Precalculus

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Preface

This lecture note is an expanded version of the material presented in my lectures at KU IWC 2025-2026 and ISC 2026.

Contents

Chapter 1

Functions and their graphs

Let us review some basic terms and notation.

1.1 Sets and functions

In this section, we define sets and functions and fix our notation.

1.1.1 Sets

A set is a collection of objects, which we call elements. A set is usually represented by a capital letter, and its elements are denoted by lowercase letters. To indicate that x is an element of set X, we write

$$x \in X.$$

This notation is typically read as "x is an element of X," "x is in X," or "x belongs to X." To indicate that x is not an element of X, we write $x \notin X$, which is read as "x is not an element of X," "x is not in X," "x does not belong to X," or "X does not contain x."

Some well-known sets already have their own symbols.

1. The set of all integers is denoted by $\mathbb{Z}$

$$\mathbb{Z} = \{\, ..., -2, -1,\ 0,\ 1,\ 2,\ 3,\ ...\,\}.$$

2. The set of all rational numbers is represented by $\mathbb{Q}$

$$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}.$$

3. The set of all real numbers is denoted by $\mathbb{R}$ and is often visualized as a horizontal line called the real line (Figure 1.1). Every real number corresponds to a unique point on this line. Real numbers that cannot be expressed as rational numbers are called irrational numbers. Here are some famous irrational numbers:

$$\sqrt{2} = 1.414..., \qquad e = 2.718..., \qquad \pi = 3.141592....$$

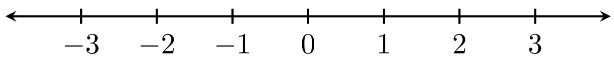

Figure 1.1: A real line.

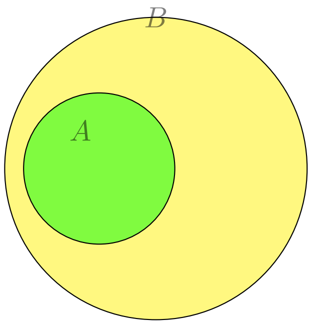

Figure 1.2: $A \subset B$.

4. The set of all complex numbers is represented by $\mathbb{C}$

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

on which we have complex addition and multiplication

$$z + z' = (a + a') + i(b + b'), \qquad zz' = (aa' - bb') + i(ab' + a'b)$$

for $z = a + ib$ and $z' = a' + ib'$.

For two sets A and B, if all elements of A are also elements of B, then we say that A is a <u>subset</u> of B (or A is included in B) and write

$$A \subset B.$$

For example,

$$\mathbb{Z} \;\subset\; \mathbb{Q} \;\subset\; \mathbb{R} \;\subset\; \mathbb{C}.$$

Most sets we encounter in calculus are subsets of $\mathbb{R}$. Here are some frequently seen subsets of $\mathbb{R}$ called <u>intervals</u>: for two real numbers a and b with $a < b$,

$$[a, b] = \{x \in \mathbb{R} : a \le x \le b\}, \qquad [a, b) = \{x \in \mathbb{R} : a \le x < b\},$$
$$(a, b] = \{x \in \mathbb{R} : a < x \le b\}, \qquad (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

We call $[a, b]$ a <u>closed</u> interval and (a, b) an <u>open</u> interval. We also define unbounded intervals:

$$[a, \infty) = \{x \in \mathbb{R} : x \ge a\}, \qquad (-\infty, b] = \{x \in \mathbb{R} : x \le b\},$$
$$(a, \infty) = \{x \in \mathbb{R} : x > a\}, \qquad (-\infty, b) = \{x \in \mathbb{R} : x < b\}.$$

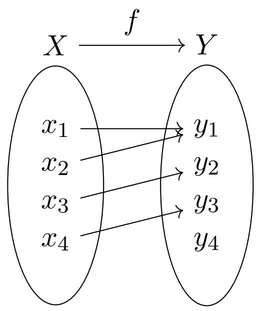

Figure 1.3: A function from X to Y.

1.1.2 Functions

A function f from set X to set Y, denoted

$$f : X \to Y,$$

is a rule that assigns exactly one element $y \in Y$ to each element $x \in X$. The "output" $y \in Y$ assigned to an "input" $x \in X$ by the rule f is denoted by

$$y = f(x),$$

which is called the value of the function f at x. The set X of all inputs is the domain of f, while the set Y of all possible outputs is the codomain of f. The image (or range) of f is the subset of the codomain consisting of all function values.

$$\mathrm{im}(f) = \{f(x) \in Y : x \in X\}$$

Example 1.1. *For the function f given in Figure 1.3, the domain is $\{x_1, x_2, x_3, x_4\}$, the codomain is $\{y_1, y_2, y_3, y_4\}$, and the image is $\{y_1, y_2, y_3\}$.*

In calculus, we are primarily interested in the following families of elementary functions mapping a subset X of $\mathbb{R}$ to $Y = \mathbb{R}$:

1. **Algebraic functions:** These include polynomial functions, rational functions, and root functions such as $y = \sqrt{x}$.

$$y = a_n x^n + \cdots + a_0, \quad y = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}, \quad y = \sqrt[n]{x}$$

2. **Exponential and logarithmic functions:**

$$y = a^x, \quad y = \log_a x \quad (\text{most notably } y = e^x,\ y = \ln x)$$

3. **Trigonometric functions and their inverses:**

$$y = \sin x, \quad y = \cos x, \quad y = \tan x, \quad y = \arcsin x$$

Example 1.2 (Domain of a rational function)**.** *The domain of the following rational function*

$$y = \frac{4(x-1)(x+1)}{(2x+1)(x-3)^2} \tag{1.1}$$

is $\{x \in \mathbb{R} : x \neq -1/2, x \neq 3\}$*.*

Example 1.3 (Domain of a square root function)**.** *The domain of the following square root function*

$$y = \sqrt{2x-5} \tag{1.2}$$

is $\{x \in \mathbb{R} : x \geq 5/2\}$*.*

Example 1.4 (Image of a quadratic function)**.** *Find the image of the following polynomial function from* $\mathbb{R}$ *to* $\mathbb{R}$

$$y = 2x^2 - 4x + 3 = 2(x-1)^2 + 1. \tag{1.3}$$

1.1.3 Sequences

A real-valued function f defined on the set of non-negative integers is called a sequence.

$$f : \mathbb{Z}_{\geq 0} \to \mathbb{R}, \quad n \mapsto f(n)$$

By convention, we write f_n instead of $f(n)$ and identify the function f with the ordered list of its values:

$$f_0,\ f_1,\ f_2,\ f_3,\ \ldots$$

The limit of a sequence $\{f_n\}_{n=0}^{\infty}$, denoted by

$$\lim_{n\to\infty} f_n = L$$

is the unique value L that the terms of the sequence approach as n approaches infinity. If such a limit exists and is finite, the sequence is called convergent (or is said to converge). A sequence that does not converge is called divergent (or is said to diverge).

Example 1.5. *For each of the following sequences, determine whether it is convergent or divergent. Find its limit if it exists.*

$$a_n = (-1)^n, \qquad b_n = \left(\frac{1}{2}\right)^n, \qquad c_n = \frac{(-1)^n}{n}, \qquad d_n = \frac{3n-5}{n^2+2n},$$
$$e_n = \frac{2n+1}{n-1}, \qquad f_n = \frac{2n^2-5}{4-3n}, \qquad g_n = \frac{3n^4-7n^2+5}{6-4n^4}.$$

1.2 Graph of a function

While algebraic formulas such as (1.1), (1.2) and (1.3) give us precise rules, graphs provide the big picture. The graph of a function serves as a vital bridge between algebra and geometry—a connection that forms the cornerstone of Calculus. By learning to translate abstract equations into geometric properties now, you will develop the essential intuition needed to understand rates of change, optimization, and behavior at infinity later.

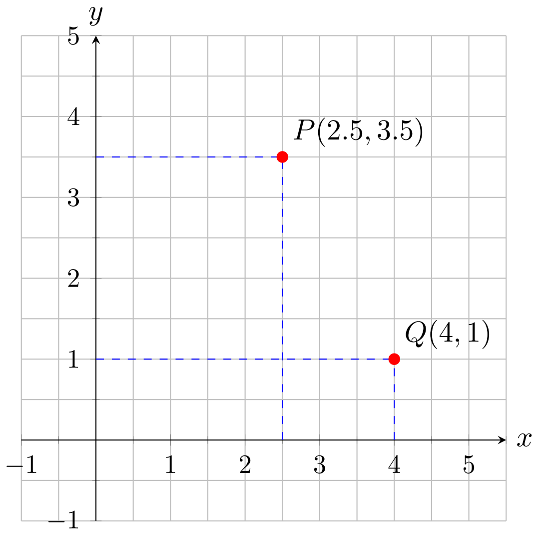

Figure 1.4: The Cartesian coordinate plane.

1.2.1 Coordinate plane

Points in a plane are identified by ordered pairs of real numbers, which together form the Cartesian coordinate plane (see Figure 1.4). This coordinate system is constructed using two perpendicular real number lines that intersect at their zero points. The horizontal line is called the x-axis, and the vertical line is called the y-axis. Their point of intersection is the origin, denoted by $(0, 0)$. For any ordered pair (c, d), the first number c represents the horizontal directed distance from the y-axis, called the x-coordinate. The second number d represents the vertical directed distance from the x-axis, called the y-coordinate.

The distance between two points (x_1, y_1) and (x_2, y_2) on the coordinate plane is found using the distance formula

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which can be derived from the Pythagorean theorem.

1.2.2 Graphs

For sets $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$, the graph of a function $f : X \to Y$ is the set of points on the coordinate plane corresponding to ordered pairs of the form $(x, f(x))$

$$G(f) = \{(x, f(x)) : x \in X\}.$$

This "curve" can be considered a visualization of f (Figure 1.5). More generally, an equation with two variables x and y defines a curve on the plane (Figure 1.6)

$$\{(x, y) : F(x, y) = 0\}.$$

The x-intercepts of a curve are the x-coordinates of the points where the curve intersects the x-axis. To find the x-intercepts, set $y = 0$ in the curve's equation and solve for x. The y-intercepts of a curve are the y-coordinates of the points where the curve

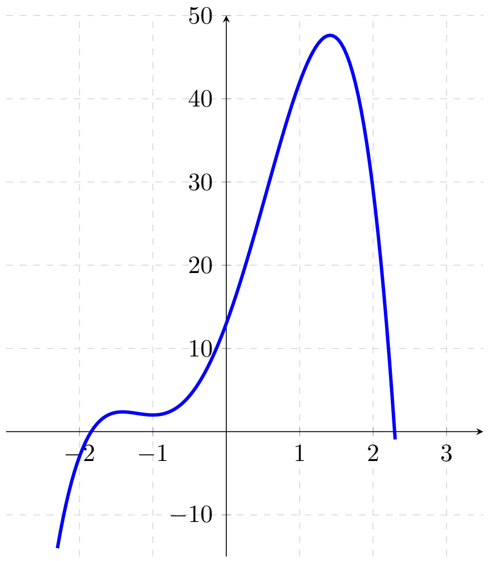

Figure 1.5: The graph of $y = -3x^4 - 4x^3 + 12x^2 + 24x + 13$.

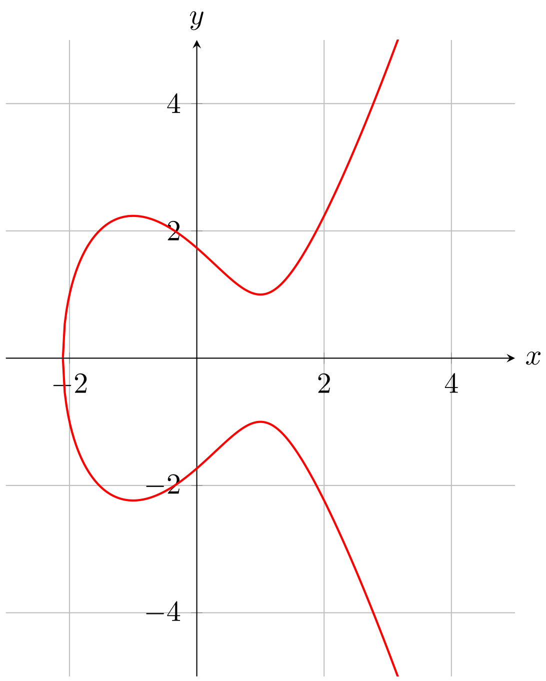

Figure 1.6: An elliptic curve $x^3 - 3x - y^2 + 3 = 0$.

intersects the y-axis. To find the y-intercepts, set $x = 0$ in the curve's equation and solve for y. If any vertical line hits the curve more than once, it cannot be the graph of a function, because one input x has multiple outputs y (Wikipedia: Vertical line test).

Example 1.6. *Find the x and y-intercepts of the line given by*

$$y = 2x - 3.$$

Example 1.7. *Find the x and y-intercepts of the parabola given by*

$$y = 2x^2 - 3x + 1.$$

Example 1.8. *A circle on the coordinate plane cannot be represented by a function (why?). Find the x and y-intercepts of the circle with radius 3 centered at $(-1, 2)$.*

$$(x + 1)^2 + (y - 2)^2 = 3^2.$$

1.2.3 Properties of a function

The "rise and fall" of a graph indicates whether a function is increasing (moving upward) or decreasing (moving downward) when read from left to right. This behavior is determined by the output values as the input values increase. More precisely, a function $y = f(x)$ is increasing on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I, it is decreasing on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

The average rate of change of $y = f(x)$ between $x = x_1$ and $x = x_2$ is the ratio of the vertical distance (rise or fall) to the horizontal distance (run) between two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph.

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \tag{1.4}$$

The "peaks" and "valleys" on a graph of a function represent where the graph changes direction—specifically, where the function shifts from increasing to decreasing (peak) or from decreasing to increasing (valley), and they are related to the maximum and minimum values of the function. To be more precise, let X be a subset of $\mathbb{R}$ containing a and $y = f(x)$ be a function from X to $\mathbb{R}$. The function value $f(a)$ is the absolute maximum of f, if

$$f(a) \geq f(x) \quad \text{for all } x \in X.$$

The function value $f(a)$ is a local maximum value of f, if

$$f(a) \geq f(x) \quad \text{for all } x \text{ near } a.$$

To state it more formally, we say that $f(a)$ is a local maximum value of f if there is an open interval (α, β) containing a such that for all x in this open interval, $f(a) \geq f(x)$. The absolute and local minimums of f can be defined by reversing inequalities.

Intuitively, a function $y = f(x)$ is continuous on an interval $[a, b]$ if its graph can be drawn in a single, continuous stroke without lifting the pen from the paper (see Definition 3.14 for a formal definition).

A continuous function is guaranteed to have both an absolute maximum and an absolute minimum value on any closed interval (Wikipedia: Extreme value theorem).

1.3 New Functions from Old Ones

Functions can be transformed and combined in various ways to generate new functions. We can transform existing functions geometrically by shifting, stretching, compressing, and reflecting their graphs. Alternatively, we can combine multiple functions algebraically using standard arithmetic operations and functional composition.

1.3.1 Geometric Transformations

Let $y = f(x)$ be a given function, and let $c > 0$ be a positive real number. The graph of f can be transformed geometrically according to the following rules:

Shifting and Scaling

- **Vertical Shifting:**
  - $y = f(x) + c$ shifts the graph of $f(x)$ **upward** by c units.
  - $y = f(x) - c$ shifts the graph of $f(x)$ **downward** by c units.
- **Horizontal Shifting:**
  - $y = f(x - c)$ shifts the graph of $f(x)$ to the **right** by c units.
  - $y = f(x + c)$ shifts the graph of $f(x)$ to the **left** by c units.
- **Vertical Scaling:** For $y = cf(x)$,
  - If $c > 1$, the graph is **stretched vertically** by a factor of c.
  - If $0 < c < 1$, the graph is **compressed vertically** by a factor of c.
- **Horizontal Scaling:** For $y = f(cx)$,
  - If $c > 1$, the graph is **compressed horizontally** by a factor of $1/c$.
  - If $0 < c < 1$, the graph is **stretched horizontally** by a factor of $1/c$.

Reflecting

- **Reflection over the x-axis:**
  $$y = -f(x)$$
  The graph is reflected vertically across the x-axis (reversing the signs of the y-coordinates).
- **Reflection over the y-axis:**
  $$y = f(-x)$$
  The graph is reflected horizontally across the y-axis (reversing the signs of the x-coordinates).

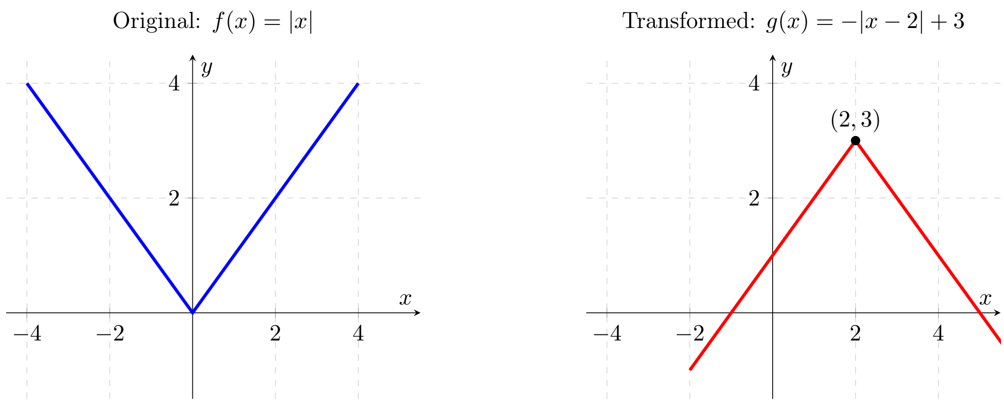

Figure 1.7: The graphs of the functions in Example 1.9

Example 1.9. *Consider the base function $f(x) = |x|$ defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases} \tag{1.5}$$

Let us find the formula for a new function $g(x)$ that is obtained by shifting the graph of $f(x)$ to the right by 2 units, reflecting it over the x-axis, and then shifting it upward by 3 units.

1. *Shift right by 2 units:* $f(x-2) = |x-2|$
2. *Reflect over the x-axis:* $-f(x-2) = -|x-2|$
3. *Shift up by 3 units:* $g(x) = -|x-2|+3$

The resulting vertex of the absolute value graph moves from $(0,0)$ *to* $(2,3)$.

1.3.2 Algebraic Operations

Two functions $f, g : \mathbb{R} \to \mathbb{R}$ can be combined using basic arithmetic operations.

- **Sum:**
$$(f+g)(x) = f(x) + g(x)$$
- **Difference:**
$$(f-g)(x) = f(x) - g(x)$$
- **Product:**
$$(fg)(x) = f(x)g(x)$$
- **Quotient:**
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{provided that } g(x) \neq 0$$

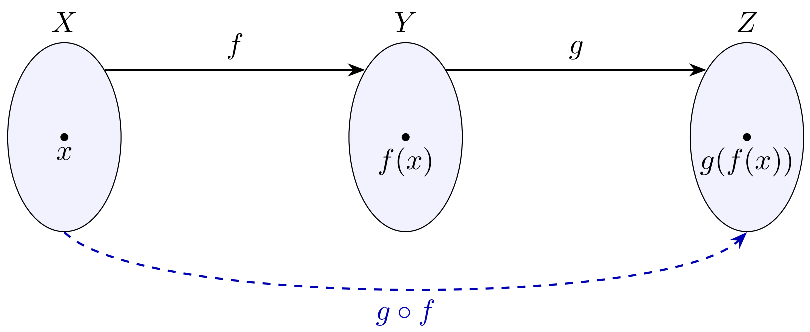

Figure 1.8: The composition of functions f and g.

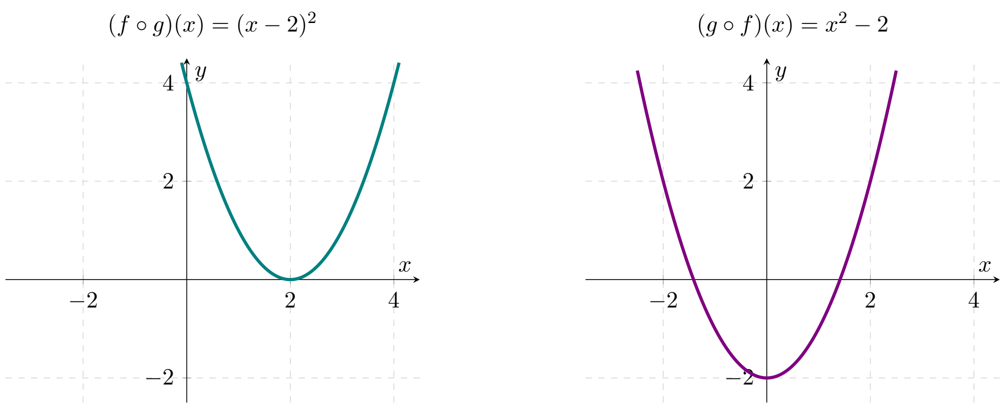

Figure 1.9: The graphs of the functions in Example 1.11.

1.3.3 Composition of Functions

Another powerful method to combine functions is through composition, where the output of one function becomes the input for another (Figure 1.8).

Definition 1.10. *Given two functions $f : X \to Y$ and $g : Y \to Z$, their composite $f \circ g$ is a function from X to Z defined by*

$$(f \circ g)(x) = f(g(x)).$$

Example 1.11. *Let $f(x) = x^2$ and $g(x) = x - 2$. Find the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$, and compare their graphs to notice that composition is not commutative.*

- $(f \circ g)(x) = f(g(x)) = f(x-2) = (x-2)^2$ *(A parabola shifted right by 2)*
- $(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 2$ *(A parabola shifted down by 2)*

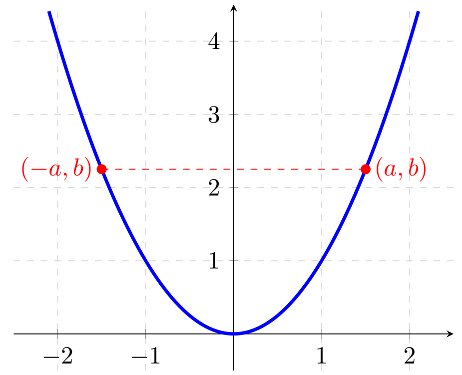

Figure 1.10: Reflection symmetry of the even function $y = x^2$ across the y-axis.

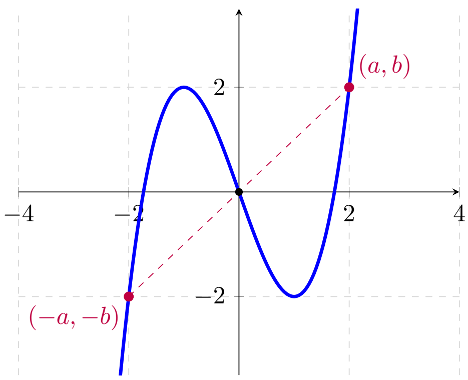

Figure 1.11: Rotational symmetry of the odd function $y = x^3 - 3x$ about the origin.

1.3.4 Even and odd functions

A function $f : \mathbb{R} \to \mathbb{R}$ is <u>even</u> if

$$f(-x) = f(x) \quad \text{for all } x$$

it is <u>odd</u> if

$$f(-x) = -f(x) \quad \text{for all } x.$$

- The graph of an even function is symmetric with respect to the y-axis. If the point (a, b) lies on the graph, its mirror image $(-a, b)$ also lies on the graph. This creates a perfect left-to-right reflectional symmetry (Figure 1.10).

- The graph of an odd function is symmetric with respect to the origin $(0, 0)$. This means that if you rotate the entire graph 180° around the origin, it will land exactly on top of itself. Visually, every point (a, b) on the graph has a corresponding point $(-a, -b)$ on the opposite side of the origin (Figure 1.11).

Note that a function $f : \mathbb{R} \to \mathbb{R}$ can be expressed as

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

and then you can verify that

$$g(x) = \frac{f(x) + f(-x)}{2}$$

is even and

$$h(x) = \frac{f(x) - f(-x)}{2}$$

is odd. This shows that $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of an even function and an odd function.

Example 1.12. *Is the following function even or odd?*

1. $y = x^3 - 2x$
2. $y = x^2 - 3$
3. $y = x^2 - 2x + 5$

Chapter 2

Polynomial functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called a polynomial function, if its function value at x is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Here, a_k are called the coefficients of f. When $a_n \neq 0$, the highest power n is called the degree of f. We will focus on polynomial functions of degree 1 and 2, and their graphs.

2.1 Linear functions

Polynomial functions of degree 1 from $\mathbb{R}$ to $\mathbb{R}$

$$y = mx + b \quad (m \neq 0)$$

are often called linear functions, because their graphs on the coordinate plane are lines.

Example 2.1. *There exists exactly one line passing through two distinct points on the plane. Find the linear function whose graph passes through two points* $(-1, 3)$ *and* $(2, -1)$.

The linear function $y = mx + b$ has constant rate of change: for any two x_1 and x_2 with $x_1 < x_2$,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = m.$$

It is equal to the coefficient m and measures the steepness and direction of the line (Figure 2.1). We call m the slope of the linear function. Note that the average rate of change of a function defined in (1.4) represents the slope of the secant line connecting two points on the graph of $y = f(x)$ (Figure 2.2).

The line passing through a point (x_0, y_0) with slope m is the graph of

$$y = m(x - x_0) + y_0$$

Think of it as a consequence of geometric transformations: starting from $y = mx$,

$$y = f(x) \Rightarrow y = f(x - x_0) \Rightarrow y = f(x - x_0) + y_0.$$

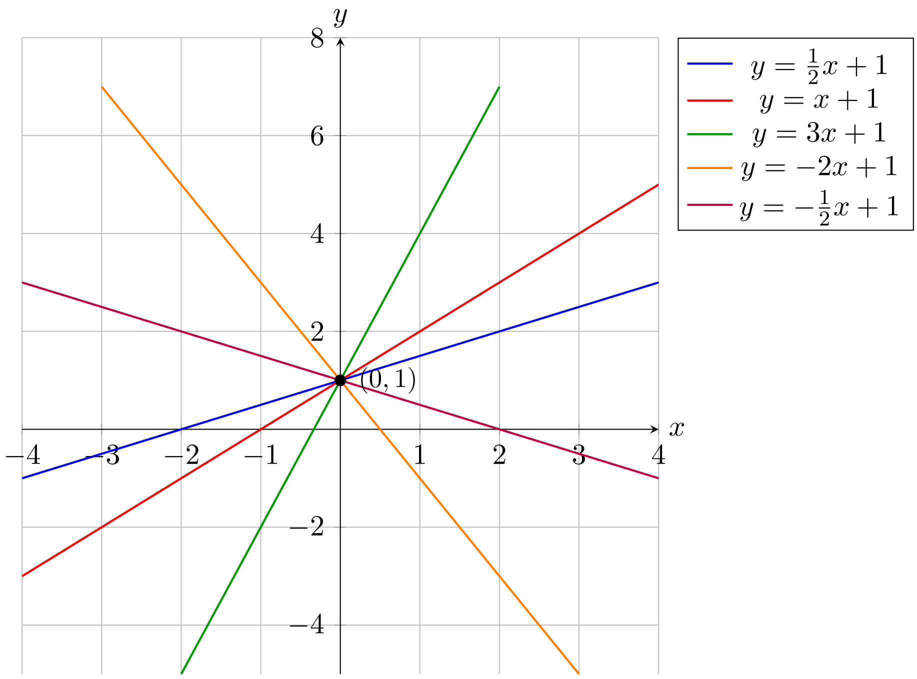

Figure 2.1: The graphs of linear functions with various slopes.

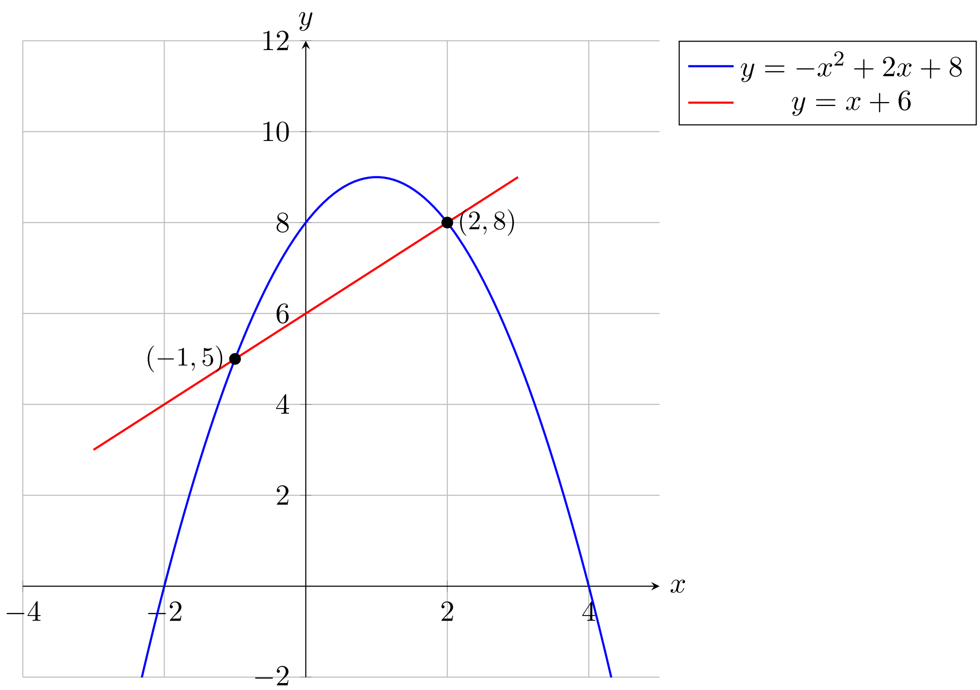

Figure 2.2: The average rate of change of f over an interval $[a, b]$ is equal to the slope of the secant line passing through $(a, f(a))$ and $(b, f(b))$.

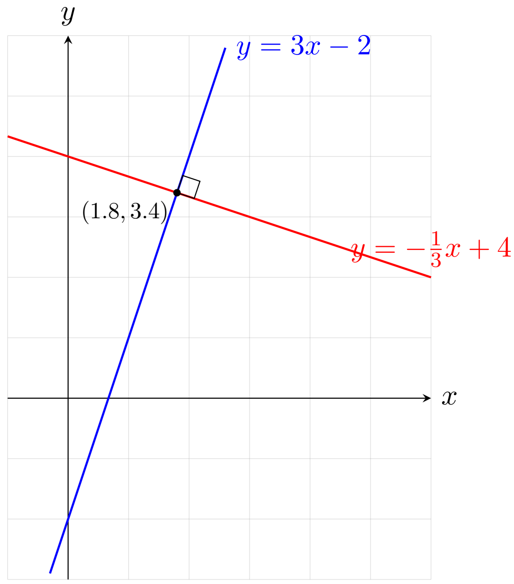

Figure 2.3: Two perpendicular lines.

In a plane, two lines can have three possible relationships:

1. they can be the same line,
2. they can be parallel, or
3. they can intersect at exactly one point.

Two lines represented by

$$y = m_1 x + b_1 \quad \text{and} \quad y = m_2 x + b_2.$$

are coincident if $m_1 = m_2$ and $b_1 = b_2$; they are parallel if $m_1 = m_2$ and $b_1 \neq b_2$.

Theorem 2.2. *Two lines represented by $y = m_1 x + b_1$ and $y = m_2 x + b_2$ are perpendicular to each other, i.e. intersecting at a right angle, if and only if the product of their slopes is -1.*

$$m_1 m_2 = -1$$

Proof. After translations, it is enough to consider $y = m_1 x$ and $y = m_2 x$ (why?). Apply the Pythagorean rule to the right-angle triangle formed by $(1, m_1)$, $(1, m_2)$, and $(0, 0)$ to derive the relation between m_1 and m_2. □

Not all lines on the coordinate plane are the graphs of linear functions. To include missing ones like vertical lines, we have to use an equation like

$$ax + by + c = 0.$$

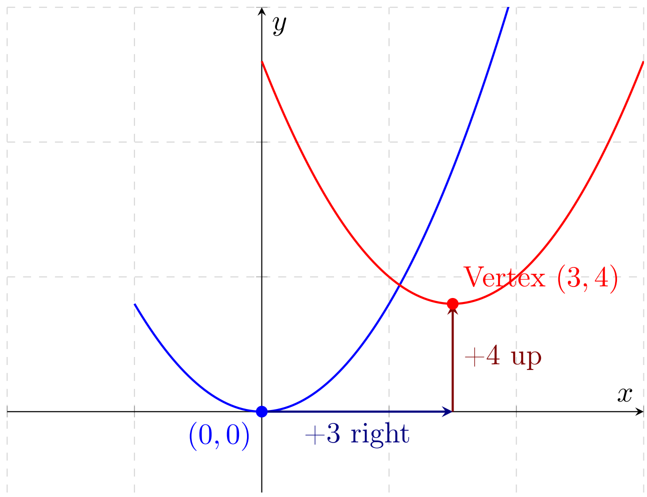

Figure 2.4: The graph of $y = (x-3)^2 + 4$ is obtained by shifting $y = x^2$ to the right by 3 and upward by 4.

Vertical lines can be represented by $x = d$ (thus $a \neq 0$ and $b = 0$) and horizontal lines can be represented by $y = d$ (thus $a = 0$ and $b \neq 0$). If $b \neq 0$ then this equation defines a linear function

$$y = -\frac{a}{b}x - \frac{c}{b}.$$

Note that $y = d$ is a constant function, that is a function with the same output d for all all inputs $x \in \mathbb{R}$.

2.2 Quadratic functions

Polynomial functions of degree 2 from $\mathbb{R}$ to $\mathbb{R}$

$$y = a_2x^2 + a_1x + a_0 \quad (a_2 \neq 0)$$

are often called quadratic functions, and its graph is a parabola.

2.2.1 Vertex form

Any quadratic function can be expressed in vertex form

$$y = a(x-p)^2 + q$$

Its graph can be obtained from the graph of $y = ax^2$ by applying geometric transformations

$$y = f(x) \Rightarrow y = f(x-p) \Rightarrow y = f(x-p) + q.$$

See Figure 2.4.

Example 2.3. *To draw the graph of $y = x^2 - 3x + 3$, using the equality*

$$(x-p)^2 = x^2 - 2px + p^2$$

we have

$$\begin{aligned}
y &= x^2 - 3x + 3 \\
&= \left[x^2 - 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2\right] - \left(\frac{3}{2}\right)^2 + 3 \\
&= \left(x - \frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 3 \\
&= \left(x - \frac{3}{2}\right)^2 + \frac{3}{4}
\end{aligned}$$

Thus, we can obtain the graph of $y = x^2 - 3x + 3$ by shifting that of $y = x^2$ to the right $3/2$ and upward $3/4$.

When a quadratic function is given as $y = ax^2 + bx + c$, using the equality

$$(x+d)^2 = x^2 + 2dx + d^2$$

we can rewrite

$$\begin{aligned}
y &= ax^2 + bx + c \\
&= a\left[x^2 + \left(\frac{b}{a}\right)x\right] + c \\
&= a\left[x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2\right] - \frac{b^2}{4a} + c \\
&= a\left[x + \frac{b}{2a}\right]^2 - \frac{b^2}{4a} + c
\end{aligned}$$

The vertex form of a quadratic function shows that the graph of $y = ax^2 + bx + c$ can be obtained by shifting that of $y = ax^2$ to the right $p = -\frac{b}{2a}$ and upward $q = -\frac{b^2}{4a} + c$.

$$\begin{aligned}
y = ax^2 \quad &\Longrightarrow \quad y = a(x-p)^2 \\
&\Longrightarrow \quad y = a(x-p)^2 + q
\end{aligned}$$

Therefore, the coefficient a determines the width and direction of the parabola, making its shape identical to that of $y = ax^2$, but shifted to the vertex

$$(p, q) = \left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right).$$

From the vertex form of a quadratic function, we can also tell the range of the function.

1. If a quadratic function from $\mathbb{R}$ to $\mathbb{R}$ is given as

$$f(x) = a(x-p)^2 + q$$

with $a > 0$ then since $m(x-p)^2 \geq 0$ for all $x \in \mathbb{R}$, we have

$$f(x) = a(x-p)^2 + q \geq q$$

for all $x \in \mathbb{R}$ and we obtain $y = q$ when $x = p$. Therefore, the minimum value of the function f is $q = f(p)$.

2. If $a < 0$ then $m(x-p)^2 \leq 0$ for all $x \in \mathbb{R}$ and thus we have

$$f(x) = a(x-p)^2 + q \leq q$$

for all $x \in \mathbb{R}$ and we obtain $y = q$ when $x = p$. Therefore, the maximum value of the function f is $q = f(p)$.

2.2.2 Factored form

Suppose that a quadratic function is in the form

$$y = a(x-\alpha)(x-\beta) \quad \text{with } \alpha < \beta$$

It is easy to see that $y = 0$ when $x = \alpha$ or β, and thus the x-intercepts are α and β. Using this observation, you can sketch its graph (Figure 2.5).

In order to rewrite a quadratic function in a factored form, you may want to recall the identity

$$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta \tag{2.1}$$

or the quadratic formula: the solution to $ax^2 + bx + c = 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Thus,

$$ax^2 + bx + c = a\left(x - \frac{-b - \sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b + \sqrt{b^2-4ac}}{2a}\right).$$

Example 2.4. *Using the identity* (2.1)

$$\begin{aligned} y &= x^2 - 3x + 2 \\ &= x^2 - (1+2)x + 1 \cdot 2 \\ &= (x-1)(x-2) \end{aligned}$$

or, by applying the quadratic formula

$$\alpha, \beta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} = \frac{3 \pm 1}{2} = 1, 2.$$

Now we can sketch the graph of $y = x^2 - 3x + 2$ using its x-intercepts. From its graph, it is easy to see that

$$x^2 - 3x + 2 > 0$$

for $x < 1$ or $x > 2$.

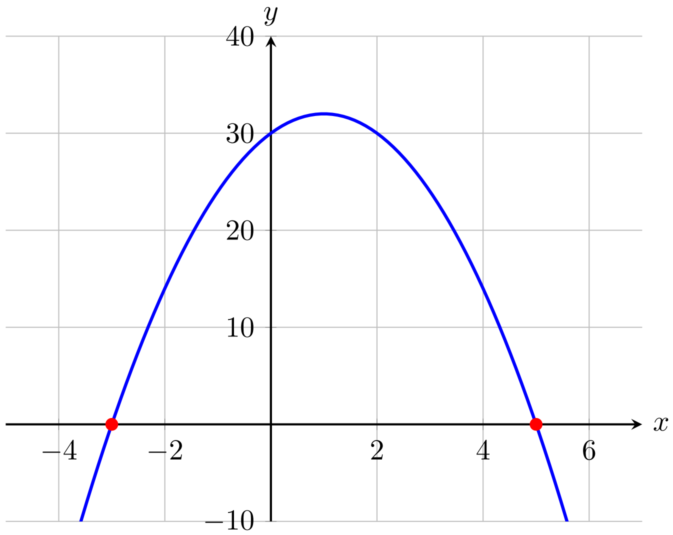

Figure 2.5: The graph of $y = -2(x+3)(x-5)$.

Example 2.5. *Consider $y = 2x^2 + 3x + 1$. Then, the quadratic equation*

$$2x^2 + 3x + 1 = 0$$

has the solutions

$$\alpha, \beta = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = -\frac{1}{2}, -1.$$

Thus, using

$$2x^2 + 3x + 1 = 2(x-\alpha)(x-\beta)$$

we have

$$y = 2x^2 + 3x + 1 = 2[x - (-1)]\left[x - \left(-\frac{1}{2}\right)\right] = 2(x+1)\left(x + \frac{1}{2}\right).$$

Now we can sketch the graph of $y = 2x^2 + 3x + 1$ using its x-intercepts. From its graph we see that

$$2x^2 + 3x + 1 \le 0$$

for $-1 \le x \le -1/2$.

2.2.3 Curve sketching

In fact the above method can be generalized: to sketch the graph of a polynomial function in factored form

$$f(x) = a(x-r_1)^{m_1}(x-r_2)^{m_2} \cdots (x-r_k)^{m_k},$$

1. first, note that the x-intercepts are r_1, r_2, ..., and r_k.

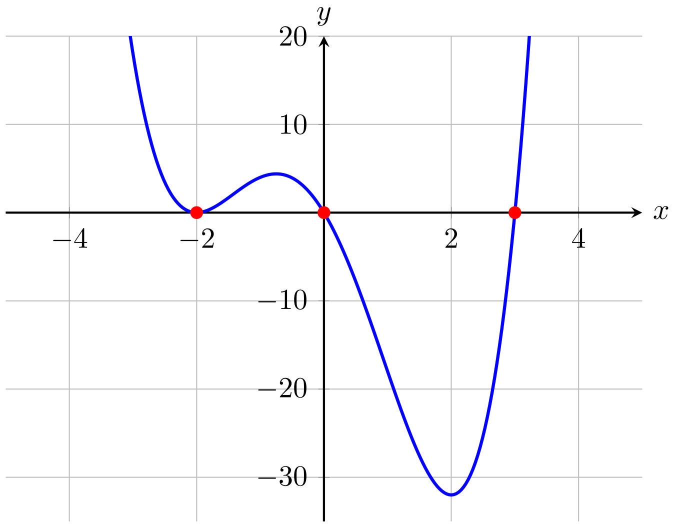

Figure 2.6: The graph of $y = x(x-3)(x+2)^2$.

2. if the multiplicity m_i of r_i is even then the graph bounces off the x-axis at $x = r_i$; if m_i is odd then the graph crosses the x-axis at $x = r_i$.

3. if the degree $(m_1 + \cdots + m_k)$ of f is odd then the ends of the graph go opposite directions (up-right, down-left or vice-versa); if the degree of f is even then the ends of the graph go same direction (both up or both down).

4. if the leading coefficient a is positive then the graph rises to the right (for odd degree) or both up (for even degree); if a is negative then the graph falls to the right (for odd degree) or both down (for even degree).

5. finally, start with the end behavior, move to the first x-intercept (crossing or bouncing), continue to the next, and so on, until the graph reaches the other end behavior.

Example 2.6. *Sketch the graphs of the following functions.*

1. $f(x) = (x+1)(x-2)(x-4)$

2. $f(x) = (x+3)(x+1)(x-2)(x-5)$

3. $f(x) = -3(x-2)^2(2x+3)$

4. $f(x) = (x+3)(x-2)^2(x+1)^3$

5. $f(x) = x^6 - 3x^4 + 2x^2$

6. $f(x) = x^3 - 5x^2 - x + 5$

2.3 Lines and parabolas

We have been studying the graphs of linear and quadratic functions. Let us draw any of these two

$$y = f(x) \quad \text{and} \quad y = g(x)$$

on the same coordinate plane:

1. two lines given by
$$y = m_1x + b_1 \quad \text{and} \quad y = m_2x + b_2,$$
2. a line and a parabola given by
$$y = mx + n \quad \text{and} \quad y = ax^2 + bx + c,$$
3. two parabolas given by
$$y = ax^2 + bx + c \quad \text{and} \quad y = a'x^2 + b'x + c'.$$

Their intersection is a common point (a, b) on the graph of $y = f(x)$ and the graph of $y = g(x)$. Therefore, finding their intersections is equivalent to finding $x = a$ and $y = b$ such that both equalities $f(a) = b$ and $g(a) = b$ hold. In order to find them, we can begin by solving the equation

$$f(x) = g(x)$$

to obtain $x = a$ and then compute $f(a)$, or $g(a)$, to obtain b.

Exercises

1. Find the intersection(s) of two lines given by
   (a) $y = 2x + 3$ and $y = -x + 5$.
   (b) $y = -2x + 5$ and $y = -2x - 3$.
   (c) $y = 3$ and $y = -4x + 6$.
2. Find the intersection(s) of a line and a parabola given by
   (a) $y = -x - 1$ and $y = -2x^2 + 2$.
   (b) $y = 2x + 3$ and $y = -x^2 + 2$.
   (c) $y = -4x - 3$ and $y = 2x^2 + 1$.
   (d) $y = 1$ and $y = -4x^2 + 2$.
   (e) $y = 2$ and $y = -4x^2 + 2$.
   (f) $y = 3$ and $y = -4x^2 + 2$.
3. Find the intersection(s) of a line and a parabola given by
   (a) $y = x^2 + 3$ and $y = -x^2 - 5x$.
   (b) $y = -x^2 + 4x - 1$ and $y = -2x^2 + 4x + 3$.
   (c) $y = -x^2 + 4x - 1$ and $y = -2x^2 + 2x - 2$.
   (d) $y = x^2 + x$ and $y = -x^2 + 3x - 1$.
4. On what intervals is the inequality satisfied? See Figure 2.7.
   (a) $x^2 \geq x + 2$
   (b) $x^2 < x + 2$
5. On what intervals is the inequality satisfied? See Figure 2.8.
   (a) $-(x + 1)^2 + 4 > x^2$
   (b) $-(x + 1)^2 + 4 \leq x^2$

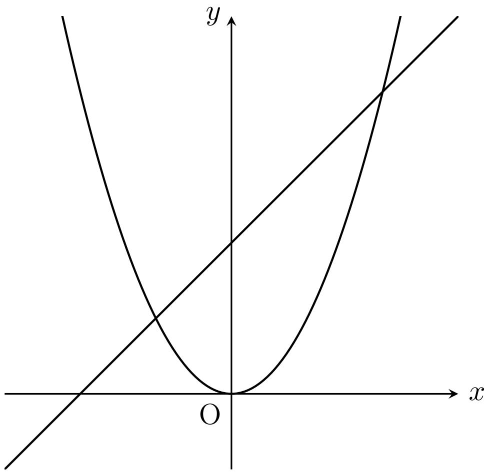

Figure 2.7: $y = x + 2$ and $y = x^2$.

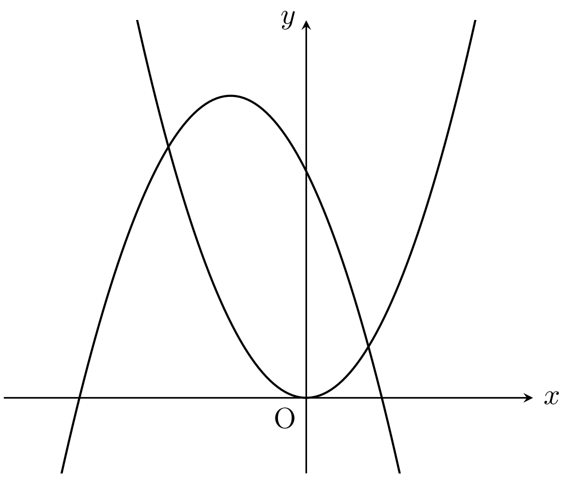

Figure 2.8: $y = -(x + 1)^2 + 4$ and $y = x^2$.

Chapter 3

Rational functions and limits

We introduce rational functions and then study the limit behavior and continuity of a function.

3.1 Polynomial long division

A function is called a rational function if it can be written in the form

$$\frac{f(x)}{g(x)}$$

where f and q are polynomial functions of x. Its domain is the set of all values of x for which the denominator $g(x)$ is not zero. When the degree of the numerator $f(x)$ is greater than or equal to the degree of the denominator $g(x)$, we can perform polynomial long division.

Theorem 3.1. *For two polynomials $f(x)$ and $g(x)$, there exist $q(x)$ and $r(x)$ such that*

$$f(x) = g(x)q(x) + r(x),$$

where $r(x) = 0$ or the degree of $r(x)$ is strictly less than n.

See Wikipedia: Polynomial long division.

Example 3.2. *Verify the following identities.*

1. $6x^2 - 26x + 12 = (x - 4)(6x - 2) + 4.$

2. $8x^4 + 6x^2 - 3x + 1 = (2x^2 - x + 2)(4x^2 + 2x) - 7x + 1.$

3. $2x^3 - 7x^2 + 5 = (x - 3)(2x^3 - x - 3) - 4.$

Corollary 3.3. *Every rational function $f(x)/g(x)$ can be rewritten as*

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)},$$

where $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

We say $x = x_0$ is a zero of a polynomial $f(x)$ if $f(x_0) = 0$.

Corollary 3.4. *If x_0 is a zero of $f(x)$ then*

$$f(x) = (x - x_0)q(x)$$

for some polynomial $q(x)$.

The above fact can be used to factorize a polynomial function. Once a polynomial function $y = f(x)$ is given in a factored form, we can use its x-intercepts and signs on intervals divided by them to sketch the graph.

Example 3.5. *Rewrite each of the following polynomials in a factored form using a given zero and then sketch its graph.*

1. $y = x^3 + 2x^2 - x - 2$ *with* $x = -1$.
2. $y = x^3 - 3x + 2$ *with* $x = 1$.
3. $y = x^3 + 6x^2 + 12x + 8$ *with* $x = -2$.

3.2 Limits

One of the most distinct characteristics of rational functions compared to polynomial functions is their behavior concerning limits.

Definition 3.6. *Let $f(x)$ be a function from a subset X of $\mathbb{R}$ to $\mathbb{R}$. We say that L is the limit of $f(x)$ as x approaches a if, as the input values of x approach a but do not equal a, the corresponding output values $f(x)$ get closer to L, and then we write*

$$\lim_{x\to a} f(x) = L.$$

Note that

$$\lim_{x\to\infty} (a_n x^n + \cdots + a_0) = \lim_{x\to\infty} a_n x^n$$

and it is ∞ if $a_n > 0$ and $-\infty$ if $a_n < 0$. What if $x \to -\infty$?

Example 3.7. *Let us sketch the graphs of*

$$y = \frac{a}{x-b} + c \quad \text{and} \quad y = \frac{a}{(x-b)^2} + c$$

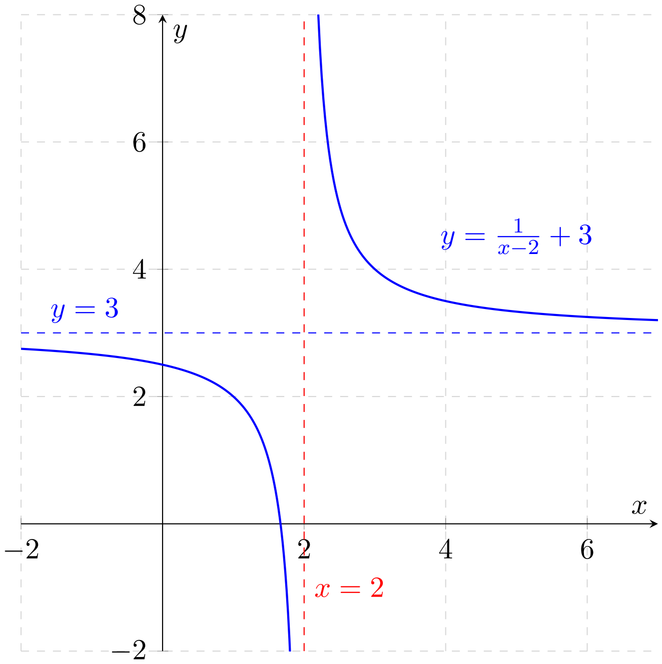

Figure 3.1: The graph of $y = \frac{1}{x-2} + 3$.

Example 3.8. *Let* $g(x) = \frac{a}{x-b} + c$. *Using its graph, compute*

$$\lim_{x\to b^+} g(x), \qquad \lim_{x\to b^-} g(x), \qquad \lim_{x\to+\infty} g(x), \qquad \lim_{x\to-\infty} g(x).$$

Example 3.9. *Evaluate the limits.*

1. $$\lim_{x\to\infty} \frac{4x+11}{5x^3-2x+1}$$

2. $$\lim_{x\to\infty} \frac{2x^3-x^2}{3x^2+4}$$

3. $$\lim_{x\to\infty} \frac{5-4x^3+2x}{3x^3-x^2+8}$$

Example 3.10. *Evaluate the limits.*

1. $$\lim_{x\to-1} \frac{x^2-2x-3}{x+1}$$

2. $$\lim_{x\to3} \frac{x^2-9}{x^2-5x+6}$$

3.

$$\lim_{x\to 0}\frac{(3-x)^2-9}{x}$$

4.

$$\lim_{x\to 0}\frac{\sqrt{5-x}-\sqrt{5}}{x}$$

5.

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x}-\sqrt{3}}$$

3.3 Continuity

Note that the graph of $y = 1/x$ consists of two curves, one for $x < 0$ and the other for $x > 0$. We can define a single function by applying different rules to different intervals of the domain. These are known as piecewise-defined functions.

Probably the most famous piecewise defined function is the absolute value function $y = |x|$ defined in (1.5). Piecewise-defined functions are useful in studying the continuity property of a function.

Example 3.11. *Evaluate the limits.*

$$\lim_{x\to 4^+}\frac{|x-4|}{4-x} \quad \textit{and} \quad \lim_{x\to 4^-}\frac{|x-4|}{4-x}.$$

Example 3.12. *Let us consider*

$$f(x) = \begin{cases} x^2+1 & \textit{if } x<1 \\ 1 & \textit{if } x=1 \\ -\frac{1}{2}x+1 & \textit{if } x>1. \end{cases}$$

Evaluate its limits

$$\lim_{x\to 1^+} f(x) \quad \textit{and} \quad \lim_{x\to 1^-} f(x).$$

Example 3.13. *Let us consider*

$$f(x) = \begin{cases} x^2 & \textit{if } x\le 1 \\ 3 & \textit{if } 1<x\le 2 \\ x & \textit{if } x>2. \end{cases}$$

Evaluate its limits

$$\lim_{x\to 2^-} f(x) \qquad \textit{and} \qquad \lim_{x\to 2^+} f(x).$$

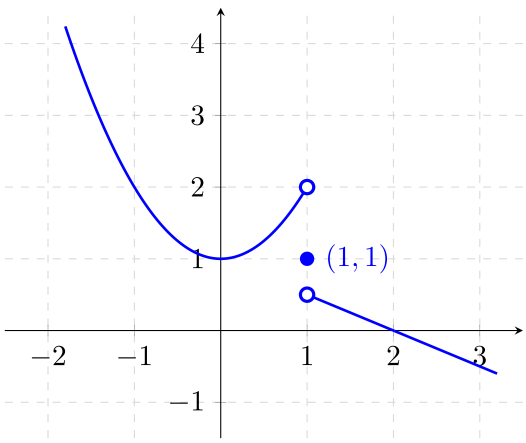

Figure 3.2: The graph of a piecewise function in Example 3.12.

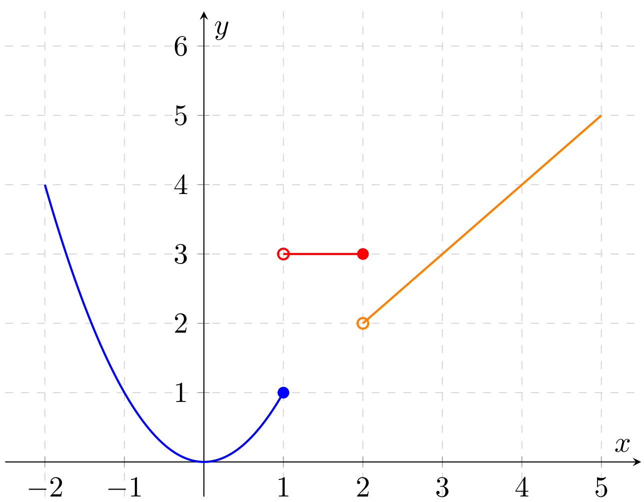

Figure 3.3: The graph of the function in Example 3.13.

Definition 3.14. *A function $y = f(x)$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous at $x = a$, if*

1. *the function value $f(a)$ at $x = a$ is well defined,*
2. *the left and right limits of $y = f(x)$ as $x \to 1$ exist and*

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(a).$$

On an interval I, $y = f(x)$ is continuous if it is continuous at every $a \in I$.

Example 3.15. *Let us consider*

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 2x + k & \text{if } x > 1. \end{cases}$$

Compute

$$\lim_{x \to 1-} f(x) \qquad \lim_{x \to 1+} f(x) \qquad \text{and} \qquad f(1).$$

Find k such that $y = f(x)$ is continuous on $\mathbb{R}$.

Example 3.16. *Find c and k such that the following function is continuous on $\mathbb{R}$.*

$$f(x) = \begin{cases} x^2 & \text{if } x > 1 \\ c & \text{if } x = 1 \\ 2x + k & \text{if } x > 1. \end{cases}$$

Chapter 4

Exponential and logarithmic functions

We introduce exponential functions and their inverses.

4.1 Exponential functions

Let a be a positive real number. We define $a^0 = 1$. For a positive integers m, define

$$a^m = a \cdot a \cdot \cdots \cdot a \quad \text{and} \quad a^{-m} = \frac{1}{a^m}.$$

For a positive integer n, we write $a^{1/n}$ (or $\sqrt[n]{a}$) for a positive real number b such that $b^n = a$. Then, for any positive integer m, define

$$a^{m/n} = \left(a^{1/n}\right)^m \quad \text{and} \quad a^{-m/n} = \frac{1}{a^{m/n}}.$$

These definitions show that a^x is well defined for all rational numbers x, and then we can extend its domain to all real numbers to obtain a function $y = a^x$ from $\mathbb{R}$ to $\mathbb{R}$.

Note that for positive real numbers a and b, and non-negative integers m and n,

1. $a^m a^n = (a \cdots a) \cdot (a \cdots a) = a \cdots a = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$, since

$$a^{m-n} = \frac{a \cdots a}{a \cdots a} = \begin{cases} a^{m-n}/1 = a^{m-n} & \text{if } m > n, \\ 1 = a^0 = a^{m-n} & \text{if } m = n, \\ 1/a^{n-m} = a^{-(n-m)} = a^{m-n} & \text{if } m < n. \end{cases}$$

3. $(a^m)^n = a^m \cdots a^m = (a \cdots a) \cdots (a \cdots a) = a \cdots a = a^{mn}$
4. $(ab)^m = (ab) \cdots (ab) = (a \cdots)(b \cdots b) = a^m b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a}{b} \cdots \frac{a}{b} = \frac{a^m}{b^m}$

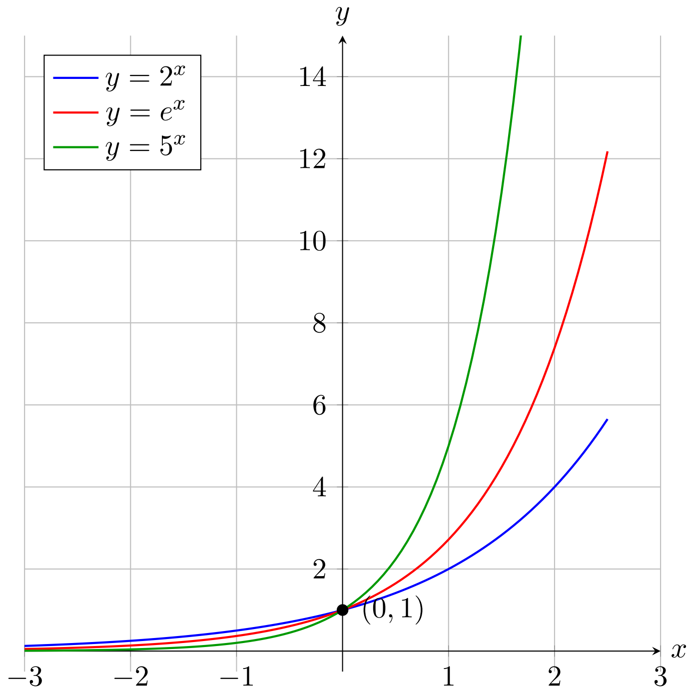

Figure 4.1: The graphs of $y = 2^x$, $y = e^x$, and $y = 5^x$.

6. $\left(\frac{a}{b}\right)^{-m} = 1/(\frac{a}{b})^m = 1/(a^m/b^m) = b^m/a^m = \left(\frac{b}{a}\right)^m$

These identities can be extended to real numbers: for all real numbers s and t,

1. $a^s a^t = a^{s+t}$
2. $\frac{a^s}{a^t} = a^{s-t}$
3. $(a^s)^t = a^{st}$
4. $(ab)^s = a^s b^s$
5. $\left(\frac{a}{b}\right)^s = \frac{a^s}{b^s}$
6. $\left(\frac{a}{b}\right)^{-s} = \left(\frac{b}{a}\right)^s$

Assume $a > 1$ and then sketch the graph $y = a^x$. This graph passes through $(0, 1)$ and $(1, a)$. The function $y = a^x$ is increasing with

$$\lim_{x\to\infty} a^x = \infty \quad \text{and} \quad \lim_{x\to-\infty} a^x = 0.$$

If the base b is between 0 and 1 ($0 < b < 1$), the function $y = b^x$ is decreasing. Using $a = 1/b > 1$, realize it as

$$y = b^x = (a^{-1})^x = a^{-x},$$

then we can draw its graph by flipping the graph of $y = a^x$ over the y-axis. We have

$$\lim_{x\to\infty} b^x = 0 \quad \text{and} \quad \lim_{x\to-\infty} b^x = \infty.$$

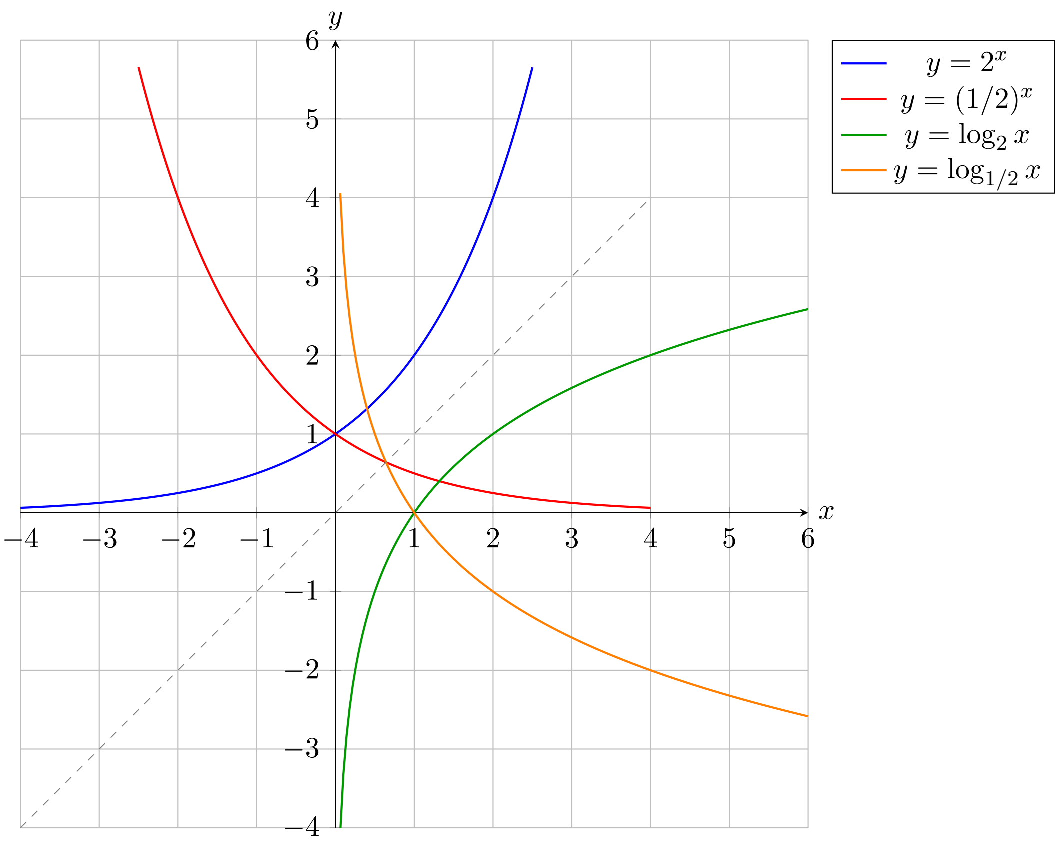

Figure 4.2: The graphs of $y = 2^x$, $y = (1/2)^x$, $y = \log_2 x$, and $y = \log_{1/2} x$.

4.2 Inverse of a function

A function $f : X \to Y$ is one-to-one if whenever $x_1 \neq x_2$ in X, we have $f(x_1) \neq f(x_2)$, or equivalently, whenever $f(x_1) = f(x_2)$, we have $x_1 = x_2$. A function $f : X \to Y$ is onto if for every $y \in Y$, there is $x \in X$ such that $y = f(x)$.

Note that every exponential function from $\mathbb{R}$ to $\mathbb{R}_{>0}$ is one-to-one and onto. When $f : X \to Y$ is one-to-one and onto, we can obtain a new function $g : Y \to X$ by reversing the correspondence arrows. Then they satisfy the following identities.

$$g(f(z)) = z \text{ for all } z \in X \quad \text{and} \quad f(g(z)) = z \text{ for all } z \in Y.$$

Thus $g \circ f$ is the identity function on X and $f \circ g$ is the identity function on Y. In this setting, we say f is invertible and g is the inverse of f, and then write f^{-1} for g.

Example 4.1. *The following functions are invertible, i.e., one-to-one and onto.*

1. *The linear function $y = 3x - 2$ from $\mathbb{R}$ to $\mathbb{R}$.*
2. *The quadratic function $y = x^2$ from the set of all non-negative real numbers to the set of all non-negative real numbers.*
3. *The rational function*
$$y = \frac{2}{x-1} + 3$$
from $\mathbb{R} \setminus \{1\}$ to $\mathbb{R} \setminus \{3\}$.

When $f : X \to Y$ is invertible, to obtain an algebraic expression of its inverse, first you can swap x and y in $y = f(x)$ and then solve for y. The graph of f^{-1} is

$$G(f^{-1}) = \{(b, a) : (a, b) \in G(f)\}$$

thus on the coordinate plane it can be obtained by flipping the graph of $y = f(x)$ over the line $y = x$.

Example 4.2. *Compute the inverses of functions in Example 4.1, and draw their graphs.*

4.3 Logarithmic functions

For a positive real number $a \neq 1$, the logarithmic function with base a is the inverse of the exponential function $y = a^x$ from $\mathbb{R}$ to $\mathbb{R}_{>0}$. We will denote it by

$$y = \log_a x$$

Note that $y = \log_a x$ is a one-to-one and onto function from $\mathbb{R}_{>0}$ to $\mathbb{R}$.

Here are some important properties of logarithmic functions. They can be verified by using properties of exponential functions.

1. $\log_a 1 = 0$
2. $\log_a a = 1$

3. $\log_a a^x = x$ for all real numbers x
4. $a^{\log_a x} = x$ for all positive real numbers x
5. $\log_a AB = \log_a A + \log_a B$

   Let $\alpha = \log_a A$ and $\beta = \log_a B$. Then $A = a^\alpha$ and $B = a^\beta$, thus $AB = a^\alpha a^\beta = a^{\alpha+\beta}$, which says that $\log AB = \alpha + \beta = \log_a A + \log_a B$.
6. $\log_a A/B = \log_a A - \log_a B$
7. $\log_a A^c = c \log_a A$
8. base change: $\log_b x = (\log_b a)(\log_a x)$, or

$$\frac{\log_b x}{\log_b a} = \log_a x.$$

   Let $\alpha = \log_a x$, $\beta = \log_b x$, and $\gamma = \log_b a$. Then,

$$a^\alpha = x, \quad b^\beta = x, \quad b^\gamma = a$$

   and thus $b^\beta = a^\alpha = (b^\gamma)^\alpha = b^{\gamma\alpha}$. Therefore, $\beta = \gamma\alpha$.

Euler's number e is an irrational number defined by

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.718...$$

If $\log_e a = c$, then $a = e^c$ and

$$a^x = (e^c)^x = e^{cx}.$$

Thus when studying properties of exponential functions $y = a^x$, we can focus on the natural exponential function

$$y = e^x.$$

Also, for any positive real number a with $a \neq 1$ we have

$$y = \log_a x = \frac{\log_e x}{\log_e a} = \frac{1}{\ln a}\ln x.$$

Thus when studying properties of logarithmic functions $y = \log_a x$, we can focus on the natural logarithm function

$$y = \ln x.$$

Exercises

1. Sketch the graph of the function.
   (a) $y = -2^{x-1}$
   (b) $y = (1/2)^x - 5$
   (c) $y = 2 - \ln(x+3)$
2. Verify the following identities:
   (a) $\log_3 81 = 4$, $\log_{1/3} 1 = 0$.
   (b) $\ln \frac{1}{e^2} = -2$.
   (c) $\log_5 1/5 = \log_{1/5} 5 = -1$.
   (d) $\log_8 2 = 1/3$, $\log 0.01 = -2$.
3. Find the domain of the following functions:
   (a) $f(x) = \ln(4 - x^2)$
   (b) $f(x) = \ln(x^2 - x)$
   (c) $f(x) = \sqrt{x-2} - \log_5(10 - x)$
4. Expand the logarithmic expression.
   (a) $\log_2(x\sqrt{x^2+1})$
   (b) $\ln \frac{\sqrt{x^2-1}}{x^2+1}$
5. Combine into a single logarithm.
   (a) $\log 6 + 4\log 2$
   (b) $\ln 2 + \ln(x+1) - \frac{1}{3}\ln(3x+7)$
6. Solve the following equations:
   (a) $8e^{3-2x} = 20$
   (b) $3xe^x + x^2e^x = 0$
   (c) $e^{2x} - e^x - 6 = 0$
   (d) $\log_2(25 - x) = 3$
   (e) $\log(x+2) + \log(x-1) = 1$

Chapter 5

Trigonometric functions

We define and study trigonometric functions.

5.1 Angle, triangle and circle

Trigonometric functions relate angles to the side lengths of a right-angled triangle or to coordinates on a unit circle.

5.1.1 Right triangles

An angle measures the amount of rotation between two lines (or rays/segments) that meet at a common endpoint (the vertex). The most familiar unit of angle measurement is the degree. One degree is 1/360 of a circular rotation, so a complete circular rotation contains 360 degrees. Then, a right angle is 1/4 turn or 90° and a straight angle is 180°.

Consider right triangles, which are triangles where two sides are perpendicular and form a 90° angle (Figure 5.1). Since similar triangles have proportional sides, these ratios depend solely on the angle θ, regardless of the triangle's size.

$$\frac{\text{Opposite}}{\text{Hypotenuse}}, \qquad \frac{\text{Adjacent}}{\text{Hypotenuse}}, \qquad \frac{\text{Opposite}}{\text{Adjacent}}$$

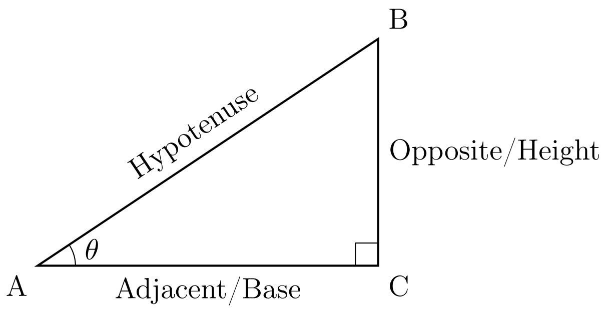

Figure 5.1: A right triangle.

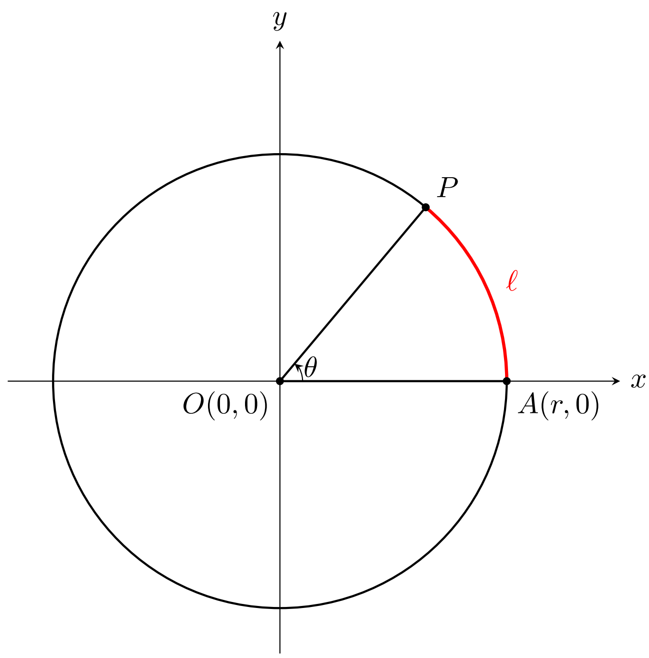

Figure 5.2: The angle θ is ℓ/r in radians.

Degree	$\cdots$	$-30°$	0	$30°$	$45°$	$60°$	$90°$	$180°$	$360°$	$390°$	$\cdots$
Radian	$\cdots$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π	$\frac{13\pi}{6}$	$\cdots$

Table 5.1: Degree to radian conversion.

Thus we can define the sine, cosine, and tangent values for their corresponding angles θ:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}.$$

5.1.2 Radians

In order to extend the previous definitions of sine, cosine and tangent to functions defined over real numbers, we need a different, more natural unit of angle measurement, called radian. The angle at the center of a circle in a plane that is subtended by an arc is, when measured in radians, equal to the ratio of the arc length ℓ to the radius of the circle r (Figure 5.2).

$$\theta = \frac{\ell}{r}$$

By convention, counterclockwise rotations are positive, while clockwise rotations are negative. Since a point can travel infinitely along the circle's circumference in either direction, the arc length s (and thus θ) can span the entire set of real numbers, $\mathbb{R}$. See Table 5.1.

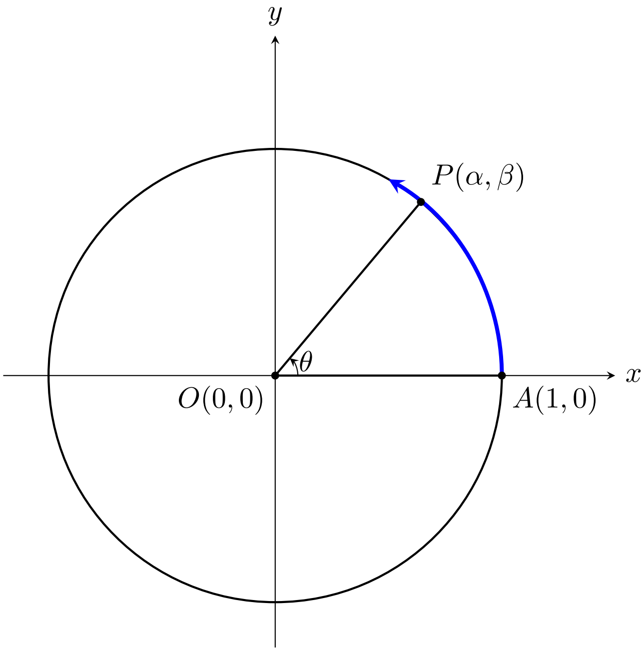

Figure 5.3: Point P rotating around the circumference of a unit circle.

5.1.3 Unit circle

Now to define sine and cosine as functions with domain $\mathbb{R}$, we let $P(\alpha, \beta)$ be a point on the unit circle

$$x^2 + y^2 = 1$$

on the coordinate plane moving counterclockwise starting from $A(1, 0)$, and θ be the angle between OA and OP (Figure 5.3). Then we define $\cos\theta$ and $\sin\theta$ using the x and y coordinates of P:

$$\cos\theta = \alpha \quad \text{and} \quad \sin\theta = \beta$$

This gives us the cosine function

$$\cos : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \cos x$$

and the sine function

$$\sin : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin x.$$

Remark 5.1. *By convention, we write* $\sin x$ *and* $\cos x$ *in place of* $\sin(x)$ *and* $\cos(x)$*. Also* $\sin^k x$ *and* $\cos^k x$ *are used to denote* $(\sin x)^k$ *and* $(\cos x)^k$*.*

Then, since P is on the unit circle, their coordinates are between -1 and 1,

$$-1 \leq \sin\theta, \cos\theta \leq 1$$

and

$$\cos^2\theta + \sin^2\theta = 1 \tag{5.1}$$

for all $\theta \in \mathbb{R}$. Thus

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}, \quad \sin\theta = \pm\sqrt{1-\cos^2\theta}.$$

We further define the tangent function

$$\tan x = \frac{\sin x}{\cos x}$$

and then the secant, cosecant and cotangent functions

$$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$

By dividing both sides of (5.1) by $\cos^2\theta$ and $\sin^2\theta$ respectively, we obtain

$$1+\tan^2\theta = \sec^2\theta \quad \text{and} \quad \cot^2\theta + 1 = \csc^2\theta.$$

Example 5.2. 1. *Compute the following values*

(a) $\sin x$ *and* $\cos x$ *for* $x = 0, \pi/2, \pi, 3\pi/2$ *and* 2π

(b) $\sin(7\pi/6)$, $\cos(7\pi/6)$

(c) $\sin(5\pi/4)$, $\cos(5\pi/4)$

(d) $\sin(-\pi/3)$, $\cos(-\pi/3)$

(e) $\sin(-5\pi/6)$, $\cos(-5\pi/6)$

2. *Solve the following equations:*

(a) $2\sin x - 1 = 0$

(b) $\tan^2 x - 3 = 0$

(c) $2\cos^2 x - 7\cos x + 3 = 0$

(d) $\sqrt{2}\sin x\cos x + \cos x = 0$

3. *Verify the following identities:*

(a) $\sin x/\cos x + \cos x/(1+\sin x) = \sec x$

(b) $\cos x(\sec x - \cos x) = \sin^2 x$

(c) $2\tan x\sec x = 1/(1-\sin x) - 1/(1+\sin x)$

(d) $\cos x/(1-\sin x) = \sec x + \tan x$

5.1.4 Graphs of trigonometric functions

We can draw the graphs of $y = \cos x$ and $y = \sin x$ by recording the x and y coordinates of P turning around the unit circle. See the animation in Wikipedia: Sine and Cosine.

From the graphs (Figure 5.4), we see that

$$\cos(-x) = \cos x, \quad \sin(-x) = -\sin x, \quad \tan(-x) = -\tan x.$$

Because we can obtain each graph by shifting itself horizontally,

$$\cos(x-2\pi) = \cos x, \quad \sin(x-2\pi) = \sin x, \quad \tan(x-\pi) = \tan x$$

and similarly

$$\cos(x-\pi/2) = \sin x, \quad \sin(x+\pi/2) = \cos x.$$

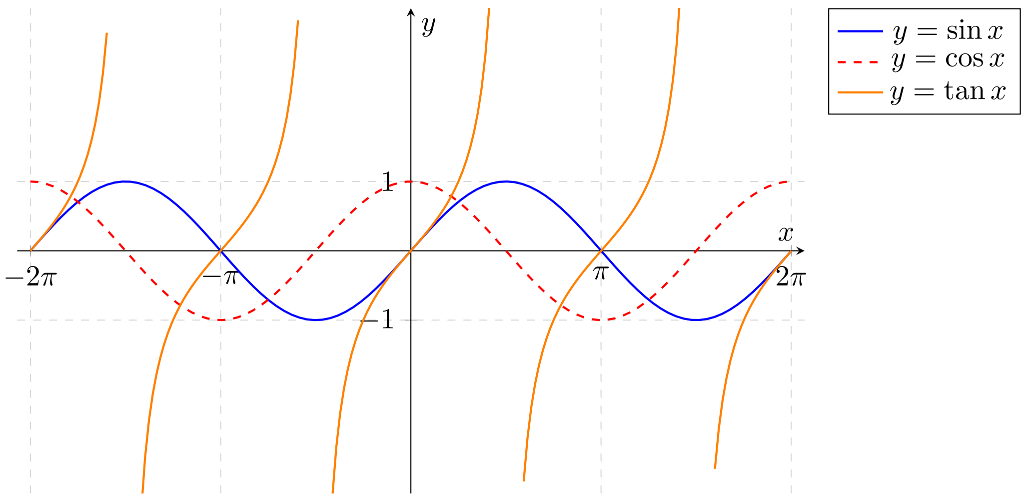

Figure 5.4: The graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$.

5.1.5 Inverse trigonometric functions

As a function from $[0, \pi]$ to $[-1, 1]$, $y = \cos x$ is invertible; as a function from $[-\pi/2, \pi/2]$ to $[-1, 1]$, $y = \sin x$ is invertible; as a function from $(-\pi/2, \pi/2)$ to $\mathbb{R}$, $y = \tan x$ is invertible. We define their inverse functions

$$y = \arccos x, \quad y = \arcsin x, \quad y = \arctan x$$

We also denote them by

$$y = \cos^{-1} x, \quad y = \sin^{-1} x, \quad y = \tan^{-1} x.$$

Example 5.3. *Compute the following values.*

1. $\sin^{-1}(\sqrt{2}/2)$
2. $\cos^{-1}(1/2)$
3. $\tan^{-1}(-\sqrt{3})$
4. $\cos^{-1}(\sin(\pi/3))$
5. $\sin^{-1}(\cos(-\pi/2))$
6. $\cos(\sin^{-1}(3/5))$
7. $\cos(\tan^{-1}(4/3))$
8. $\sin(\cos^{-1}(3/5))$
9. $\tan(\sin^{-1}(x-1))$
10. $\cos(\tan^{-1}(3x-1))$

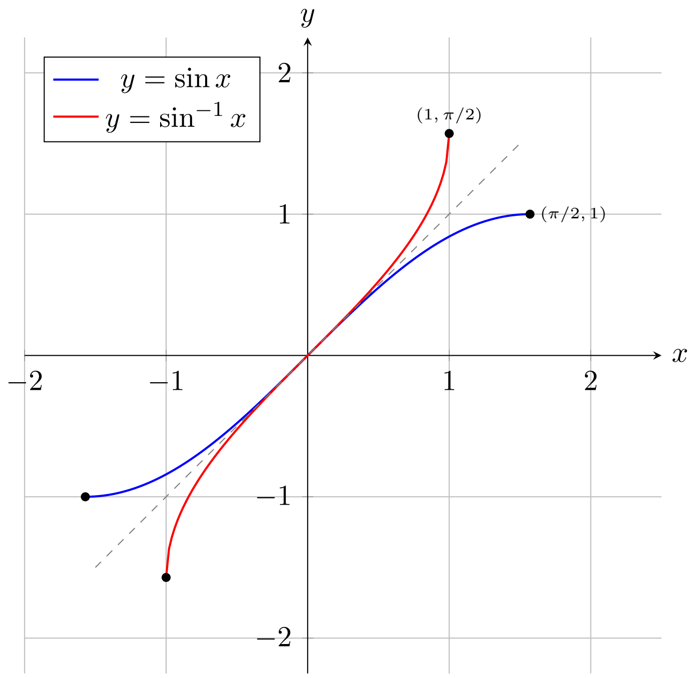

Figure 5.5: The graphs of $y = \sin x$ and $y = \sin^{-1} x$.

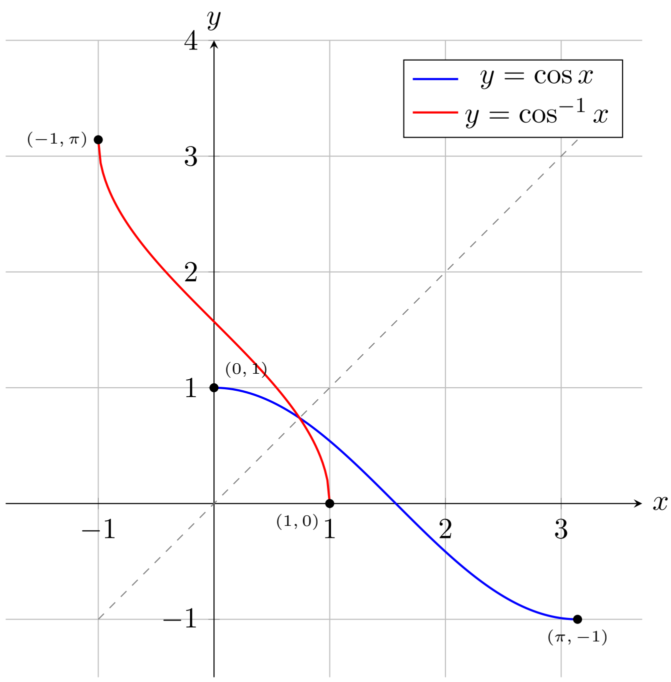

Figure 5.6: The graphs of $y = \cos x$ and $y = \cos^{-1} x$.

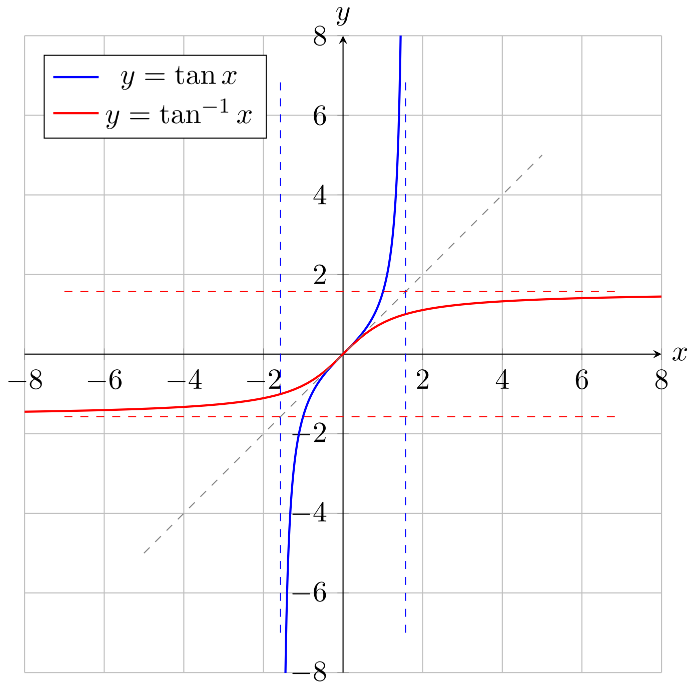

Figure 5.7: The graphs of $y = \tan x$ and $y = \tan^{-1} x$.

5.2 Trigonometric identities

Here is a list of useful identities.

1. Angle sum and difference formulas

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta\\ \cos(\alpha + \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta\\ \tan(\theta \pm \phi) &= \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}\end{aligned}$$

2. Double angle formulas

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ \sin 2\theta &= 2\sin\theta\,\cos\theta\\ \tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

3. Triple angle formulas

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) = 4\cos^3\theta - 3\cos\theta\\ \sin 3\theta &= \sin(2\theta + \theta) = 3\sin\theta - 4\sin^3\theta\\ \tan 3\theta &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\end{aligned}$$

4. Half angle formulas

$$\sin^2 \frac{\varphi}{2} = (1-\cos\varphi)/2$$
$$\cos^2 \frac{\varphi}{2} = (1+\cos\varphi)/2$$
$$\tan^2 \frac{\varphi}{2} = \frac{1-\cos\varphi}{1+\cos\varphi}$$

5. Product-to-sum formulas

$$\begin{aligned} \cos\theta\,\cos\varphi &= \tfrac{1}{2}\big(\cos(\theta-\varphi)+\cos(\theta+\varphi)\big) \\ \sin\theta\,\sin\varphi &= \tfrac{1}{2}\big(\cos(\theta-\varphi)-\cos(\theta+\varphi)\big) \\ \sin\theta\,\cos\varphi &= \tfrac{1}{2}\big(\sin(\theta+\varphi)+\sin(\theta-\varphi)\big) \\ \cos\theta\,\sin\varphi &= \tfrac{1}{2}\big(\sin(\theta+\varphi)-\sin(\theta-\varphi)\big) \end{aligned}$$

6. Sum-to-product formulas

$$\begin{aligned} \sin\theta+\sin\varphi &= 2\sin\tfrac{1}{2}(\theta+\varphi)\,\cos\tfrac{1}{2}(\theta-\varphi) \\ \sin\theta-\sin\varphi &= 2\cos\tfrac{1}{2}(\theta+\varphi)\,\sin\tfrac{1}{2}(\theta-\varphi) \\ \cos\theta+\cos\varphi &= 2\cos\tfrac{1}{2}(\theta+\varphi)\,\cos\tfrac{1}{2}(\theta-\varphi) \\ \cos\theta-\cos\varphi &= -2\sin\tfrac{1}{2}(\theta+\varphi)\,\sin\tfrac{1}{2}(\theta-\varphi) \end{aligned}$$

Example 5.4. *Verify the following identities using suggested formulas.*

1. $\cos(\pi/2 - x) = \sin x$, *using an angle different formula.*

2. $(1+\tan x)/(1-\tan x) = \tan(\pi/4 + x)$, *using an angle sum formula.*

3. $\cos 3x = 4\cos^3 x - 3\cos x$, *using addition and double angle formulas.*

4. $(\sin 3x - \sin x)/(\cos 3x + \cos x) = \tan x$, *using a sum-to-product formula.*

5.3 Matrix function and rotation†

So far we studied functions from $\mathbb{R}$ (or its subset) to $\mathbb{R}$. We can also consider functions from $\mathbb{R}^2$ to $\mathbb{R}^2$, sending a point (x, y) to a point (x', y') on the coordinate plane.

5.3.1 Matrix functions

We write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for the function sending a point (x, y) to $(ax+by, cx+dy)$. For example, when $a = 0, b = 1, c = 1, d = 0$, this function sends (x, y) to (y, x).

It is often clearer to use vector notation.

$$A : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{bmatrix} x \\ y \end{bmatrix} \mapsto A \begin{bmatrix} x \\ y \end{bmatrix}$$

Then,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

When $\det(A) = ad - bc \neq 0$, such a function is one-to-one and onto, and its inverse function is

$$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}.$$

5.3.2 Rotation matrix and angle sum formulas

In particular, the following function

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

sends a point (x, y) to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} (\cos\theta)x - (\sin\theta)y \\ (\sin\theta)x + (\cos\theta)y \end{bmatrix}.$$

Its geometric meaning is the rotation of points on the coordinate plane counterclockwise through an angle θ about the origin. Note that

$$\det(A_\theta) = \cos^2\theta + \sin^2\theta = 1$$

for all θ, and the inverse function is given by

$$A_{-\theta} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

We have

$$\begin{aligned} A_\theta \left(A_\phi \begin{bmatrix} x \\ y \end{bmatrix} \right) &= A_\theta \begin{bmatrix} \cos\phi\, x - \sin\phi\, y \\ \sin\phi\, x + \cos\phi\, y \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta(\cos\phi\, x - \sin\phi\, y) - \sin\theta(\sin\phi\, x + \cos\phi\, y) \\ \sin\theta(\cos\phi\, x - \sin\phi\, y) + \cos\theta(\sin\phi\, x + \cos\phi\, y) \end{bmatrix} \\ &= \begin{bmatrix} (\cos\theta\cos\phi - \sin\theta\sin\phi)\, x - (\cos\theta\sin\phi + \sin\theta\cos\phi)\, y \\ (\sin\theta\cos\phi + \cos\theta\sin\phi)\, x + (-\sin\theta\sin\phi + \cos\theta\cos\phi)\, y \end{bmatrix}, \end{aligned}$$

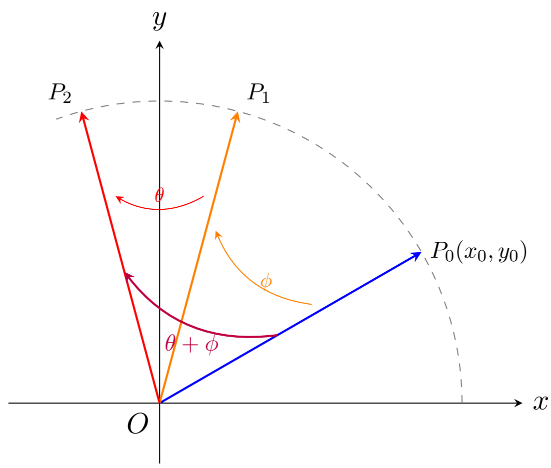

Figure 5.8: A two-step transformation: a first rotation of ϕ reaching P_1, followed by a second rotation of θ reaching P_2. The cumulative arc demonstrates that the composition of these two rotations is equivalent to a single rotation through an angle of $\phi + \theta$.

which should be equal to

$$A_{\theta+\phi}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi)\,x - \sin(\theta+\phi)\,y \\ \sin(\theta+\phi)\,x + \cos(\theta+\phi)\,y \end{bmatrix}$$

for all (x, y) on the coordinate plane (see Figure 5.8).

Therefore, we have the **angle sum formulas**:

1. $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$
2. $\sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi$

For the tangent function,

$$\tan(\theta+\phi) = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} = \frac{\cos\theta\sin\phi + \sin\theta\cos\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

and then by dividing the numerator and denominator by $\cos\theta\cos\phi$, we obtain

$$\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

The **angle difference formulas** can be derived from the angle sum identities by applying

$$\cos(-\phi) = \cos\phi, \qquad \sin(-\phi) = -\sin\phi, \qquad \tan(-\phi) = -\tan\phi.$$

5.3.3 Double, half and triple angle formulas

By setting $\phi = \theta$ in the angle sum identities, we obtain the **double angle formulas**:

1. $\cos 2\theta = \cos^2\theta - \sin^2\theta$.
2. $\sin 2\theta = 2\sin\theta\,\cos\theta$.

For the tangent function,

$$\begin{aligned}\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} &= \frac{2\sin\theta\,\cos\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{2\,\frac{\sin\theta}{\cos\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}} = \frac{2\tan\theta}{1-\tan^2\theta}.\end{aligned}$$

Rewriting the cosine double angle formula using $\cos^2\theta + \sin^2\theta = 1$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 1 - 2\sin^2\theta\\ &= 2\cos^2\theta - 1,\end{aligned}$$

we have

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

and then, by setting $\varphi = 2\theta$, we obtain the **half angle formula**:

1. $\sin^2\frac{\varphi}{2} = (1-\cos\varphi)/2$.
2. $\cos^2\frac{\varphi}{2} = (1+\cos\varphi)/2$.

For the tangent function,

$$\tan^2\frac{\varphi}{2} = \frac{\sin^2\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}} = \frac{1-\cos\varphi}{1+\cos\varphi}.$$

Using the angle sum identities and double angle formulas, we obtain the **triple angle formulas**:

1. $\cos 3\theta = \cos(2\theta+\theta) = 4\cos^3\theta - 3\cos\theta$.
2. $\sin 3\theta = \sin(2\theta+\theta) = 3\sin\theta - 4\sin^3\theta$.

For the tangent function,

$$\begin{aligned}\tan 3\theta = \tan(2\theta+\theta) &= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\,\tan\theta}\\ &= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\tan\theta} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.\end{aligned}$$

Chapter 6

Differentiation

We study differentiation of functions and their applications.

6.1 Derivative and tangent line

The average rate of change of $y = f(x)$ on interval $[a, b]$ given in (1.4) can be rewritten as

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

using $h = b - a$. The "instantaneous rate of change" of $y = f(x)$ at $x = a$ is measured by its limit as $h \to 0$.

The derivative of a function $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 6.1. *Compute the derivative of the following function at a given point.*

1. $y = mx + b$ *at* $x = a$.
2. $y = 2x^2 - 3x + 5$ *at* $x = -1, 0, 1, 2$.
3. $y = (3 + x)/(2 - x)$ *at* $x = a$.
4. $y = 9\sqrt{x}$ *at* $x = 9$.

The tangent line to $y = f(x)$ at $x = a$ is the line passing through $(a, f(a))$ on the graph of $y = f(x)$ with the same rate of change at $x = a$. Thus its equation is

$$y = f'(a)(x - a) + f(a).$$

Example 6.2. *Find the equation of a line tangent to the graph of the following function at a given point.*

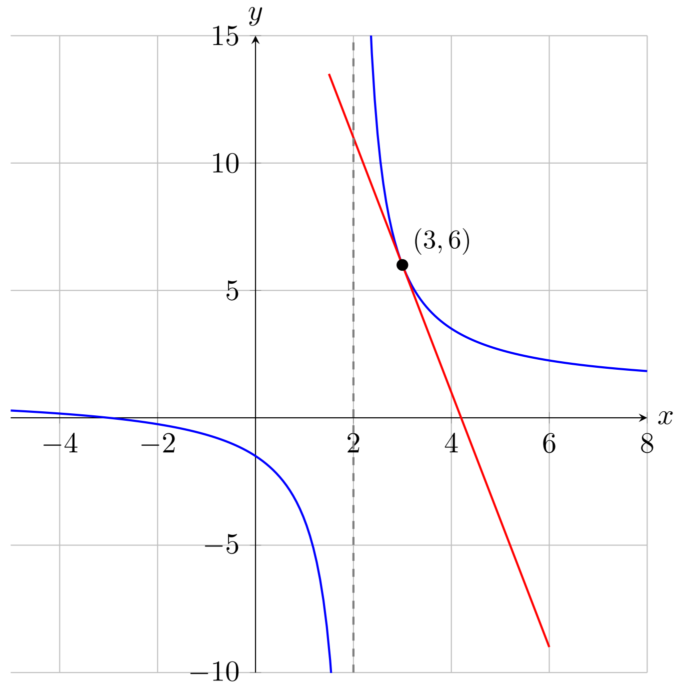

Figure 6.1: The graph of $y = \frac{3+x}{2-x}$ and its tangent line at $x = 3$.

1. $y = 5x^2 - x + 4$ *at* $x = 2$.
2. $y = -16x^2 + 36x + 200$ *at* $x = 2$.
3. $y = (3 + x)/(2 - x)$ *at* $x = 3$.
4. $y = 9\sqrt{x}$ *at* $x = 9$.

6.2 Derivative rules

The <u>derivative</u> as a function, denoted $f'(x)$ or $\frac{df}{dx}$ or $\frac{dy}{dx}$, is a new function that gives the instantaneous rate of change (or slope of the tangent line) of the original function $y = f(x)$ at any point x. It can be obtained by combining the following rules:

1. $(cg)'(x) = cg'(x)$
2. $(g \pm h)'(x) = g'(x) \pm h'(x)$
3. $(gh)'(x) = g'(x)h(x) + g(x)h'(x)$
4. $(\frac{g}{h})'(x) = \frac{g'(x)h(x)-g(x)h'(x)}{h(x)^2}$
5. $(g \circ h)'(x) = g'(h(x))h'(x)$ (the chain rule)

and formulas:

1. $\frac{d\,c}{dx} = 0$.
2. $\frac{d\,x^r}{dx} = rx^{r-1}$.
3. $\frac{d\,a^x}{dx} = a^x \ln a$, thus $\frac{d\,e^x}{dx} = e^x$.
4. $\frac{d\,\log_a x}{dx} = \frac{1}{x \ln a}$, thus $\frac{d\,\ln x}{dx} = \frac{1}{x}$ for $x > 0$.
5. $\frac{d\,\sin x}{dx} = \cos x$.
6. $\frac{d\,\cos x}{dx} = -\sin x$.
7. $\frac{d\,\tan x}{dx} = \sec^2 x$.

Examples

1. Find an equation of the tangent line to the curve $y = \frac{x^2-1}{x^2+1}$ at the point $(1, 0)$.

   Answer: First, let us compute the derivative: using the quotient rule,

$$y' = \frac{2x(x^2+1) - (x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}.$$

   Therefore, at $x = 1$, $m = y'(1) = \frac{4}{4} = 1$, and the tangent line is $y = 1(x-1) + 0$, or $y = x - 1$.

2. Find the x-coordinate(s) of all points on the graph of $f(x) = \sqrt{2x+1}$ where the tangent line is parallel to the line $x - 3y = 6$.

   Answer: By rewriting $x - 3y = 6$, we have $y = \frac{1}{3}x + \frac{6}{2}$ and thus the line has slope $m = \frac{1}{3}$. The derivative of f is, by the chain rule,

$$f'(x) = \frac{d\,(2x+1)^{\frac{1}{2}}}{dx} = \frac{1}{2}(2x+1)^{\frac{1}{2}-1}\,2 = (2x+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{2x+1}}.$$

   Now, setting

$$f'(x) = \frac{1}{\sqrt{2x+1}} = \frac{1}{3}$$

   we have $\sqrt{2x+1} = 3$. Therefore, $2x + 1 = 9$, or $x = 4$.

3. Consider the function $f(x) = x^2 e^{-x}$. Find the coordinates of all points on the graph of f where the tangent line is horizontal.

   Answer: First, we compute the derivative: using the product rule and the chain rule,

$$f'(x) = 2xe^{-x} - x^2 e^{-x} = xe^{-x}(2 - x).$$

   Then, setting $f'(x) = 0$ we obtain $x = 0$ or $x = 2$. Since $f(0) = 0$ and $f(2) = 4e^{-2}$, we have $(0, 0)$ and $(2, 4e^{-2})$.

4. Find an equation of the line tangent to the curve $y = \ln(x^2 - 3)$ at the point where $x = 2$.

   Answer: Let us compute the derivative: using the chain rule,
$$y' = \frac{d\ln(x^2-3)}{dx} = \frac{2x}{x^2-3}.$$
   Therefore, $m = y'(2) = \frac{4}{1} = 4$. At $x = 2$, $y = \ln(4-3) = 0$. The tangent line is $y = 4(x-2) + 0$, or $y = 4x - 8$.

5. Find an equation of the tangent line to the curve $f(x) = \tan(2x)$ at $x = \frac{\pi}{8}$.

   Answer: Let us compute the derivative: using the chain rule
$$f'(x) = 2\sec^2(2x).$$
   Therefore, $m = f'(\frac{\pi}{8}) = 2\sec^2(\frac{\pi}{4}) = 2(\sqrt{2})^2 = 4$. At $x = \frac{\pi}{8}$, $y = \tan(\frac{\pi}{4}) = 1$. The tangent line is $y = 4(x - \frac{\pi}{8}) + 1$, or $y = 4x + 1 - \frac{\pi}{2}$.

6. Determine the values of x for which the rate of change of $g(x) = x\ln x - x$ is equal to 2.

   Answer: Let us compute the derivative: using the product rule,
$$g'(x) = \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) - 1 = \ln x.$$
   Setting $g'(x) = 2$, we have $\ln x = 2$, and thus $x = e^2$.

7. Find an equation of the tangent line to the curve $y = e^x \cos x$ at the y-intercept $(0, 1)$.

   Answer: Let us compute the derivative. By the product rule,
$$y' = e^x \cos x - e^x \sin x = e^x(\cos x - \sin x).$$
   At $x = 0$, $m = y'(0) = 1(1 - 0) = 1$. Thus the tangent line is $y = 1(x-0) + 1$, or $y = x + 1$.

8. Find the equation of the line tangent to $y = \frac{e^x}{2x+1}$ at $x = 0$.

   Answer: The derivative is, by the quotient rule,
$$y' = \frac{e^x(2x+1) - e^x(2)}{(2x+1)^2} = \frac{e^x(2x-1)}{(2x+1)^2}.$$
   At $x = 0$, $m = \frac{e^0(-1)}{1} = -1$ and $y = \frac{e^0}{1} = 1$. Therefore, the tangent line is $y = -1(x-0) + 1$, or $y = -x + 1$.

9. Find all points on the interval $[0, 2\pi]$ where the tangent line to $f(x) = 2\sin x + \cos^2 x$ is horizontal.

   Answer: The derivative is, by the chain rule,
$$f'(x) = 2\cos x - 2\cos x \sin x = 2\cos x\,(1 - \sin x).$$
   From $f'(x) = 0$, we have $\cos x = 0$ or $\sin x = 1$. Therefore, $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

10. Find the point on the curve $y = \sqrt{x}$ where the tangent line passes through the point $(-1, 0)$.

    Answer: Let the unknown point on the curve be $(a, \sqrt{a})$. Then, the slope of tangent line at a is
    $$y'(a) = \frac{1}{2\sqrt{a}}.$$
    The slope of the line connecting $(-1, 0)$ and $(a, \sqrt{a})$ is
    $$\frac{\sqrt{a} - 0}{a - (-1)} = \frac{\sqrt{a}}{a+1}.$$
    Then, from
    $$\frac{1}{2\sqrt{a}} = \frac{\sqrt{a}}{a+1}$$
    we obtain $a + 1 = 2a$, or $a = 1$. Therefore, the point we are looking for is $(1, 1)$.

6.3 Maximum and minimum values

By the Extreme Value Theorem, a continuous function f on a closed interval $[a, b]$ is guaranteed to attain both an absolute maximum and an absolute minimum. Furthermore, by Fermat's Theorem, if f has a local extremum (maximum or minimum) at c and $f'(c)$ exists, then $f'(c) = 0$.

Intuitively, a local maximum occurs where a function stops increasing and begins decreasing—that is, where its derivative $f'(x)$ changes sign from positive to negative ($+$ to $-$). Conversely, a local minimum occurs where $f'(x)$ changes sign from negative to positive ($-$ to $+$). See Figure 6.2. For a function to change direction in this way, its slope must either momentarily hit zero or fail to exist entirely. This simple observation reveals exactly where to look: points where $f'(x) = 0$ or where $f'(x)$ is undefined are our primary candidates for local extrema. This insight leads directly to Fermat's Theorem and sets up our general strategy for finding maximum and minimum values.

> A number c in the open interval (a, b) is called a critical number of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Consequently, the local extrema—and by extension, the absolute extrema—can only occur at critical numbers or at the boundary endpoints. The values of f at these critical numbers serve as our prime candidates for the absolute maximum and minimum. To find the absolute maximum and minimum values of a continuous function f on $[a, b]$:

1. **Find the critical numbers:** Find all critical numbers of f in the open interval (a, b).

2. **Evaluate f at the critical numbers:** Calculate the function value $f(c)$ for each critical number c found in Step 1.

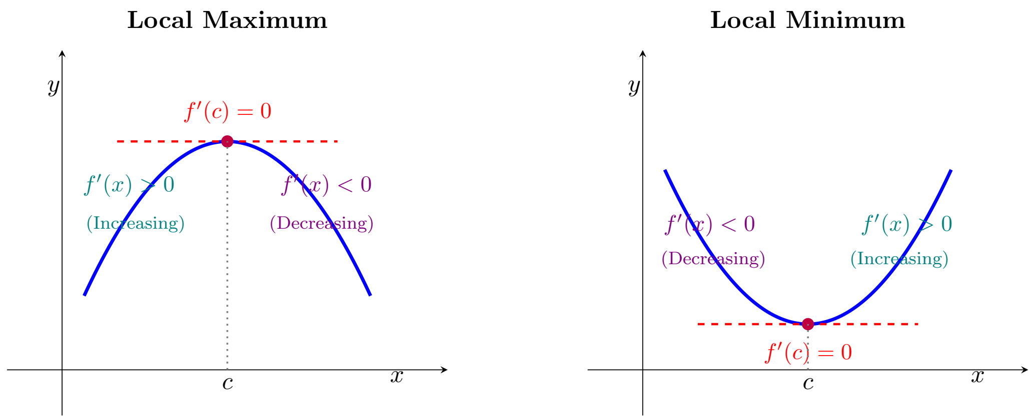

Figure 6.2: Sign change of $f'(x)$ at critical points: A local maximum occurs when f' changes from $+$ to $-$, and a local minimum occurs when f' changes from $-$ to $+$.

3. **Evaluate f at the endpoints:** Calculate the function values at the boundary endpoints: $f(a)$ and $f(b)$.

4. **Compare the values:** Compare all the function values collected in Steps 2 and 3:

   - The **largest** value is the **absolute maximum value** of f on $[a, b]$.
   - The **smallest** value is the **absolute minimum value** of f on $[a, b]$.

Examples

1. Find the absolute maximum and minimum values of $f(x) = x^3 - 3x^2 + 1$ on the interval $[-1, 3]$.

   Answer: To find its critical numbers,

   $$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0.$$

   It give us $x = 0, 2 \in (-1, 3)$. Then, we evaluate $f(-1) = -3$, $f(0) = 1$, $f(2) = -3$, and $f(3) = 1$. Therefore, we have Absolute Max: 1 (at $x = 0, 3$), Absolute Min: -3 (at $x = -1, 2$).

x	-1	$(-1, 0)$	0	$(0, 2)$	2	$(2, 3)$	3
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	-3	$\nearrow$	1	$\searrow$	-3	$\nearrow$	1

Table 6.1: First derivative test and interval behavior for $f(x) = x^3 - 3x^2 + 1$.

2. Find the absolute maximum and minimum values of $f(x) = x^4 - 2x^2 + 3$ on the interval $[-2, 2]$.

   Answer: To find its critical numbers, we need to solve

   $$f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0$$

It gives us $x = -1, 0, 1 \in (-2, 2)$. Next, let us evaluate $f(-2) = 11$, $f(-1) = 2$, $f(0) = 3$, $f(1) = 2$, and $f(2) = 11$. Therefore, we have Absolute Max: 11 (at $x = \pm 2$), Absolute Min: 2 (at $x = \pm 1$).

x	-2	$(-2,-1)$	-1	$(-1,0)$	0	$(0,1)$	1	$(1,2)$	2
$f'(x)$	$-$	$-$	0	$+$	0	$-$	0	$+$	$+$
$f(x)$	11	$\searrow$	2	$\nearrow$	3	$\searrow$	2	$\nearrow$	11

Table 6.2: First derivative test and interval behavior for $f(x) = x^4 - 2x^2 + 3$.

3. Find the absolute extrema of $f(x) = \frac{x}{x^2+1}$ on the interval $[0, 2]$.

   Answer: To find its critical numbers, we need to solve

   $$f'(x) = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0$$

   It gives us $x = 1 \in (0, 2)$. Then, let us evaluate: $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(2) = \frac{2}{5}$. $\therefore$ Absolute Max: $\frac{1}{2}$ (at $x = 1$), Absolute Min: 0 (at $x = 0$).

4. Find the absolute maximum and minimum values of $f(x) = x\sqrt{4-x^2}$ on the interval $[-1, 2]$.

   Answer: To find its critical numbers, we need to solve

   $$f'(x) = \sqrt{4-x^2} + x \cdot \frac{-x}{\sqrt{4-x^2}} = \frac{4-2x^2}{\sqrt{4-x^2}} = 0$$

   It gives $x = \sqrt{2} \in (-1, 2)$. Next, we evaluate: $f(-1) = -\sqrt{3}(\approx -1.732)$, $f(\sqrt{2}) = 2$, and $f(2) = 0$. $\therefore$ Absolute Max: 2 (at $x = \sqrt{2}$), Absolute Min: $-\sqrt{3}$ (at $x = -1$).

5. Find the absolute maximum and minimum values of $f(x) = x^{2/3}(5-x)$ on the interval $[-1, 4]$.

   Answer: The derivative is

   $$f'(x) = \frac{10}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{5(2-x)}{3x^{1/3}}.$$

   Then, from $f'(x) = 0$, we have $x = 2$; $f'(x)$ does not exist at $x = 0$. Both are in $(-1, 4)$. Let us evaluate $f(-1) = 6$, $f(0) = 0$, $f(2) = 3\sqrt[3]{4}(\approx 4.76)$, and $f(4) = \sqrt[3]{16}(\approx 2.52)$. $\therefore$ Absolute Max: 6 (at $x = 1$), Absolute Min: 0 (at $x = 0$).

6. Find the absolute maximum and minimum values of $f(x) = 2\cos x + \sin(2x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

   Answer: Let us compute the derivative:

   $$f'(x) = -2\sin x + 2\cos(2x) = -2\sin x + 2(1 - 2\sin^2 x) = -2(2\sin x - 1)(\sin x + 1).$$

   Then, on $\left(0, \frac{\pi}{2}\right)$, we have $f'(x) = 0$ only when $x = \frac{\pi}{6}$. Next, we evaluate $f(0) = 2$, $f\left(\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}(\approx 2.598)$, and $f\left(\frac{\pi}{2}\right) = 0$. $\therefore$ Absolute Max: $\frac{3\sqrt{3}}{2}$ (at $x = \frac{\pi}{6}$), Absolute Min: 0 (at $x = \frac{\pi}{2}$).

7. Find the absolute extrema of $f(x) = xe^{-2x}$ on the interval $[0, 1]$.

   Answer: The derivative is

   $$f'(x) = e^{-2x} - 2xe^{-2x} = e^{-2x}(1 - 2x).$$

   Thus, $f'(x) = 0$ gives $x = \frac{1}{2} \in (0, 1)$. Let us evaluate $f(0) = 0$, $f\left(\frac{1}{2}\right) = \frac{1}{2e}(\approx 0.184)$, and $f(1) = \frac{1}{e^2}(\approx 0.135)$. $\therefore$ Absolute Max: $\frac{1}{2e}$ (at $x = \frac{1}{2}$), Absolute Min: 0 (at $x = 0$).

8. Find the absolute maximum and minimum values of $f(x) = \frac{\ln x}{x}$ on the interval $[1, e^2]$.

   Answer: To find its critical numbers,

   $$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2} = 0,$$

   which gives $\ln x = 1$, or $x = e \in (1, e^2)$. Then, let us evaluate $f(1) = 0$, $f(e) = \frac{1}{e}(\approx 0.368)$, and $f(e^2) = \frac{2}{e^2}(\approx 0.271)$. $\therefore$ Absolute Max: $\frac{1}{e}$ (at $x = e$), Absolute Min: 0 (at $x = 1$).

9. Find the absolute maximum and minimum values of

   $$f(x) = \begin{cases} 4 - x^2 & \text{if } -2 \le x < 1 \\ 2x + 1 & \text{if } 1 \le x \le 3 \end{cases}$$

   on the closed interval $[-2, 3]$.

   Answer: Piece 1 gives $f'(x) = -2x = 0 \implies x = 0 \in (-2, 1)$. Piece 2 gives $f'(x) = 2 \neq 0$. Let us check boundary point $x = 1$. Then, we evaluate candidate values: $f(-2) = 0$, $f(0) = 4$, $f(1) = 3$, and $f(3) = 7$. $\therefore$ Absolute Max: 7 (at $x = 3$), Absolute Min: 0 (at $x = -2$).

10. A particle moves along a straight line with position function $s(t) = 2t^3 - 9t^2 + 12t + 1$ for $0 \le t \le 3$. Find the particle's maximum and minimum distance from the origin during this time interval.

    Answer: The derivative is

    $$s'(t) = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2) = 6(t - 1)(t - 2)$$

    Thus, $s'(t) = 0$ gives $t = 1, 2 \in (0, 3)$. Now we evaluate position: $s(0) = 1$, $s(1) = 6$, $s(2) = 5$, and $s(3) = 10$. $\therefore$ Maximum distance: 10 (at $t = 3$), Minimum distance: 1 (at $t = 0$).

Chapter 7

Integration

We study integration of functions and their applications.

7.1 Definite integrals and area

The definite integral of a function f over a closed interval $[a, b]$

$$\int_a^b f(x)\,dx$$

represents the net signed area bounded by the curve $y = f(x)$ and the x-axis for $a \le x \le b$ (Figure 7.1), and it is defined as the limit of a Riemann sum (Figure 7.2).

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\frac{b-a}{n}.$$

In practice, however, we compute it using an antiderivative of f, that is, $y = F(x)$ defined on $[a, b]$ such that $F'(x) = f(x)$ for all x in (a, b).

Theorem 7.1. *Let f be a continuous function on a closed interval $[a, b]$. If F is an an antiderivative of f, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Examples

Evaluate the following definite integrals.

1. $\int_{-1}^{1} x^3 - 2x^2 + 4x\,dx$

   Answer: We can find the antiderivative using the Power Rule:

$$\left(\frac{x^4}{4} - \frac{2x^3}{3} + 2x^2\right)' = x^3 - 2x^2 + 4x.$$

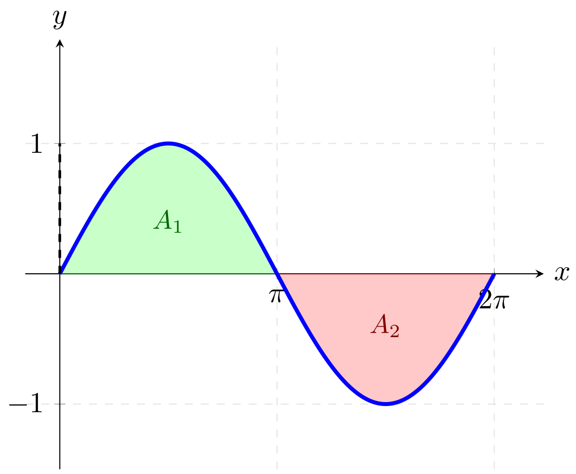

Figure 7.1: $\int_0^{2\pi} \sin x \, dx = A_1 - A_2$.

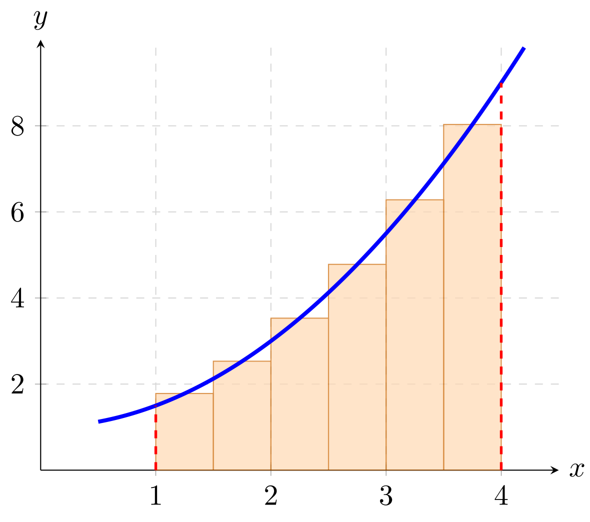

Figure 7.2: A Riemann sum for $\int_1^4 \frac{x^2+2}{2} \, dx$.

Apply the Fundamental Theorem of Calculus:

$$\left[\frac{x^4}{4} - \frac{2x^3}{3} + 2x^2\right]_{-1}^{1} = \left(\frac{1}{4} - \frac{2}{3} + 2\right) - \left(\frac{1}{4} + \frac{2}{3} + 2\right) = -\frac{4}{3}.$$

2. $\int_1^4 \sqrt{x} + \frac{1}{x}\, dx$

   Answer: We can rewrite $\sqrt{x}$ as $x^{1/2}$ and integrate:

$$\begin{aligned}\int_1^4 (x^{1/2} + x^{-1})\, dx &= \left[\frac{2}{3}x^{3/2} + \ln x\right]_1^4 \\ &= \left(\frac{2}{3}(4)^{3/2} + \ln 4\right) - \left(\frac{2}{3}(1)^{3/2} + \ln 1\right)\end{aligned}$$

   Since $4^{3/2} = 8$ and $\ln 1 = 0$,

$$= \left(\frac{16}{3} + \ln 4\right) - \frac{2}{3} = \frac{14}{3} + \ln 4.$$

3. $\int_2^4 \frac{x^2+3x+5}{x+1}\, dx$

   Answer: Use polynomial long division on the integrand:

$$\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}$$

   Now integrate term-by-term:

$$\int_2^4 \left(x + 2 + \frac{3}{x+1}\right) dx = \left[\frac{x^2}{2} + 2x + 3\ln(x+1)\right]_2^4$$

   Evaluate at the bounds $x = 4$ and $x = 2$:

$$= \left(\frac{16}{2} + 8 + 3\ln 5\right) - \left(\frac{4}{2} + 4 + 3\ln 3\right) = 10 + 3\ln\left(\frac{5}{3}\right).$$

Examples

Compute the area of the region enclosed by the given curves.

1. $y = x^3 - 5x^2 + 4x$ and the x-axis.

   Answer: Find the x-intercepts:

$$x^3 - 5x^2 + 4x = 0 \implies x(x-1)(x-4) = 0 \implies x = 0,\ 1,\ 4.$$

   Determine the sign of $f(x)$ on each sub-interval:

   - On $[0, 1]$, $f(x) \geq 0$ (above x-axis).
   - On $[1, 4]$, $f(x) \leq 0$ (below x-axis).

Find the antiderivative: $F(x) = \frac{x^4}{4} - \frac{5x^3}{3} + 2x^2$. Calculate the areas:

$$A_1 = \int_0^1 (x^3 - 5x^2 + 4x)\,dx = \left[\frac{x^4}{4} - \frac{5x^3}{3} + 2x^2\right]_0^1 = \frac{1}{4} - \frac{5}{3} + 2 = \frac{7}{12},$$
$$A_2 = (-1)\left(\int_1^4 (x^3 - 5x^2 + 4x)\,dx\right) = -\left(F(4) - F(1)\right) = (-1)\left(-\frac{32}{3} - \frac{7}{12}\right) = \frac{135}{12}.$$

Therefore, the total area is

$$A_1 + A_2 = \frac{7}{12} + \frac{135}{12} = \frac{142}{12} = \frac{71}{6}.$$

2. $y = \sin x$, $y = \cos x$, and the y-axis (See Figure 7.3).

   Answer: Find the first positive intersection point:

$$\sin x = \cos x \implies \tan x = 1 \implies x = \frac{\pi}{4}$$

   On $\left[0, \frac{\pi}{4}\right]$, $\cos x \geq \sin x$. Set up and evaluate the integral:

$$\begin{aligned}(Area) &= \int_0^{\pi/4} (\cos x - \sin x)\,dx = [\sin x + \cos x]_0^{\pi/4}.\\ &= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0) = \sqrt{2} - 1.\end{aligned}$$

3. $y = 2x$ and $y = x^2$.

   Answer: Find the intersection points by setting the equations equal:

$$x^2 = 2x \implies x^2 - 2x = 0 \implies x(x-2) = 0 \implies x = 0,\ x = 2.$$

   On $[0, 2]$, $2x \geq x^2$. Set up and evaluate the integral:

$$(Area) = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}.$$

4. $y = e^x$, $y = e^{-x}$, and $x = \ln 3$.

   Answer: Find the intersection point of $y = e^x$ and $y = e^{-x}$:

$$e^x = e^{-x} \implies e^{2x} = 1 \implies x = 0.$$

   On $[0, \ln 3]$, $e^x \geq e^{-x}$. Set up and evaluate the integral:

$$\begin{aligned}(Area) &= \int_0^{\ln 3} (e^x - e^{-x})\,dx = \left[e^x + e^{-x}\right]_0^{\ln 3}\\ &= \left(e^{\ln 3} + e^{-\ln 3}\right) - \left(e^0 + e^0\right) = \left(3 + \frac{1}{3}\right) - 2 = \frac{4}{3}.\end{aligned}$$

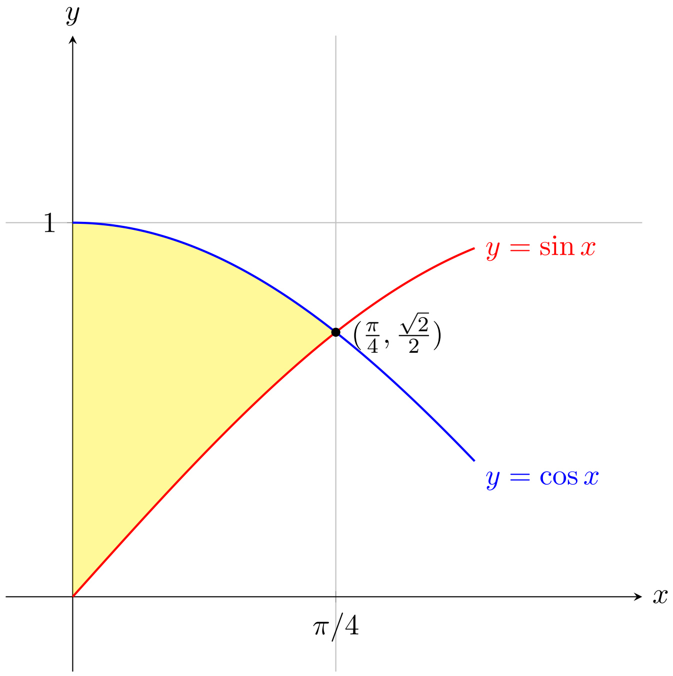

Figure 7.3: The region enclosed by $y = \sin x$, $y = \cos x$, and the y-axis.

7.2 Integration rules

In all formulas below, C represents an arbitrary constant of integration. The antiderivative of a function $f(x)$ is represented using the indefinite integral symbol:

$$\int f(x)\, dx = F(x) + C.$$

7.2.1 Fundamental Rules

$$\begin{aligned}
\int 0\, dx &= C \\
\int k\, dx &= kx + C \quad (k \text{ is a constant}) \\
\int kf(x)\, dx &= k \int f(x)\, dx \\
\int [f(x) \pm g(x)]\, dx &= \int f(x)\, dx \pm \int g(x)\, dx
\end{aligned}$$

7.2.2 Power Rules and Logarithmic Forms

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$
$$\int \frac{1}{x} \, dx = \ln|x| + C$$
$$\int \frac{g'(x)}{g(x)} \, dx = \ln|g(x)| + C$$

7.2.3 Exponential Functions

$$\int e^x \, dx = e^x + C$$
$$\int a^x \, dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$$

7.2.4 Trigonometric Functions

$$\int \sin x \, dx = -\cos x + C$$
$$\int \cos x \, dx = \sin x + C$$
$$\int \sec^2 x \, dx = \tan x + C$$
$$\int \tan x \, dx = \ln|\sec x| + C = -\ln|\cos x| + C$$

7.2.5 Core Integration Techniques

Integration by parts is a useful technique to compute the integral of a product of functions.

$$\int uv' \, dx = uv - \int u'v \, dx$$

Theorem 7.2. *For two continuously differentiable functions u and v,*

$$\int_a^b u(x)v'(x) \, dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x) \, dx$$
$$= u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x) \, dx.$$

Integration by substitution is a counterpart to the chain rule for differentiation.

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du \quad \text{where } u = g(x)$$

Theorem 7.3. *Let g be a differentiable function from $[a,b]$ to an interval I with a continuous derivative. If $f : I \to \mathbb{R}$ is a continuous function then*

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\ du.$$

Given $\int_a^b f(g(x))h(x)\, dx$ with $h(x) = cg'(x)$ for some constant c, we set $u = g(x)$ and then from

$$\frac{du}{dx} = g'(x)$$

we obtain $dx = 1/g'(x)\, du$. Thus, rewriting everything in terms of u, we have

$$\int_{g(a)}^{g(b)} f(u)\, h(x)\, \frac{1}{g'(x)}\, du = \int_{g(a)}^{g(b)} c\, f(u) du.$$

Examples

Evaluate the following definite integrals.

1. $\int_0^{\pi/2} \cos^2\theta\, d\theta$

   Answer: Using the identity $\cos 2\theta = 2\cos^2\theta - 1$, rearrange to get $\cos^2\theta = \frac{1+\cos 2\theta}{2}$,

   $$\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

   Evaluate at $\frac{\pi}{2}$ and 0

   $$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{2} + 0\right) = \frac{\pi}{4}.$$

2. $\int_0^{\pi/2} \frac{\sin x}{1+\cos x}\, dx$

   Answer: Let $u = 1 + \cos x$, so $\frac{du}{dx} = -\sin x$, or $\sin x\, dx = -du$. Update the integration interval: When $x = 0$, $u = 1+\cos 0 = 2$; When $x = \frac{\pi}{2}$, $u = 1+\cos\left(\frac{\pi}{2}\right) = 1$. Substitute into the integral:

   $$\int_2^1 \frac{-du}{u} = \int_1^2 \frac{1}{u}\, du = \Big[\ln u\Big]_1^2 = \ln 2 - \ln 1 = \ln 2.$$

3. $\int_0^1 xe^{4x^2+3}\, dx$

   Answer: Let $u = 4x^2 + 3$, so $\frac{du}{dx} = 8x$, or $x\, dx = \frac{1}{8}du$. Update the integration interval: When $x = 0$, $u = 4(0)^2 + 3 = 3$; When $x = 1$, $u = 4(1)^2 + 3 = 7$. Substitute into the integral:

   $$\int_3^7 \frac{e^u}{8}\, du = \frac{1}{8}\left[e^u\right]_3^7 = \frac{e^7 - e^3}{8}.$$

4. $\int_0^1 x^2 \cos(\frac{\pi x^3}{2})\, dx$

   Answer: Let $u = \frac{\pi x^3}{2}$, so $\frac{du}{dx} = \frac{3\pi}{2}x^2$, or $x^2\, dx = \frac{2}{3\pi}\, du$. Update the integration interval: When $x = 0$, $u = 0$; When $x = 1$, $u = \frac{\pi}{2}$. Substitute into the integral:

$$\int_0^{\pi/2} \cos u \left(\frac{2}{3\pi}\right) du = \frac{2}{3\pi}\Big[\sin u\Big]_0^{\pi/2} = \frac{2}{3\pi}\Big(\sin\frac{\pi}{2} - \sin 0\Big) = \frac{2}{3\pi}$$

5. $\int_0^{\pi/2} x \cos x\, dx$

   Answer: Use integration by parts with $u = x$ and $v' = \cos x$:

$$\begin{aligned} u = x \quad &\Longrightarrow \quad u' = 1 \\ v' = \cos x \quad &\Longrightarrow \quad v = \sin x \end{aligned}$$

   Apply $\int uv'\, dx = uv - \int u'v\, dx$

$$\int_0^{\pi/2} x\cos x\, dx = \Big[x\sin x\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin x\, dx$$
$$= \Big(\frac{\pi}{2}\sin\frac{\pi}{2} - 0\Big) - \Big[-\cos x\Big]_0^{\pi/2} = \frac{\pi}{2}(1) + \Big(\cos\frac{\pi}{2} - \cos 0\Big) = \frac{\pi}{2} - 1.$$

6. $\int_0^1 xe^{-2x}\, dx$

   Answer: Use integration by parts with $u = x$ and $v' = e^{-2x}$:

$$\begin{aligned} u = x \quad &\Longrightarrow \quad u' = 1 \\ v' = e^{-2x} \quad &\Longrightarrow \quad v = -\tfrac{1}{2}e^{-2x} \end{aligned}$$

   Apply $\int uv'\, dx = uv - \int u'v\, dx$

$$\int_0^1 xe^{-2x}\, dx = \left[-\frac{1}{2}xe^{-2x}\right]_0^1 - \int_0^1 \left(-\frac{1}{2}e^{-2x}\right) dx$$
$$= \left(-\frac{1}{2}e^{-2} - 0\right) - \left[\frac{1}{4}e^{-2x}\right]_0^1 = \frac{1}{4} - \frac{3}{4}e^{-2} = \frac{1 - 3e^{-2}}{4}.$$