Heine-Borel 정리

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Heine-Borel 정리란 유클리드 공간에서 어떤 집합이 compact이기 위한 조건이 그 집합이 closed and bounded이기 위한 조건과 동치라는 정리이다. 이 정리는 Rudin 책의 2.41 Theorem에 나타나 있다.

Theorem 2.41. *If a set E in R^k has one of the following three properties, then it has the other two :*

(a) E is closed and bounded.
(b) E is compact.
(c) Every infinite subset of E has a limit point in E.

Proof. If (a) holds, then $E \subset I$ for some k-cell, and (b) follows from Theorem 2.40 and 2.35. Theorem 2.37 shows that (b) implies (c). It remains to be shown that (c) implies (a).

If E is not bounded, then E contains points $\mathbf{x_n}$ with

$$|\mathbf{x_n}| > n \quad (n = 1, 2, 3, \ldots)$$

The set S consisting of $\mathbf{x_n}$ is infinite and clearly has no limit point in R^k, hence has none in E. Thus (c) implies that E is bounded.

If E is not closed, then there is a point $\mathbf{x_0} \in R^k$ which is a limit point of E but not a point of E. For $n = 1, 2, 3, \ldots$, there are points $\mathbf{x_n} \in E$ such that $|\mathbf{x_n} - \mathbf{x_0}| < 1/n$. Let S be the set of these $\mathbf{x_n}$. Then S is infinite(otherwise $|\mathbf{x_n} - \mathbf{x_0}|$ would have a constant positive value, for infinitely many n), S has x_0 as a limit point, and S has no other limit point in R^k. For if $\mathbf{y} \in R^k$, $\mathbf{y} \neq \mathbf{x_0}$, then

$$\begin{aligned}|\mathbf{x_n} - \mathbf{y}| &\geq |\mathbf{x_0} - \mathbf{y}| - |\mathbf{x_n} - \mathbf{x_0}| \\ &\geq |\mathbf{x_0} - \mathbf{y}| - \frac{1}{n} \geq \frac{1}{2}|\mathbf{x_0} - \mathbf{y}|\end{aligned}$$

for all but finitely many n ; this shows that $\mathbf{y}$ is not a limit point of S(Theorem 2.20).

Thus S has no limit point in E ; hence E must be closed if (c) holds. □

하지만 유클리드 공간이 아닐 경우 이 정리는 성립하지 않는다.

임의의 무한집합 K에 거리함수 d를

$$d(x,y) = \begin{cases} 1 & (x \neq y) \\ \\ 0 & (x = y) \end{cases}$$

로 주고, K의 무한부분집합 E를 생각하면 E는 closed이고 bounded이지만 compact는 아니다.