-1}(D_r(a)).$$

Integrating over $A_s \cap \varphi^{-1}(D_r(a))$ both sides of the above against the measure $dG_q(z)$, and interchanging the order of integrations, we obtain

$$\begin{aligned}
\left(\widehat{\omega_{q,s}^{(\varphi,u)}}\right)_r(a) &= \int_{A_s\cap\varphi^{-1}(D_r(a))} \left|u(z)e^{\frac{1}{2}|\varphi(z)|^2}\right|^q \, dG_q(z) \\
&\lesssim \int_{D_t(a)} \left\{ \int_{A_s\cap\varphi^{-1}(D_r(a))} |Tk_\xi(z)|^q \, dG_q(z) \right\} dA(\xi) \\
&\lesssim \sup_{\xi\in D_t(a)} \int_{A_s\cap\varphi^{-1}(D_r(a))} |Tk_\xi(z)|^q \, dG_q(z). \qquad (4.5)
\end{aligned}$$

One may check that the constants suppressed in the above estimates depend only on r and t.

Estimate on N_s. Let $z \in N_s$. Since $|\delta| \le s$, we note $|\delta| \approx |1 - e^{\epsilon\delta}|$ and thus

$$\begin{aligned}
\left|\delta v(z)e^{\bar{b}\delta}\right| &\approx \left|v(z)e^{\bar{b}\delta}\right|\left|1 - e^{\epsilon\delta}\right| \\
&= \left|v(z)e^{\bar{b}\delta} - v(z)e^{(\bar{b}+\epsilon)\delta}\right| \\
&\le \left|u(z) - v(z)e^{\bar{b}\delta}\right| + \left|u(z) - v(z)e^{(\bar{b}+\epsilon)\delta}\right|.
\end{aligned}$$

Accordingly,

$$\left|u(z) - v(z)e^{\bar{b}\delta}\right| + \left|\delta v(z)e^{\bar{b}\delta}\right| \lesssim \left|u(z) - v(z)e^{\bar{b}\delta}\right| + \left|u(z) - v(z)e^{(\bar{b}+\epsilon)\delta}\right|.$$

We thus obtain by (4.1)

$$\left[\left|u(z) - v(z)e^{\bar{b}\delta}\right| + \left|\delta v(z)e^{\bar{b}\delta}\right|\right]e^{\frac{1}{2}|b|^2} \lesssim |Tk_b(z)| + |Tk_{b+\epsilon}(z)| \qquad (4.6)$$

with hidden constant depending only on s, r and t.

Now, having (4.6), one may proceed as in the estimate on A_s to conclude

$$\left(\widehat{\sigma_{q,s}^{(\varphi,u;\psi,v)}}\right)_r(a) \lesssim \sup_{\xi\in D_t(a)} \int_{N_s\cap\varphi^{-1}(D_r(a))} |Tk_\xi(z)|^q \, dG_q(z)$$

with hidden constant depending only on s, r and t. Finally, combining this and (4.5), we conclude the lemma. The proof is complete. □

Lemma 4.3. *Let $0 < q < \infty$ and $0 < s < r$. Let $\varphi, \psi \in \mathcal{O}(\mathbb{C})$ and u, v be weights. Then there is a constant $C = C(q, r, s) > 0$ such that*

$$\|Tf\|_q^q \le C \int_{\mathbb{C}} |f|^q \widehat{\mu}_r \, dG_q \qquad \left(\mu := \omega_{q,s} + \sigma_{q,s}^{(\varphi,u;\psi,v)}\right)$$

for $f \in \mathcal{O}(\mathbb{C})$. In particular,

$$\|Tk_a\|_q^q \le C \int_{\mathbb{C}} e^{-\frac{q}{2}|z-a|^2} \widehat{\mu}_r(z) \, dA(z)$$

for $a \in \mathbb{C}$.

Proof. The second part being immediate from the first part, we only need to prove the first part of the lemma.

Let $f \in \mathcal{E}(\mathbb{C})$. We write

$$\|\!|Tf|\!\|_q^q = \int_{A_s} + \int_{N_s} |uf(\varphi) - vf(\psi)|^q \, dG_q =: I + II.$$

We will estimate integrals I and II separately.

For the integral I, we have

$$I \lesssim \int_{A_s} \left(|uf(\varphi)|^q + |vf(\psi)|^q\right) \, dG_q.$$

Note

$$\begin{aligned}\int_{A_s} |uf(\varphi)|^q \, dG_q &= \int_{\mathbb{C}} \left|f(\varphi)e^{-\frac{1}{2}|\varphi|^2}\right|^q \left|\chi_{A_s} u e^{\frac{1}{2}|\varphi|^2}\right|^q \, dG_q \\ &= \int_{\mathbb{C}} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^q \, d\omega_{q,s}^{(\varphi,u)}(z).\end{aligned}$$

The same holds with (ψ, v) in place of (φ, u). Consequently, we obtain by Corollary 3.4

$$I \lesssim \int_{\mathbb{C}} |f|^q \widehat{(\omega_{q,s})}_r \, dG_q \tag{4.7}$$

with hidden constant depending only on allowed parameters.

For the estimate of the integral II, we decompose

$$uf(\varphi) - vf(\psi) = \left[u - ve^{\overline{\varphi}(\psi-\varphi)}\right] f(\varphi) + v[e^{\overline{\varphi}(\psi-\varphi)} f(\varphi) - f(\psi)]$$

so that

$$\begin{aligned}II &\lesssim \int_{N_s} \left|\left[u - ve^{\overline{\varphi}(\psi-\varphi)}\right] f(\varphi)\right|^q dG_q \\ &\qquad + \int_{N_s} \left|v\left[e^{\overline{\varphi}(\psi-\varphi)} f(\varphi) - f(\psi)\right]\right|^q dG_q \\ &=: II_1 + II_2.\end{aligned}$$

By the same argument as above, we obtain

$$II_1 = \int_{\mathbb{C}} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^q \, d\sigma_{q,s}^{(\varphi,u;\psi,v)}(z) \lesssim \int_{\mathbb{C}} |f|^p \left(\widehat{\sigma_{q,s}^{(\varphi,u;\psi,v)}}\right)_r dG_q \tag{4.8}$$

with hidden constant depending only on allowed parameters.

To estimate II_2, we note by Theorem 3.6 (with $p = q$ and $\beta = \frac{1}{2}$)

$$\begin{aligned}&\left|e^{\overline{\varphi}(z)(\psi(z)-\varphi(z))} f(\varphi(z)) - f(\psi(z))\right|^q \\ &\qquad \lesssim \left|d(z)\left[e^{\overline{\varphi}(z)(\psi(z)-\varphi(z))}\right] e^{\frac{1}{2}|\varphi(z)|^2}\right|^q \int_{D_r(\varphi(z))} |f(w)|^q \, dG_q(w)\end{aligned}$$

for $z \in N_s$; the assumption $r > s$ is used in this estimate. Now, integrating over N_s both sides of the above against the measure $|v(z)|^q \, dG_q(z)$ and setting

$$W(z) := d(z)v(z)\left[e^{\overline{\varphi}(z)(\psi(z)-\varphi(z))}\right]$$

for short, we obtain

$$\begin{aligned} II_2 &\lesssim \int_{N_s} \left\{ \int_{D_r(\varphi(z))} |f(w)|^q \, dG_q(w) \right\} \left| W(z) e^{\frac{1}{2}|\varphi(z)|^2} \right|^q \, dG_q(z) \\ &= \int_{\mathbb{C}} |f(w)|^q \left\{ \int_{N_s \cap \varphi^{-1}(D_r(w))} \left| W(z) e^{\frac{1}{2}|\varphi(z)|^2} \right|^q \, dG_q(z) \right\} dG_q(w) \\ &\le \int_{\mathbb{C}} |f(w)|^q \Big(\widehat{\sigma_{q,s}^{(\varphi,u;\psi,v)}} \Big)_r(w) \, dG_q(w). \end{aligned} \tag{4.9}$$

Finally, we see that the estimates in (4.7), (4.8) and (4.9) altogether implies the asserted inequality in the first part of the lemma. The proof is complete. □

Having obtained two key lemmas Lemma 4.2 and Lemma 4.3, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The implication (a) $\Longrightarrow$ (b) holds by the boundedness (see [14, Theorem 2.10]) of the embedding $\mathcal{F}^p \hookrightarrow \mathcal{F}^q$. Since $\|k_a\|_p = 1$ and $k_a \to 0$ as $|a| \to \infty$ uniformly on compact subsets of $\mathbb{C}$, we see that the implication (b) $\Longrightarrow$ (c) holds. The implication (c) $\Longrightarrow$ (d) is a consequence of Lemma 4.2 and Theorem 2.1. The implication (d) $\Longrightarrow$ (e) is trivial. The implication (e) $\Longrightarrow$ (a) for boundedness is a consequence of Lemma 4.3 and Theorem 2.1 by symmetry.

It remains to verify the implication (e) $\Longrightarrow$ (a) for compactness. Put

$$\mu := \omega_{q,s} + \sigma_{q,s}^{(\varphi,u;\psi,v)}$$

for simplicity. By symmetry we may assume that μ is a compact 1-Carleson measure.

Consider an arbitrary sequence $\{f_j\}$ in $\mathcal{F}^q$ such that $\|f_j\|_q \le 1$ and $f_j \to 0$ uniformly on compact subsets of $\mathbb{C}$. We need to show

$$Tf_j \to 0 \quad \text{in} \quad \mathcal{L}^q \tag{4.10}$$

as $j \to \infty$. Fix $r > s > 0$. We have by Lemma 4.3

$$\|Tf_j\|_q^q \lesssim \int_{\mathbb{C}} |f_j|^q \widehat{\mu}_r \, dG_q = \int_{D_R(0)} + \int_{\mathbb{C}\setminus D_R(0)} =: I_j + II_j$$

for all j and $R > 0$. Since $f_j \to 0$ uniformly on $D_R(0)$ and $\|\widehat{\mu}_r\|_{L^\infty} < \infty$, we note $I_j \to 0$ as $j \to \infty$. Meanwhile, since $\|f_j\|_q \le 1$ for all j, we note

$$II_j \le \sup_{\mathbb{C}\setminus D_R(0)} \widehat{\mu}_r$$

for all j. It follows that

$$\limsup_{j\to\infty} \|Tf_j\|_q^q \lesssim \sup_{\mathbb{C}\setminus D_R(0)} \widehat{\mu}_r$$

and this holds for arbitrary $R > 0$. One may check that the constant hidden above is independent of R. Now, since μ is a compact 1-Carleson measure by assumption, taking the limit as $R \to \infty$, we conclude (4.10), as desired. So, the implication (e) $\Longrightarrow$ (a) holds for compactness. The proof is complete. □

Remark 4.4. Closely examining the proof of Theorem 1.1, one may see by the norm estimate in Theorem 2.1 that the quantities

$$\|T\|^q_{\mathscr{F}^p \to \mathscr{L}^q} \quad \text{and} \quad \|(\widehat{\omega_{q,s} + \sigma_{q,s}})_r\|_{L^\infty}$$

are comparable to each other.

Corollary 4.7 below is the content of [12, Theorem 1.4]. Here, applying our Carleson measure characterization in Theorem 1.1, we provide a new proof. Before proceeding, we note the following consequence of Theorem 1.1 (with $v \equiv 0$).

Corollary 4.5. *Let $0 < p \le q < \infty$. Let $\varphi, u \in \mathcal{O}(\mathbb{C})$. Then $C_{\varphi,u} : \mathscr{F}^p \to \mathscr{L}^q$ is bounded (compact, resp.) if and only if $(\left|ue^{\frac{1}{2}|\varphi|^2}\right|^q dG_q) \circ \varphi^{-1}$ is a (compact, resp.) 1-Carleson measure.*

The next lemma is a consequence of [12, Theorem 1.3 and Theorem 1.4]. Here, we include a direct proof for convenience of readers.

Lemma 4.6. *For $\varphi, \psi, u, v \in \mathcal{O}(\mathbb{C})$ with $uv \not\equiv 0$, assume that $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded for some $0 < p, q < \infty$. Then $\varphi - \psi$ is either a constant or a linear polynomial.*

Proof. Assume that $\varphi - \psi$ is not a constant. We will show that $\varphi - \psi$ is a linear polynomial. To begin with, we note from Lemma 3.1

$$|Tk_w(z)|e^{-\frac{1}{2}|z|^2} \lesssim \|Tk_w\|_q \le \|T\|_{\mathscr{F}^p \to \mathscr{L}^q}.$$

for all $z, w \in \mathbb{C}$. With this in mind, we take $w = \varphi(z)$ to obtain by (1.2) and (1.3)

$$\begin{aligned} |Tk_{\varphi(z)}(z)| &= e^{\frac{1}{2}|\varphi(z)|^2} \left| u(z) - v(z)e^{-|\varphi(z)|^2 + \overline{\varphi(z)}\psi(z)} \right| \\ &= \left| u(z)e^{\frac{|\varphi(z)|^2}{2}} - v(z)e^{\overline{\varphi(z)}\psi(z) - \frac{1}{2}|\varphi(z)|^2} \right| \\ &\ge |u(z)|e^{\frac{1}{2}|\varphi(z)|^2} - |v(z)|e^{\frac{1}{2}(|\psi(z)|^2 - d^2(z))}; \end{aligned}$$

recall $d = |\varphi - \psi|$. Thus, setting

$$X(z) := |u(z)|e^{\frac{1}{2}(|\varphi(z)|^2 - |z|^2)} \quad \text{and} \quad Y(z) := |v(z)|e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)},$$

we obtain

$$X - Ye^{-\frac{1}{2}d^2} \lesssim \|T\|_{\mathscr{F}^p \to \mathscr{L}^q}.$$

So, we obtain by symmetry

$$(X + Y)\big(1 - e^{-\frac{1}{2}d^2}\big) \lesssim \|T\|_{\mathscr{F}^p \to \mathscr{L}^q}$$

and, hence by the elementary inequality $1 - e^{-x} \ge \frac{x}{1+x} \ge \frac{x}{5}e^{-\frac{x}{5}}$,

$$d^2 e^{-\frac{1}{10}d^2}(X + Y) \lesssim \|T\|_{\mathscr{F}^p \to \mathscr{L}^q} \quad \text{on } \mathbb{C}.$$

Now, since $4XY \le (X + Y)^2$ and $d^2 \le 2(|\varphi|^2 + |\psi|^2)$, we obtain

$$\begin{aligned} |uv|d^4 e^{\frac{1}{4}d^2 - \frac{1}{5}d^2 - |z|^2} &\le |uv|d^4 e^{\frac{1}{2}(|\varphi|^2 + |\psi|^2) - \frac{1}{5}d^2 - |z|^2} \\ &= d^4 e^{-\frac{1}{5}d^2} XY \\ &\lesssim [d^2 e^{-\frac{1}{10}d^2}(X + Y)]^2 \\ &\lesssim \|T\|^2_{\mathscr{F}^p \to \mathscr{L}^q}. \end{aligned}$$

Accordingly, setting $H := \frac{\varphi - \psi}{\sqrt{20}}$ and $G := z^{-m}uv(\varphi - \psi)^4$ where m is the multiplicity of zero of $uv(\varphi - \psi)^4$ at 0, we may take logarithm to obtain

$$\log|G| + \log|H|^2 + m\log|z| - |z|^2 \le C \quad \text{on} \quad \mathbb{C}$$

for some constant $C > 0$ independent of z. Integrating over the circle of radius $r > 0$, we obtain by subharmonicity of $\log|G|$ and Parseval's Identity

$$\log|G(0)| + \sum_{k=0}^{\infty} \left| \frac{H^{(k)}(0)}{k!} \right|^2 r^{2k} + m \log r - r^2 \leq C.$$

Since this holds for all $r > 0$, we see that $H^{(k)}(0) = 0$ for $k \geq 2$. Thus, we conclude that $\varphi - \psi$ is a linear polynomial. The proof is complete. □

Applying Theorem 1.1, we provide a new proof of the next corollary.

Corollary 4.7 ([12]). *Let $0 < p \leq q < \infty$ and $\varphi, \psi, u, v \in \mathcal{O}(\mathbb{C})$. Assume $\varphi \not\equiv \psi$. Then $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded (compact, resp.) if and only if $C_{\varphi,u}$ and $C_{\psi,v} : \mathscr{F}^p \to \mathscr{L}^q$ are individually bounded (compact, resp.).*

Proof. To avoid triviality assume $uv \not\equiv 0$. The sufficiency being trivial, we only need to prove the necessity. So, assuming that $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded (compact, resp.), we need to show $C_{\varphi,u} : \mathscr{F}^p \to \mathscr{L}^q$ is bounded (compact, resp.). Fixing $s > 0$, note from Theorem 1.1 and Corollary 4.5 that $C_{\varphi,u\chi_{A_s}} : \mathscr{F}^p \to \mathscr{L}^q$ is bounded (compact, resp.). We will complete the proof by showing that $C_{\varphi,u\chi_{N_s}} : \mathscr{F}^p \to \mathscr{L}^q$ is compact, or equivalently by Corollary 4.5,

$$\mu := \left(\left| \chi_{N_s} u e^{\frac{1}{2}|\varphi|^2} \right|^q dG_q \right) \circ \varphi^{-1} \tag{4.11}$$

is a compact 1-Carleson measure.

Note from Lemma 4.6 that $\varphi - \psi$ is either a nonzero constant or a linear polynomial. In the former case, we may take a small s for which $N_s = \emptyset$. So, we may assume that $\varphi - \psi$ is a linear polynomial and hence that N_s is contained in a compact set, say K. Pick $t > 0$. Note

$$\begin{aligned} \widetilde{\mu}_t(a) :&= \int_{\mathbb{C}} e^{-\frac{t}{2}|w-a|^2} \, d\mu(w) \\ &= \frac{q}{2\pi} \int_{N_s} e^{-\frac{t}{2}|\varphi(z)-a|^2} \left| u(z) e^{\frac{1}{2}(|\varphi(z)|^2 - |z|^2)} \right|^q dA(z); \end{aligned}$$

recall that $\widetilde{\mu}_t$ denotes the t-Berezin transform of μ. Now, since

$$|\varphi(z) - a| \geq |a| - |\varphi(z)| \geq \frac{1}{2}|a|, \qquad z \in K,$$

for all a sufficiently large, we obtain

$$\widetilde{\mu}_t(a) \lesssim e^{-\frac{t}{8}|a|^2} \int_K \left| u(z) e^{\frac{1}{2}(|\varphi(z)|^2 - |z|^2)} \right|^q dA(z)$$

for all a sufficiently large. Since $|u| e^{\frac{1}{2}|\varphi|^2}$ is bounded on K, this yields

$$\widetilde{\mu}_t(a) \leq C e^{-\frac{t}{8}|a|^2}, \qquad a \in \mathbb{C}, \tag{4.12}$$

for some constant $C > 0$ independent of a. It follows from Theorem 2.1 that μ is a compact 1-Carleson measure, as desired. The proof is complete. □

5. The Case $q < p$

In this section we prove Theorem 1.2 and, as an application, provide a new proof of the rigidity phenomenon for the range $q < p$.

The implications (b) $\Longrightarrow$ (a) and (c) $\Longrightarrow$ (d) are clear. We will complete the proof by establishing the implications

$$\text{(d)} \Longrightarrow \text{(a)+(b)} \quad \text{and} \quad \text{(a)} \Longrightarrow \text{(c)}.$$

Proof of (d) $\Longrightarrow$ (a)+(b) in Theorem 1.2. Assume (d). Fix $r > s$. By symmetry we may assume that

$$\mu := \omega_{q,s} + \sigma_{q,s}^{(\varphi,u;\psi,v)}$$

is a λ-Carleson measure so that

$$\widehat{\mu}_r \in L^{\frac{p}{p-q}}(\mathbb{C}, A) \tag{5.1}$$

by Theorem 2.2.

We see from Lemma 4.3

$$\|\!|Tf|\!\|_q^q \lesssim \int_{\mathbb{C}} |f|^q \widehat{\mu}_r \, dG_q \tag{5.2}$$

for $f \in \mathcal{O}(\mathbb{C})$ with suppressed constant depending only on q, r and s. Since $\lambda > 1$, we now apply Hölder's Inequality to conclude that $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded with norm estimate

$$\|T\|_{\mathscr{F}^p \to \mathscr{L}^q}^q \le C \|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}}$$

for some constant $C = C(p, r, s) > 0$.

We now proceed to the proof of compactness. Consider an arbitrary sequence $\{f_j\}$ in $\mathscr{F}^p$ such that $\sup_j \|\!|f_j|\!\|_p \le 1$ and $f_j \to 0$ uniformly on compact subsets of $\mathbb{C}$. We need to show

$$Tf_j \to 0 \quad \text{in} \quad \mathscr{L}^q \tag{5.3}$$

as $j \to \infty$.

Let $R > 0$. We have by (5.2)

$$\int_{\mathbf{B}} |Tf_j|^q \, dG_q \lesssim \int_{D_R(0)} + \int_{\mathbb{C}\setminus D_R(0)} |f_j|^q \widehat{\mu}_r \, dG_q =: I_{j,R} + II_{j,R}$$

for all j. We note

$$I_{j,R} \le \left(\sup_{D_R(0)} |f_j|^q \right) \int_{D_R(0)} \widehat{\mu}_r \, dG_q$$

and, moreover, the above integral is finite by Hölder's Inequality and (5.1). Since $f_j \to 0$ as $j \to \infty$ uniformly on $D_R(0)$, we see that

$$I_{j,R} \to 0 \quad \text{(with } R \text{ fixed)}$$

as $j \to \infty$. For $II_{j,R}$, applying Hölder's Inequality, we obtain

$$II_{j,R} \lesssim \|\!|f_j|\!\|_p^q \, \|\chi_{\mathbb{C}\setminus D_R(0)} \widehat{\mu}_r\|_{L^{\frac{p}{p-q}}} \le \|\chi_{\mathbb{C}\setminus D_R(0)} \widehat{\mu}_r\|_{L^{\frac{p}{p-q}}}$$

for all j. Accordingly, we obtain

$$\limsup_{j\to\infty} \int_{\mathbb{C}} |Tf_j|^q \, dG_q \lesssim \|\chi_{\mathbb{C}\setminus D_R(0)} \widehat{\mu}_r\|_{L^{\frac{p}{p-q}}};$$

the constant suppressed in this estimate is independent of R. Note from (5.1) that the right hand side of the above tends to 0 as $R \to \infty$. Thus, taking the limit as $R \to \infty$, we conclude (5.3), as required. The proof is complete. □

For the proof of the implication (a) $\Longrightarrow$ (c), we first recall the well-known Khinchine's Inequality. The sequence of Rademacher functions $\{R_j\}$ is defined by

$$R_j(x) := \text{sgn}\{\sin(2^j \pi x)\}, \qquad 0 \le x \le 1,$$

for positive integers j. For a proof of the following Khinchine's Inequality, we refer to, for example, [8, Appendix C].

Lemma 5.1 (Khinchine's Inequality)**.** *Given $0 < p < \infty$, there is a constant $C = C(p) > 0$ such that*

$$C^{-1} \left(\sum_{j=1}^{\infty} |c_j|^2 \right)^{p/2} \le \int_0^1 \left| \sum_{j=1}^{\infty} c_j R_j(x) \right|^p dx \le C \left(\sum_{j=1}^{\infty} |c_j|^2 \right)^{p/2}$$

for all $\{c_j\} \in \ell^2$.

In what follows we use the notation

$$r\mathbb{Z}^2 := \{r(m + ni) : m, n \in \mathbb{Z}\}$$

for $r > 0$. Here, and elsewhere, $i = \sqrt{-1}$ and $\mathbb{Z}$ denotes the set of all integers. Note that $r\mathbb{Z}^2$ is an r-lattice.

Lemma 5.2. *Let $0 < p < \infty$, $r > 0$ and $t > 0$. Let $\{a_j\}$ be an enumeration of all points in $r\mathbb{Z}^2$. Let $\{c_j\} \in \ell^p$ and $\{\xi_j\}$ be a sequence such that $\xi_j \in D_t(a_j)$ for each j. Put*

$$f := \sum_{j=1}^{\infty} c_j k_{\xi_j}. \tag{5.4}$$

Then $f \in \mathscr{F}^p$ and

$$\|f\|_p \le C \|\{c_j\}\|_{\ell^p}$$

for some constant $C = C(p, r, t) > 0$.

Proof. The proof here is a slight modification of the first part of the proof of [14, Theorem 2.34].

First, consider the case $0 < p \le 1$. In this case, we have

$$|f|^p \le \sum_{j=1}^{\infty} |c_j|^p |k_{\xi_j}|^p.$$

Thus, integrating both sides over $\mathbb{C}$ against the measure dG_p, we obtain by (1.1)

$$\|f\|_p^p \le \sum_{j=1}^{\infty} |c_j|^p.$$

Next, consider the case $1 < p < \infty$. The proof will be completed by interpolation between $p = 1$ and $p = \infty$. The case $p = 1$ is done above. To consider the case $p = \infty$, we introduce the space $\mathscr{F}^\infty$ consisting of all $g \in \mathcal{O}(\mathbb{C})$ such that

$$\|g\|_\infty := \sup_{z \in \mathbb{C}} \left| g(z) e^{-\frac{1}{2}|z|^2} \right| < \infty.$$

Clearly, $\mathscr{F}^\infty$ is a Banach space. Assume that f is given by (5.4) with $\{c_j\} \in \ell^\infty$, we have by (1.2)

$$\|\!|f|\!\|_\infty \le \|\{c_j\}\|_{\ell^\infty} \left(\sup_{z\in\mathbb{C}} \sum_{j=1}^\infty e^{-\frac{1}{2}|z-\xi_j|^2} \right).$$

Meanwhile, since

$$|z-a_j| \le |z-\xi_j| + |\xi_j - a_j| < |z-\xi_j| + t,$$

we have

$$|z-a_j|^2 \le 2(|z-\xi_j|^2 + t^2)$$

and hence

$$\sum_{j=1}^\infty e^{-\frac{1}{2}|z-\xi_j|^2} \le e^{\frac{1}{2}t^2} \sum_{j=1}^\infty e^{-\frac{1}{4}|z-a_j|^2} \le Ce^{\frac{1}{2}t^2}$$

for all $z \in \mathbb{C}$ and for some constant $C = C(r) > 0$. The last inequality holds by [14, Lemma1.12]. Accordingly, we conclude

$$\|\!|f|\!\|_\infty \le C\|\{c_j\}\|_{\ell^\infty}$$

for some constant $C = C(r,t) > 0$. So, as mentioned above, the lemma also holds for $1 < p < \infty$ by interpolation; see [14, Section 2.4]. The proof is complete. □

We are now ready to prove Theorem 1.2.

Proof of (a) $\Longrightarrow$ (c) in Theorem 1.2. Assume (a) and put

$$\mu := \omega_{q,s}^{(\varphi,u)} + \sigma_{q,s}^{(\varphi,u;\psi,v)}$$

for simplicity. Pick $r > 0$ and fix an enumeration $\{a_j\}$ of all points in $\frac{r}{2}\mathbb{Z}^2$. By Theorem 2.2 and symmetry it suffices to show

$$\|\{\widehat{\mu}_r(a_j)\}\|_{\ell^{\frac{p}{p-q}}} \le C\|T\|^q_{\mathscr{F}^p\to\mathscr{L}^q} \tag{5.5}$$

for some constant $C = C(p,q,r,s) > 0$. Pick $t = t(r) > r$. Let $\epsilon > 0$ and choose a sequence $\{\epsilon_k\}$ of positive numbers such that $\|\{\epsilon_j\}\|_{\ell^{\frac{p}{p-q}}} \le \epsilon$. By Lemma 4.2 we can find $\xi_j \in D_t(a_j)$ such that

$$\widehat{\mu}_r(a_j) \lesssim \epsilon_j + \int_{\varphi^{-1}(D_r(a_j))} |Tk_{\xi_j}|^q \, dG_q \tag{5.6}$$

for all j. We will establish the estimate

$$\left\| \left\{ \int_{\varphi^{-1}(D_r(a_j))} |Tk_{\xi_j}|^q \, dG_q \right\} \right\|_{\ell^{\frac{p}{p-q}}} \lesssim \|T\|^q_{\mathscr{F}^p\to\mathscr{L}^q}. \tag{5.7}$$

With this estimate granted we see from (5.6) that

$$\|\{\widehat{\mu}_r(a_j)\}\|_{\ell^{\frac{p}{p-q}}} \lesssim \epsilon + \|T\|^q_{\mathscr{F}^p\to\mathscr{L}^q}.$$

One may check that the constant suppressed above depends only on allowed parameters. Since ϵ is arbitrary, this yields (5.5), as desired.

We now proceed to the proof of (5.7). Our proof will be based on the duality $(\ell^\lambda)^* = \ell^{\frac{p}{p-q}}$. So, pick an arbitrary $\{\eta_j\} \in \ell^\lambda$ with $\|\{\eta_j\}\|_{\ell^\lambda} = 1$. Given $x \in [0,1]$, put

$$f := \sum_{j=1}^\infty R_j(x)|\eta_j|^{\frac{1}{q}} k_{\xi_j}$$

where R_j is the j-th Rademacher function. We note $f \in \mathscr{F}^p$ by Lemma 5.2 with $\|\!|f|\!\|_p$ bounded by some constant depending only on p and r. Now, since $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded by assumption, we obtain

$$\int_{\mathbb{C}} \left| \sum_{j=1}^{\infty} R_j(x) |\eta_j|^{\frac{1}{q}} T k_{\xi_j} \right|^q \, dG_q \lesssim \|T\|^q_{\mathscr{F}^p \to \mathscr{L}^q}$$

for all $x \in [0,1]$. Integrating both sides over $[0,1]$ against the measure dx and interchanging the order of integrations, we obtain by Khinchine's Inequality

$$\int_{\mathbb{C}} \left(\sum_j |\eta_j|^{\frac{2}{q}} |T k_{\xi_j}|^2 \right)^{\frac{q}{2}} \, dG_q \lesssim \|T\|^q_{\mathscr{F}^p \to \mathscr{L}^q} \tag{5.8}$$

with hidden constant depending only on allowed parameters.

We now pause to show

$$\sum_{j=1}^{\infty} |\eta_j| \left| T k_{\xi_j} \right|^q \chi_{D_r(a_j)}(\varphi) \lesssim \left(\sum_{j=1}^{\infty} |\eta_j|^{\frac{2}{q}} |T k_{\xi_j}|^2 \right)^{\frac{q}{2}} \tag{5.9}$$

with hidden constant depending only on allowed parameters. To see this we consider two cases separately: (i) $q \geq 2$ and (ii) $0 < q < 2$. In Case (i), this is clear, because $\frac{q}{2} \geq 1$. In Case (ii), setting

$$M := \sup_{\mathbb{C}} \sum_{j=1}^{\infty} \chi_{D_r(a_j)} < \infty,$$

we have by Hölder's Inequality

$$\sum_{j=1}^{\infty} |\eta_j| \left| T k_{\xi_j} \right|^q \chi_{D_r(a_j)}(\varphi) \leq \left(\sum_{j=1}^{\infty} |\eta_j|^{\frac{2}{q}} \left| T k_{\xi_j} \right|^2 \right)^{\frac{q}{2}} M^{1-\frac{q}{2}}$$

and hence (5.9) holds. Now, having (5.8) and (5.9), we obtain

$$\sum_{j=1}^{\infty} |\eta_j| \int_{\varphi^{-1}(D_r(a_j))} \left| T k_{\xi_j} \right|^q \, dG_q \lesssim \|T\|^q_{\mathscr{F}^p \to \mathscr{L}^q}$$

with suppressed constant depending only on allowed parameters. This yields (5.7) by duality, completing the proof. □

Remark 5.3. Closely examining the proof of Theorem 1.2, one may see by the norm estimate in Theorem 2.2 that the quantities

$$\|T\|^q_{\mathscr{F}^p \to \mathscr{L}^q} \quad \text{and} \quad \left\| \widehat{(\omega_{q,s} + \sigma_{q,s})_r} \right\|_{L^{\frac{p}{p-q}}}$$

are comparable to each other.

Applying Theorem 1.2, we provide a new proof of the next corollary. Recall from Theorem 1.2 that boundedness and compactness are the same when $q < p$.

Corollary 5.4 ([13])**.** *Let $0 < q < p < \infty$ and $\varphi, \psi, u, v \in \mathcal{O}(\mathbb{C})$. Then $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded if and only if $C_{\varphi,u}$ and $C_{\psi,v} : \mathscr{F}^p \to \mathscr{L}^q$ are individually bounded.*

Proof. To avoid triviality assume $uv \not\equiv 0$. We continue using the notation used in the proof of Corollary 4.7. As in the proof of Corollary 4.7, assuming that $T : \mathscr{F}^p \to \mathscr{L}^q$ is bounded, we only need to prove that μ is $\frac{p}{q}$-Carleson measure, which is clear from (4.12) and Theorem 2.2. The proof is complete. □

6. EXAMPLES

In this section we provide various examples related with our results. Throughout the section we use the notation

$$x := \operatorname{Re} z \quad \text{and} \quad y := \operatorname{Im} z$$

for $z \in \mathbb{C}$.

• Compact differences via cancellation

We provide a couple of examples showing that differences are compact, while individual operators are not bounded. These examples reveal that good cancellation between two unbounded operators may result in compactness of their difference.

Example 6.1. *Let*

$$\varphi(z) := 2z, \quad \psi(z) := -2z$$

and consider the weights

$$u(z) = v(z) := \frac{\chi_E(z)}{|z|} \quad \textit{where} \quad E := D_1(0) \setminus \{0\}.$$

Then the following assertions hold:

(a) *Each of the operators C_φ, C_ψ, $C_{\varphi,u}$ and $C_{\psi,v}$ does not map $\mathscr{F}^2$ into $\mathscr{L}^2$;*
(b) *$T : \mathscr{F}^2 \to \mathscr{L}^q$ is compact for any $0 < q < \infty$.*

Proof. It is easily seen from Corollary 4.5 (or see [6]) that C_φ and C_ψ are not bounded on $\mathscr{F}^2$. Thus they both do not map $\mathscr{F}^2$ into itself by the Closed Graph Theorem. Note $C_{\varphi,u}1 = C_{\psi,v}1 = u \notin \mathscr{L}^2$. So, (a) holds.

We now prove (b). For $a \in \mathbb{C}$, we have

$$k_a(2z) - k_a(-2z) = e^{2\overline{a}z - \frac{1}{2}|a|^2} \left(1 - e^{-4\overline{a}z}\right).$$

Thus, using the elementary inequality $|1 - e^w| \le |w|e^{|w|}$, we obtain

$$\left|Tk_a(z)\right| \le e^{2|\overline{a}z| - \frac{1}{2}|a|^2} |4\overline{a}| e^{|4\overline{a}z|} \le 4|a| e^{6|a| - \frac{1}{2}|a|^2} \lesssim e^{-\frac{1}{3}|a|^2}$$

for all $z \in E$ and thus, given $0 < q < \infty$,

$$\left\|\!\left|Tk_a\right|\!\right\|_q \lesssim e^{-\frac{1}{3}|a|^2}$$

for all a. Furthermore, given $s > 0$ and $r > 0$, we have by Lemma 4.2

$$\widehat{\mu}_r(a) \lesssim \sup_{\xi \in D_t(a)} \left\|\!\left|Tk_\xi\right|\!\right\|_q^q \lesssim \sup_{\xi \in D_t(a)} e^{-\frac{q}{3}|\xi|^2} \lesssim e^{-\frac{q}{6}|a|^2}$$

for all a where $\mu := \omega_{q,s} + \sigma_{q,s}$. The constants suppressed in these estimates are independent of a. Thus we see that (b) holds by Theorem 1.1 and Theorem 1.2. The proof is complete. □

Example 6.2. *Let*

$$\varphi(z) := 2z, \quad \psi(z) := 2z + e^{z^3}$$

and consider the weights

$$u = v := \chi_S \quad \textit{where} \quad S := \left\{ z \in \mathbb{C} : |\operatorname{Arg} z - \pi| \le \frac{\pi}{9} \right\}.$$

Then the following assertions hold for $0 < p < \infty$*:*

(a) $T : \mathscr{F}^p \to \mathscr{L}^q$ *is compact for any* $0 < q < \infty$*;*
(b) $\lim_{t\to\infty} \left\| C_{\varphi,u} k_{-t} \right\|_p = \infty$ *and* $\lim_{t\to\infty} \left\| C_{\psi,v} k_{-t} \right\|_p = \infty$.

Proof. First, we prove (a). Let $a \in \mathbb{C}$ and $z = -|z|e^{i\theta} \in S$ where $|\theta| \le \frac{\pi}{9}$. Note

$$\operatorname{Re}(z^3) = -|z|^3 \cos 3\theta \le -|z|^3 \cos\frac{\pi}{3} = -\frac{|z|^3}{2}$$

and hence

$$\left|1 - e^{\overline{a}e^{z^3}}\right| \le \left|\overline{a}e^{z^3}\right| e^{|\overline{a}e^{z^3}|} \le |a| e^{-\frac{1}{2}|z|^3} e^{|a|e^{-\frac{1}{2}|z|^3}}. \tag{6.1}$$

For $z \in S$, we also note by (1.2) and (1.3)

$$|Tk_a(z)| = |k_a(2z)| \left|1 - e^{\overline{a}e^{z^3}}\right| = e^{\frac{1}{2}(4|z|^2 - |2z-a|^2)} \left|1 - e^{\overline{a}e^{z^3}}\right|.$$

Thus, given $0 < q < \infty$, setting

$$S_a := \left\{ z \in S : |z| \le \frac{|a|}{4} \right\}$$

and integrating over S against the measure $dG_q(z) = \frac{q}{2\pi} e^{-\frac{q}{2}|z|^2} dA(z)$, we obtain

$$\begin{aligned} \frac{2\pi}{q} \left\| Tk_a \right\|_q^q &= \int_S e^{\frac{q}{2}(3|z|^2 - |2z-a|^2)} \left|1 - e^{\overline{a}e^{z^3}}\right|^q dA(z) \\ &= \int_{S_a} + \int_{S\setminus S_a} =: I + II. \end{aligned}$$

We first estimate I. For $z \in S_a$, since $\left|1 - e^{\overline{a}e^{z^3}}\right| \le |a|e^{|a|}$ by (6.1) and $|2z - a| \ge \frac{1}{2}|a|$, we obtain

$$\begin{aligned} I &\le |a|^q e^{\frac{q}{8}(8|a| - |a|^2)} \int_{S_a} e^{\frac{3q}{2}|z|^2} dA(z) \\ &= |a|^q e^{\frac{q}{8}(8|a| - |a|^2)} \cdot \frac{2\pi}{27q} \left(e^{\frac{3q}{32}|a|^2} - 1 \right) \\ &\lesssim e^{-\frac{q}{33}|a|^2} \end{aligned} \tag{6.2}$$

for all a.

We now estimate II. Since $|a| \le 4|z|$, we note from (6.1) again

$$\left|1 - e^{\overline{a}e^{z^3}}\right| \le 4|z| e^{-\frac{1}{2}|z|^3} \lesssim e^{-\frac{1}{3}|z|^3}$$

and hence

$$\left|1 - e^{\overline{a}e^{z^3}}\right|^q e^{\frac{3q}{2}|z|^2} \lesssim e^{\frac{q}{2}(3|z|^2 - \frac{2}{3}|z|^3)} \lesssim e^{-\frac{q}{6}|z|^3} \lesssim e^{-\frac{q}{33}|a|^2} e^{-\frac{q}{12}|z|^3}.$$

It follows that

$$II \lesssim e^{-\frac{q}{33}|a|^2}$$

for all a. This, together with (6.2), we obtain

$$\|Tk_a\|_q \lesssim e^{-\frac{1}{33}|a|^2} \tag{6.3}$$

for all a with hidden constant independent of a. So, we conclude (a) as in the proof of Example 6.1(b).

Now, we prove (b). By (6.3) (with $p = q$) it is enough to show the first part only. Put

$$R := \Big\{ z \in \mathbb{C} : x \le -1 \ \text{ and } \ |y| \le \tan\frac{\pi}{9} \Big\} \subset S.$$

For $t > 0$, we note

$$|k_{-t}(2z)|^2 e^{-|z|^2} = \big|e^{-4tz-t^2}\big| e^{-|z|^2} = e^{-(t^2+4tx+x^2+y^2)} = e^{3t^2-(2t+x)^2-y^2}$$

and hence

$$\begin{aligned} \frac{2\pi}{p} \|C_{\varphi,u} k_{-t}\|_p^p &\ge \int_R \Big| k_{-t}(2z) e^{-\frac{1}{2}|z|^2} \Big|^p \, dA(z) \\ &= 2e^{\frac{3p}{2}t^2} \int_0^{\tan\frac{\pi}{9}} \left\{ \int_{-\infty}^{-1} e^{-\frac{p}{2}(2t+x)^2} \, dx \right\} e^{-\frac{p}{2}y^2} \, dy \\ &= Ce^{\frac{3p}{2}t^2} \int_{-\infty}^{2t-1} e^{-\frac{p}{2}x^2} \, dx \end{aligned}$$

where $C := 2\int_0^{\tan\frac{\pi}{9}} e^{-\frac{p}{2}y^2} \, dy$. This implies the first equality of (b), as desired. The proof is complete. □

• Failure of the Bergman-like approach

Closely looking at the proof of Lemma 4.3, one may see that the pullback measure $\sigma_{q,s}^{(\varphi,u;\psi,v)}$ is originated from the decomposition

$$uf(\varphi) - vf(\psi) = \big[u - ve^{\overline{\varphi}(\psi-\varphi)}\big] f(\varphi) + v[e^{\overline{\varphi}(\psi-\varphi)} f(\varphi) - f(\psi)].$$

The key idea to this decomposition is of course to insert the factor $e^{\overline{\varphi}(\psi-\varphi)}$ intentionally. If one uses the plain decomposition

$$uf(\varphi) - vf(\psi) = (u - v) f(\varphi) + v[f(\varphi) - f(\psi)],$$

the resulting pullback measure will be something like

$$\tau_{q,s}^{(\varphi,u;\psi,v)} := \Big(\chi_{N_s} \Big| \big[|u - v| + |dv|\big] e^{\frac{1}{2}|\varphi|^2} \Big|^q \, dG_q \Big) \circ \varphi^{-1}.$$

In fact this approach is completely parallel to the one carried out for the Bergman setting over the disk; see [3]. At the early stage of the present research, we naturally expected that this approach would also work for the Fock setting, but failed. Here, we provide a couple of concrete examples to demonstrate the failure in both directions with measure $\omega_{q,s} + \tau_{q,s}^{(\varphi,u;\psi,v)}$.

Before proceeding, we make some preliminary observations which will be used in both examples. Let

$$\varphi(z) := z, \quad \psi(z) := z + e^z \tag{6.4}$$

and

$$\Omega := \Big\{ z \in \mathbb{C} : x < 0 \ \text{ and } \ e^{|x|+4} < y < e^{|x|+6} \Big\}. \tag{6.5}$$

For $z \in \Omega$, note that $|z| \to \infty$ if and only if $x \to -\infty$. Since

$$d(z) = e^x \to 0 \quad \text{as} \ \ |z| \to \infty \ \ \text{with} \ \ z \in \Omega, \tag{6.6}$$

we note that A_s is contained in a compact set. In the subsequent examples, the weights u and v will be bounded on compact sets. Given $s > 0$ we thus see that $\omega_{q,s}$ is a finite measure supported on a compact set and hence is a compact 1-Carleson measure. Finally, we note $\overline{\varphi}(z)\big(\psi(z) - \varphi(z)\big) = \overline{z}e^z$ and

$$\mathrm{Re}\,(\overline{z}e^z) = e^x(x\cos y + y\sin y) \le -xe^x + e^6 \lesssim 1 \tag{6.7}$$

for all $z \in \Omega$. On the other hand, we note

$$\mathrm{Re}\,(\overline{z}e^z) \ge e^x(x + y\sin y) \ge xe^x + e^4 \sin y \tag{6.8}$$

for all $z \in \Omega$ with $\sin y \ge 0$.

First, we provide an example demonstrating "measure compact, but operator not".

Example 6.3. *Let φ, ψ, Ω be as in* (6.4) *and* (6.5)*, and consider the weights*

$$u = v := \chi_\Omega.$$

Then the following assertions hold for $s > 0$ and $0 < q < \infty$:

(a) *$C_{\varphi,u} : \mathscr{F}^q \to \mathscr{L}^q$ and $C_{\psi,v} : \mathscr{F}^q \to \mathscr{L}^q$ are bounded;*
(b) *$T : \mathscr{F}^q \to \mathscr{L}^q$ is not compact;*
(c) *$\omega_{q,s} + \tau_{q,s}^{(\varphi,u;\psi,v)}$ is a compact 1-Carleson measure;*

Proof. First, we prove (a). The boundedness of $C_{\varphi,u}$ is clear. For boundedness of $C_{\psi,v}$, we will apply Corollary 4.5. Namely, we will show that

$$\mu := \left(\left|ve^{\frac{1}{2}|\psi|^2}\right|^q \, dG_q\right) \circ \psi^{-1}$$

is a 1-Carleson measure.

Pick $r > 0$ and let $a \in \mathbb{C}$. Note

$$\widehat{\mu}_r(a) = \frac{q}{2\pi} \int_{\Omega \cap \psi^{-1}(D_r(a))} e^{\frac{q}{2}(|\psi(z)|^2 - |z|^2)} \, dA(z)$$

and, moreover, by (6.7)

$$\big||\psi(z)|^2 - |z|^2\big| = \big|e^{2x} + 2\mathrm{Re}\,(\overline{z}e^z)\big| \lesssim 1, \qquad z \in \Omega.$$

It follows that

$$\widehat{\mu}_r(a) \lesssim A[\Omega \cap \psi^{-1}(D_r(a))] \tag{6.9}$$

for all a. Meanwhile, for $z \in \Omega \cap \psi^{-1}(D_r(a))$, we have

$$|z - a| \le |z - \psi(z)| + |\psi(z) - a| \le e^x + r \le 1 + r$$

and thus $\Omega \cap \psi^{-1}(D_r(a)) \subset D_{r+1}(a)$. This, together with (6.9), shows that μ is a 1-Carleson measure, as desired.

Now, we prove (b). By Theorem 1.1 it suffices to show that

$$\begin{aligned} \nu :&= \left(\chi_{N_s}\left|u - ve^{\overline{\varphi}(\psi-\varphi)}e^{\frac{1}{2}|\varphi|^2}\right|^q dG_q\right) \circ \varphi^{-1} \\ &= \frac{q}{2\pi}\chi_{\Omega\cap N_s}\big|1 - e^{\overline{\varphi}(\psi-\varphi)}\big|^q dA \end{aligned}$$

is not a compact 1-Carleson measure. For $y_0 > 0$ to be specified later, pick $x_0 = x_0(y_0) < 0$ with

$$\sin(y_0) = 1 \quad \text{and} \quad y_0 = e^{|x_0|+5}$$

so that $a_0 =: x_0 + iy_0 \in \Omega$. Note

$$D_{\frac{\pi}{6}}(a_0) \subset \left\{ z \in \mathbb{C} : |x - x_0| < \frac{1}{2} \text{ and } |y - y_0| < \frac{1}{2} \right\} \subset \Omega$$

and, in addition,

$$\frac{1}{2} \le \sin y \le 1 \quad \text{and} \quad e^{|x|+4} < y, \qquad z \in D_{\frac{\pi}{6}}(a_0).$$

Thus, for $z \in D_{\frac{\pi}{6}}(a_0)$, we obtain by (6.8)

$$\begin{aligned} \operatorname{Re}\left[\overline{\varphi}(z)\big(\psi(z) - \varphi(z)\big)\right] &\ge \frac{1}{2}e^4 + \left(x_0 - \frac{1}{2}\right) e^{x_0 - \frac{1}{2}} \\ &= \frac{1}{2}e^4 - \left(|x_0| + \frac{1}{2}\right) e^{-(|x_0| + \frac{1}{2})}. \end{aligned}$$

Now, we fix a large y_0 (and hence $|x_0|$ also large) such that

$$\operatorname{Re}\left[\overline{\varphi}(z)\big(\psi(z) - \varphi(z)\big)\right] \ge 2, \qquad z \in D_{\frac{\pi}{6}}(a_0), \tag{6.10}$$

and hence

$$\left|1 - e^{\overline{\varphi}(\psi - \varphi)}\right| \ge \left|e^{\overline{\varphi}(\psi - \varphi)}\right| - 1 \ge 1 \quad \text{on } D_{\frac{\pi}{6}}(a_0).$$

By (6.6) we may further assume

$$D_{\frac{\pi}{6}}(a_0) \subset N_s.$$

Accordingly, we obtain

$$\widehat{\nu}_{\frac{\pi}{6}}(a_0) = \frac{q}{2\pi} \int_{D_{\frac{\pi}{6}}(a_0)} \left|1 - e^{\overline{\varphi}(\psi - \varphi)}\right|^q dA \ge \frac{q}{2\pi} \cdot \left(\frac{\pi}{6}\right)^2.$$

Since $|a_0|$ can be arbitrarily large, this shows that ν is not a compact 1-Carleson measure, as desired.

Finally, we prove (c). Recall from the preliminary observations that $\omega_{q,s}$ is already a compact 1-Carleson measure. So, since $u = v$, we need to prove that

$$\tau := \tau_{q,s}^{(\varphi,u;\psi,v)} = \left(\left| \chi_{\Omega \cap N_s} d e^{\frac{1}{2}|\varphi|^2} \right|^q \, dG_q \right) \circ \varphi^{-1} = \frac{q}{2\pi} \chi_{\Omega \cap N_s} d^q \, dA$$

is a compact 1-Carleson measure. Pick $r > 0$. Since $d \le 1$ on Ω, it is clear that $\widehat{\tau}_r$ is bounded on $\mathbb{C}$. Since $d(z) \to 0$ as $|z| \to \infty$ inside Ω, we note $\Omega \cap D_r(a) \subset N_s$ for all a with $|a|$ sufficiently large. For such a,

$$\frac{q}{2\pi} \widehat{\tau}_r(a) = \int_{\Omega \cap D_r(a)} e^{qx} \, dA(z) \le e^{q(r + \operatorname{Re} a)} A[\Omega \cap D_r(a)].$$

One may check that the rightmost quantity of the above tends to 0 as $|a| \to \infty$. Consequently, τ is a compact 1-Carleson measure. The proof is complete. □

Next, we provide an example demonstrating "operator compact, but measure not".

Example 6.4. *Let φ, ψ, Ω be as in* (6.4) *and* (6.5)*, and consider the weights*

$$u := e^{\overline{\varphi}(\psi - \varphi)} \chi_\Omega, \quad v := \chi_\Omega.$$

Then the following assertions hold for $s > 0$ and $0 < q < \infty$:

(a) *$C_{\varphi,u} : \mathscr{F}^q \to \mathscr{L}^q$ and $C_{\psi,v} : \mathscr{F}^q \to \mathscr{L}^q$ are bounded;*
(b) *$T : \mathscr{F}^q \to \mathscr{L}^q$ is compact;*
(c) *$\tau_{q,s}^{(\varphi,u;\psi,v)}$ is not a compact 1-Carleson measure.*

Proof. First, we prove (a). The boundedness of $C_{\psi,v}$ is contained in Example 6.3(a). Since $u(z) = e^{\overline{z}e^z}\chi_\Omega(z)$ is bounded on $\mathbb{C}$ by (6.7), it is clear that

$$\left(\left|ue^{\frac{1}{2}|\varphi|^2}\right|^q dG_q\right)\circ\varphi^{-1} = \frac{q}{2\pi}|u|^q\,dA$$

is a 1-Carleson measure. Thus, $C_{\varphi,u}$ is bounded by Corollary 4.5.

Now, we prove (b). Fix $s>0$ for the rest of the proof. Put

$$\mu := \sigma_{q,s}^{(\varphi,u;\psi,v)} = \left(\left|\chi_{N_s}dvue^{\frac{1}{2}|\varphi|^2}\right|^q dG_q\right)\circ\varphi^{-1} = \frac{q}{2\pi}\chi_{\Omega\cap N_s}\left|du\right|^q dA$$

for simplicity. Note from (a) and Theorem 1.1 that $\omega_{q,s}+\mu$ is a 1-Carleson measure. In addition, $\omega_{q,s}$ is already a compact 1-Carleson measure, as mentioned in the preliminary observations. Furthermore, since u is bounded on $\mathbb{C}$, we have

$$\mu \lesssim \chi_{\Omega\cap N_s}d^q\,dA$$

and hence, as in the proof of Example 6.3(c), conclude that μ is a compact 1-Carleson measure.

Finally, we prove (c). Choose a_0 as in the proof of Example 6.3 with x_0 satisfying (6.10). We have by (6.10)

$$|u(z)-v(z)| \geq e^2-1, \qquad z\in D_{\frac{\pi}{6}}(a_0).$$

Now, setting

$$\nu := \left(\chi_{N_s}\left|\left|u-v\right|e^{\frac{1}{2}|\varphi|^2}\right|^q dG_q\right)\circ\varphi^{-1} = \frac{q}{2\pi}\chi_{N_s}|u-v|^q\,dA,$$

we see, as in the proof of Example 6.3(b), that ν is not a compact 1-Carleson measure, which implies (c). The proof is complete. □

7. Remark

As mentioned in the Introduction, this research was initially motivated by [2, 3, 4], which consider weighted Bergman spaces in a setting where the weights on the domain and target spaces are allowed to differ. Our results can also be extended in a similar manner. To be more precise, let $\alpha>0$, $0<p<\infty$ and $\mu\geq 0$. We denote by $\mathscr{F}_\alpha^p$ the space of all $f\in\mathcal{O}(\mathbb{C})$ such that

$$\|\!|f|\!\|_{p,\alpha} := \left\{\frac{p\alpha}{2\pi}\int_{\mathbb{C}}\left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p dA(z)\right\}^{1/p} < \infty;$$

the reference [14] is entirely devoted to these spaces and their operators. Let $\mathscr{L}_\alpha^p$ be the Fock-Lebesgue space associated with $\mathscr{F}_\alpha^p$.

The arguments developed in this paper can be adapted, mutatis mutandis, to the operator $T:\mathscr{F}_\alpha^p\to\mathscr{L}_\beta^q$ by modifying the pullback measures appropriately. We leave the details to the interested reader.

Acknowledgement. First, crucial parts of this research were performed during the first author's visit, hosted by the third author, to UC Irvine in July 2026. He thanks the Department of Mathematics, UC Irvine for their hospitality and support. Next, the first two authors thank Dr. Han-Wool Lee for helpful conversations at the very early stage of this research.

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