1/f Noise

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Abstract—1/f noise is a nonstationary random process suitable for modeling evolutionary or developmental systems. It combines the strong influence of past events on the future and, hence somewhat predictable behavior, with the influence of random events. Nonstationary autocorrelation functions for 1/f noise are developed to demonstrate that its present behavior is equally correlated with both the recent and distant past. The minimum amount of memory for a system that exhibits 1/f noise is shown to be one state variable per decade of frequency. The system condenses its past history into the present values of its state variables, one of which represents an average over the most recent 1 unit of time, one for the last 10 time units, 100 units, 1000, 10 000, and so on. Each such state variable has an equal influence on present behavior.

I. Introduction

THE $1/f$ noise is a random process [1] defined in terms of the shape of its power spectral density $S(f)$. The power or the square of some variable associated with the random process, measured in a narrow bandwidth, is roughly proportional to reciprocal frequency (Fig. 1):

$$S(f) = \frac{\text{constant}}{|f|^{\gamma}} \quad \text{where: } 0 < \gamma < 2$$

and γ is usually close to 1.

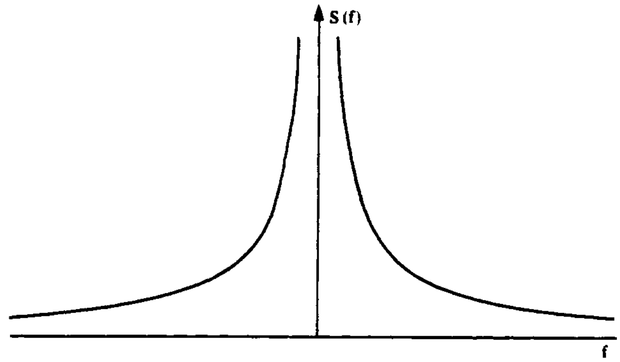

Fig. 1. The power spectral density of $1/f$ noise. When compared with white noise which has equal power at all frequencies, $1/f$ noise has an abundance at low frequencies.

$1/f$ was noticed first as an excess low-frequency noise in vaccuum tubes and then, much later, in semiconductors. Models of $1/f$ noise, based on detailed physical mechanisms, were developed by Bernamont [2] in 1937 for vacuum tubes and by McWhorter [3] in 1955 for semiconductors. Since the midfifties, $1/f$ noise has been observed as fluctuations in the parameters of many systems. Many are completely unrelated to either tubes or semiconductors and their $1/f$ noise cannot be explained with either model. For example, $1/f$ noise has been observed as fluctuations in [4], [5]

- the voltages or currents of:
  - vacuum tubes
  - diodes
  - transistors
- the resistance of:
  - carbon microphones
  - semiconductors
  - metallic thin-films
  - aqueous ionic solutions
- the frequency of quartz crystal oscillators
- average seasonal temperature
- annual amount of rainfall
- rate of traffic flow
- the voltage across nerve membranes and synthetic membranes
- the rate of insulin uptake by diabetics [6]
- economic data [7]
- the loudness and pitch of music [8].

The presence of $1/f$ noise in such a diverse group of systems (plus others not mentioned) has led researchers to speculate that there exists some profound law of nature that applies to all nonequilibrium systems and results in $1/f$ noise. Numerous specific models have been proposed, but not one can account for the presence of $1/f$ noise in even most of the systems listed. Perhaps the only similarity among these systems is the mathematical description that leads to $1/f$ noise.

The applicability of $1/f$ noise to the description of the fluctuations of pitch and loudness in many types of music strongly suggests that $1/f$ noise is somehow less random than other noise, that there is a relationship or correlation between events that is lacking in other noise. But, to date, that relationship has remained a vague notion. Over what time scales are events related? A system for which present events are influenced by the system's past history must have some mechanism for information storage, i.e., for memory. What kind of memory is required for $1/f$ noise? How much and for how long does the system remember? This paper will address these questions and will demonstrate that $1/f$ noise is appropriate for the description of fluctuations in systems that evolve with time.

In Section II, a linear system that produces $1/f$ noise is analyzed and used to derive an exact nonstationary autocorrelation function. If the time over which the process is observed is short compared with the time elapsed since the process began, then the exact nonstationary autocorrelation function can be approximated. The result is an autocorrelation function that is almost stationary and from which we can describe the memory of systems exhibiting $1/f$ noise. In Section III, the power spec-

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Fig. 2. The path of a particle undergoing Brownian motion.

tral density corresponding to the approximate autocorrelation function is derived. The result is consistent with experimental observations of $1/f$ noise in that the power spectral density is stationary except for its apparent steady value which depends logarithmically on the time elapsed since the process was started. In Section IV, we consider another linear system and, from it, arrive at limits for how much memory the system must have. Although the derivations are based on physical systems, the same ideas are applicable to strictly informational systems. This will be discussed in Section V.

II. $1/f$ Noise: A Nonstationary Random Process

The integral of the power spectral density for $1/f$ noise with $1 < \gamma < 2$ (the most common case) is infinite. If $1/f$ noise were a stationary random process, one would conclude that its variance is infinite and, therefore, that the values of the process often will be extremely large. One way to avoid this problem has been to postulate the existence of a lowest frequency, below which the shape of the power spectral density changed so that the integral would converge.

Experimenters studying $1/f$ noise have searched carefully for any evidence of a change in the shape of $S(f)$ at very low frequencies. The lowest frequency at which they can accurately determine the power spectral density is limited by the long-time stability of their equipment and by the length of time over which they are willing to observe the random process. In spite of these limitations, one group of experimenters measured the $1/f$ noise in MOSFET's down to $10^{-6.3}$ Hz or 1 cycle in 3 weeks [9]! No change in shape was observed. Using geological techniques, the $1/f$ noise in weather data has been computed down to 10^{-10} Hz or 1 cycle in 300 years [10]. Again, no change was observed. Among all the experimental observations reported, only in two cases has a change been seen: in the resistance fluctuations of thin-films of tin at the temperature of the superconducting transition [11] and in the voltage fluctuations across nerve membranes [12].

An alternate solution to the problem of infinite variance was proposed by Mandelbrot [13]. He suggested that $1/f$ noise $(1 < \gamma < 2)$ should be treated as a nonstationary random process. As such, its variance and also its power spectral density would be time dependent. Observing the process for a finite time would always yield a finite variance. But, only by knowing the exact state of the process at some prior time, could one make any statements about present behavior.

To illustrate this idea, consider another well-known nonstationary random process: a particle undergoing Brownian motion (Fig. 2). One can make statements about the present position of the particle only if its position at some prior time was known. For example, assume that the particle was in the center at time $t = 0$. At some later time, $t = T$, it is likely to be within a sphere at radius R centered around its starting position. At a still later time, the radius will be larger. The mean position of the particle is a constant, equal to its starting position. But the variance in the position of the particle, corresponding to the square of the radius of the sphere in which it is likely to be found, increases linearly with time. If one insisted on describing this process as stationary and collected data from all time, one would be forced to conclude that the variance was infinite; the particle could be anywhere with equal probability. But, describing the process as nonstationary and knowing where the particle was at some prior time, one can state where it will be later as a random variable with finite variance.

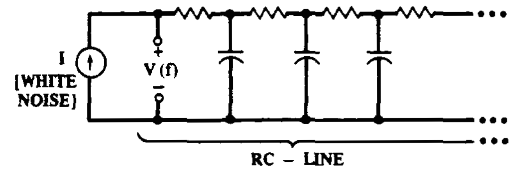

Fig. 3. A lumped representation of a continuous resistor–capacitor transmission line excited by a white noise current source.

In apparent conflict with Mandelbrot's suggestion is the fact that experimenters measure the power spectral density of $1/f$ noise, mostly without difficulty, without assuming that the process is nonstationary, and without knowledge of its initial state. If the process were nonstationary, would not the power spectral density change as a function of time and initial conditions? Experimenters sometimes find the amplitude of the power spectral density to vary among identical systems measured at different times, but the shape and, in particular, the value of the exponent γ are quite consistent. (For example, see the data presented by Hooge [14].)

We begin by considering in detail a simple system that gives $1/f$ noise exactly, with $\gamma = 1$, and deriving its nonstationary autocorrelation function. This procedure can be generalized to derive the nonstationary autocorrelation functions for all $1/f$ noise $0 < \gamma < 2$ [15]. The simple system to be considered is illustrated in Fig. 3.

The system consists of a noise current source with a white power spectral density of magnitude I driving the input of a one-dimensional continuous resistor–capacitor transmission line (RC line) of infinite length. The impedance of an infinite RC line is

$$Z(f) = \left[\frac{R}{j2\pi fC}\right]^{1/2}$$

where: R resistance of the line per unit length

C capacitance of the line per unit length

$j = \sqrt{-1}$.

The power spectral density of the voltage at the input of the line is

$$S(f) = I^2 \frac{R}{2\pi fC}.$$

Provided that the line is infinite in length, $S(f)$ is exactly proportional to $1/f$ down to zero frequency. However, if the line is finite (and terminated in a finite resistance), then $S(f)$ will have a lowest frequency below which $S(f)$ is white. The value of the lowest frequency is determined by the length of the line (l)

$$f_l = \frac{1}{2\pi R C l^2}.$$

To derive the nonstationary autocorrelation function, we will allow the line to be infinite and will construct the system such that at time $t = 0$ the voltage on the line is zero everywhere.

The white noise current source will be constructed in a way that greatly facilitates calculations, but actually is a quite general form of shot noise with an average value of zero. Consider a process that once during the interval of time T produces a unit impulse of current which is either positive or negative with equal probability, and which can occur anywhere within that interval with uniform probability.

$I = u_0(t - t_0)$ where: $u_0(t - t_0)$ denotes an impulse at time $t = t_0$;

$\mathrm{pdf}(t_0) = \frac{1}{T}$, $0 < t_0 < T$ — $\mathrm{pdf}(t_0)$ denotes the probability density function of t_0;

$= 0$, otherwise.

Summing a large number (N) of independent, identical processes would create a shot noise current, with a white power spectral density of magnitude

$$S(f) = \frac{2N}{T} = I_N^2.$$

For our computation, an increase in the interval T will be accompanied by a corresponding increase in the number of independent processes, so that the magnitude of $S(f)$ will remain constant. This is exactly equivalent to requiring a constant arrival rate of a single Poisson process.

A current impulse at $t = t_0$ excites the RC line to give a voltage response of

$$v(t) = \pm \left[\frac{R}{\pi C}\right] \frac{1}{\sqrt{t - t_0}} u_{-1}(t - t_0)$$

where

$$u_{-1}(t - t_0) = \begin{cases} 1, & \text{for } t > t_0 \\ 0, & \text{for } t < t_0. \end{cases}$$

The autocorrelation function of the voltage resulting from a single impulse is

$$R(t_1, t_2) = E\{v(t_1)\, v(t_2)\}$$

where $E\{\ \}$ denotes expected value

$$= \frac{R}{\pi C} E\left[\frac{1}{\sqrt{t_1 - t_0}} u_{-1}(t_1 - t_0) \frac{1}{\sqrt{t_2 - t_0}} u_{-1}(t_2 - t_0)\right]$$

$$= \frac{R}{\pi C} \int_0^{\infty} \frac{1}{\sqrt{t_1 - t_0}} u_{-1}(t_1 - t_0) \frac{1}{\sqrt{t_2 - t_0}} \cdot u_{-1}(t_2 - t_0)\, \mathrm{pdf}(t_0)\, dt_0.$$

Noting that both terms are nonzero only if the impulse occurs before t_1 and t_2, and assuming $t_2 > t_1$ we obtain

$$R(t_2, t_1) = \int_0^{t_1} \frac{R}{\pi C} \frac{1}{\sqrt{t_1 - t_0}} \frac{1}{\sqrt{t_2 - t_0}} \frac{1}{T}\, dt_0.$$

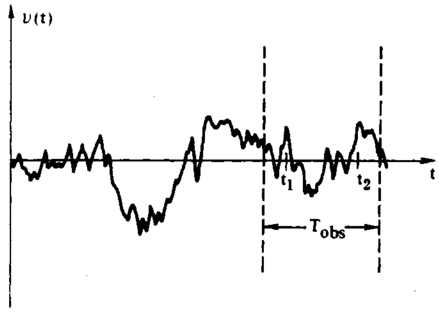

Fig. 4. A sample of $1/f$ noise where the time of observation is short when compared with the time elapsed since the process began.

For N identical and independent processes, the final result is just N times the above, which is

$$R(t_2, t_1) = \frac{N}{T} \frac{R}{\pi C} \cosh^{-1}\left[\frac{1 + t_1/t_2}{1 - t_1/t_2}\right].$$

Imagine that the RC line was constructed a very long time ago, i.e., t_2 is a big number, and that we will observe the process for a comparatively short time $T_{obs} \ll t_2$. Both times t_1 and t_2 will be within the observation interval, but the starting time, $t = 0$, will not be. As a result, the ratio of t_1 and t_2 will be nearly equal to one (see Fig. 4). Thus we can approximate the previous result by

$$R(t_2, t_1) \sim I_N^2 \frac{R}{2\pi C} \ln 2\left[\frac{1 + t_1/t_2}{1 - t_1/t_2}\right]$$

$$\sim I_N^2 \frac{R}{2\pi C} \ln 2\left[\frac{2}{1 - t_1/t_2}\right]$$

$$\sim I_N^2 \frac{R}{2\pi C} \ln 4 t_2 - I_N^2 \frac{R}{2\pi C} \ln (t_2 - t_1).$$

This procedure can be easily generalized to derive the autocorrelation functions for $1/f$ noises for which $\gamma \neq 1$, but is anywhere in the range $0 < \gamma < 2$. The details are presented elsewhere [15], but a summary of the results is presented below. For the purpose of comparing the nonstationary with stationary autocorrelation functions, the quantity $(t_2 - t_1)$ has been replaced by (τ).

$R(\tau) \propto |\tau|^{\gamma - 1}$ for $\tau > 0$, $0 < \gamma < 1$

$R(t_2, \tau) \propto \ln 4 t_2 - \ln |\tau|$ $\quad 0 < \tau \ll t_2$, $\gamma = 1$

$R(t_2, \tau) \propto t_2^{\gamma - 1} - (\text{constants})\, |\tau|^{\gamma - 1}$ $\quad 1 < \gamma < 2$.

III. The Corresponding Power Spectral Density

Assume that an experimenter observed the system for a time T_{obs} much shorter than the time elapsed since the system was assembled and the process begun. Further assume that the experimenter thought he was observing a stationary random process and computed $S(f)$ based on that assumption. What would be the result?

In the previous section, we derived the nonstationary autocorrelation functions for $1/f$ noise. These functions were then approximated for the case where the time of observation (T_{obs}) was much shorter than the total time (t_2) elapsed since the process began. The result was that each approximate autocorrelation function could be written as the sum of two terms, one dependent only on t_2 and one dependent only on τ.

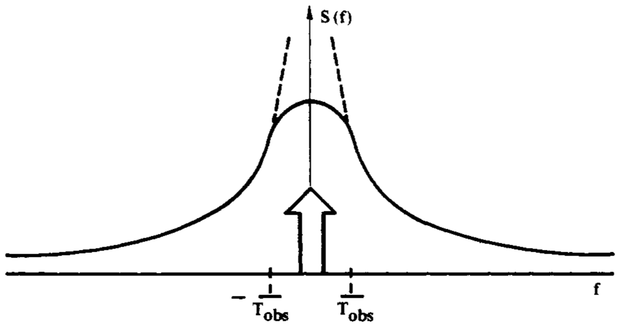

Fig. 5. The power spectral density one would observe in measuring the noise from the RC line model.

$$R(t_2, \tau) = f(t_2) + f(\tau).$$

In effect, the nonstationary autocorrelation functions are approximately stationary. The nonstationary behavior is entirely contained in the added term that depends only on t_2. By taking the Fourier transform with respect to the variable τ we will compute an approximately stationary $S(f)$ and evaluate the effect of the added term. Since one cannot observe correlations over times larger than the total observation time T_{obs}, we will limit the value of τ to the range

$$0 < |\tau| \leq T_{obs}.$$

For $1/f$ noise with $\gamma = 1$, by dividing the arguments of both terms by T_{obs}, we have

$$R(t_2, \tau) = \begin{cases} I_N^2 \dfrac{R}{2\pi C}\left[\ln \dfrac{4t_2}{T_{obs}} - \ln \dfrac{|\tau|}{T_{obs}}\right], & 0 < |\lambda| \leq T_{obs} \\ I_N^2 \dfrac{R}{2\pi C}\left[\ln \dfrac{4t_2}{T_{obs}}\right], & T_{obs} < |\tau|. \end{cases}$$

Using transform pair (1) for Erdelyi (p. 17) [16] and the scaling properties of Fourier transforms, we obtain for the second term

$$S(f) = I_N^2 \frac{R}{2\pi C} \frac{1}{\pi f} \int_0^{2\pi f T_{obs}} \frac{\sin \mu}{\mu}\, d\mu \quad \text{(Bilateral Transform).}$$

The transform of the first term with respect to τ is just the transform of a constant, which yields an impulse at zero frequency.

$$S(f) = \left[I_N^2 \frac{R}{2\pi C} \ln \frac{4t_2}{T_{obs}}\right] u_0(f)$$

where: $u_0(f)$ denotes the unit impulse located at $f = 0$. The overall result is

$$S(f) = \begin{cases} I_N^2 \dfrac{R}{2\pi C}\left[\dfrac{1}{2|f|}\right], & \text{for } |f| \gg \dfrac{1}{T_{obs}} \\ I_N^2 \dfrac{R}{2\pi C}\left[\ln\left(\dfrac{4t_2}{T_{obs}}\right) u_0(f) + 2T_{obs}\right], & |f| \ll \dfrac{1}{T_{obs}}. \end{cases}$$

This power spectral density is illustrated in Fig. 5. $S(f)$ is proportional to $1/f$ down to the lowest frequency allowed by the limited observation time. Both the exponent and the magnitude of $S(f)$ for frequencies above this lowest frequency are independent of the value of both t_2 and T_{obs} and are, therefore, stationary. The apparent steady value of the process over the interval T_{obs} would be the integral of $S(f)$ for

$$-\frac{1}{T_{obs}} < f < \frac{1}{T_{obs}}$$

and would equal

$$\int_{-1/T_{obs}}^{1/T_{obs}} S(f)\, df = I_N^2 \frac{R}{2\pi C}\left[\ln \frac{4t_2}{T_{obs}} + 2\right].$$

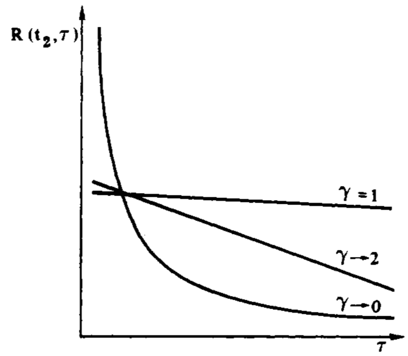

Fig. 6. Autocorrelation functions for $(1/f) ** \gamma$ noise. For $\gamma = 1$, the recent and the distant past have almost equal correlation with the present.

The nonstationary autocorrelation function derived in Section II corresponds to a power spectral density that appears to be stationary. Measurements of $S(f)$ made without knowledge of initial conditions would be consistent within the range of frequencies allowed by the limited observation time. The variance of the process grows logarithmically with time, but observing for a finite interval will always yield a finite variance.

It is common practice in the literature to normalize $S(f)$ by dividing by the square of the steady value of the process. When an independent constant term is not dominant, the magnitude of $S(f)$ would then depend logarithmically on t_2. This may explain some experimental observations where the value of the exponent varies slightly but the value of the magnitude sometimes varies enormously.

IV. The Memory of 1/f Processes

White noise has no memory of the past; current values of the process are independent of past values. Most processes are not white out to infinite frequency, but their memory is limited, often to one time constant or state variable. $1/f$ processes do have memory. The questions are how much and for how long are these processes influenced by their past? In this section, we will explore these questions by first examining the approximate autocorrelation functions from Section II to illustrate how long the process remembers and then by constructing another linear system that yields $1/f$ noise to illustrate how many initial conditions the process remembers.

$1/f$ noise is a random process with a very long memory. The derived autocorrelation functions, illustrated in Fig. 6, decay as a power of time, slower than any exponential, except for the case of γ exactly equal to 1, for which the decay is logarithmic and slower than any power of time. The closer γ is to 1, the greater the influence of the distant past when compared with

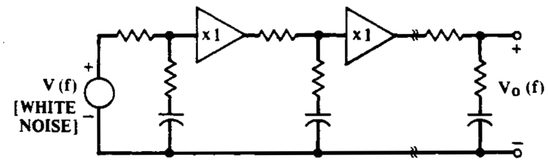

Fig. 7. Another linear system that yields $1/f$ noise, approximately. Each section has one state variable (the capacitor voltage) and can remember one number.

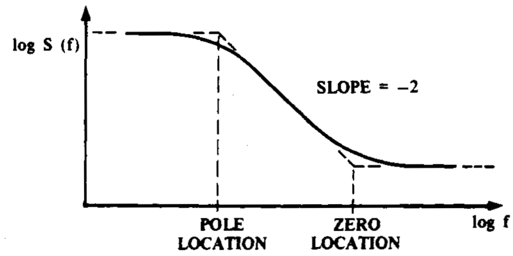

Fig. 8. The frequency response of a single section of the linear system illustrated in Fig. 7.

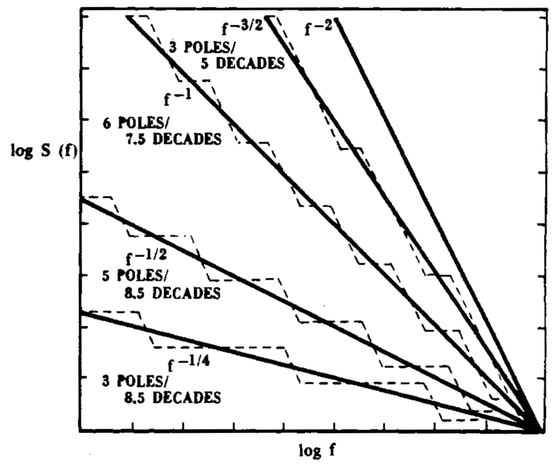

Fig. 9. A curve fit of the power spectral densities of approximating linear systems to obtain $1/f$ noise with various values of γ. The marks on the abscissa and the ordinate, are one decade apart.

the influence of the recent past. When γ equals 1, present events are approximately equally correlated with events from the recent and the very distant past. For γ near either 0 or 2, the process is influenced by the recent past much more strongly than by the distant past.

Since it is clear from the autocorrelation functions that a $1/f$ process is strongly influenced by the past, especially for γ nearly 1, one might inquire how much information the process remembers. Assume that the process is modeled with a linear system, and that its entire past history is represented by the present values of its state variables. How many state variables are required for such a system when its fluctuations have a $1/f$ power spectral density? Or how many numbers are required to summarize the influence of the past on the present? For white noise, the number is zero. For Brownian motion, it is one; only the initial position is required. But for $1/f$ noise, hundreds of numbers are required.

The continuous RC line model that yields an exact fit to a $1/f$ noise power spectral density has an infinite number of state variables. So does the sum-of-distributed-time-constants model introduced by Bernamont [2] in 1937 to explain $1/f$ noise in vacuum tubes. However, if we require only an approximate fit provided by a lumped, linear system, then only hundreds of state variables are required and the number increases slowly with the degree of precision required. A lumped linear system, quite different from the RC line used in Section II, will be analyzed to place both a lower and an upper limit on the required number of state variables for a $1/f$ process. It will also be used to illustrate the similarity among $1/f$ processes with slightly different values of the parameter γ.

A simple linear system can be constructed to have a power spectral density that differs from $(1/f)^\gamma$ by a constant percentage error which can be made arbitrarily small. It is illustrated in Fig. 7. It bears little relation to the previous model except that it is also a linear system driven by a white noise source and that $S(f)$ of some variable $V_0(f)$ will be $1/f$. It consists of resistor–capacitor sections separated and isolated by unity-gain buffer amplifiers. Each section contributes one pole and one zero to the overall response and has one state variable, the voltage on the capacitor. The response of a single section is shown in Fig. 8.

Of course, the required number of sections depends on how closely the approximate and exact curves must agree. An exact fit would require an infinite number. However, it is interesting to determine how many are required for a ±5 percent fit, and then to use that value to discuss the number of state variables and the number of values needed by a $1/f$ process to summarize the past. A construction based on the foregoing concept is presented in Fig. 9. The continuous lines are the power spectral densities for various values of γ plotted on a log versus log scale. Superimposed on each of these lines are dotted lines representing the asymptotes for $S(f)$ of an approximating linear system. The positions of the poles and zeros were chosen for a maximum error of ±5 percent.

The results may be somewhat surprising. Approximately 1 section (also 1 state variable) is required per decade of frequency to fit a $1/f$ power spectral density. Slightly fewer are required for $1/f^{1/2}$ and $1/f^{3/2}$, with the value decreasing towards zero per decade as $S(f)$ approaches white noise ($\gamma = 0$) or Brownian motion ($\gamma = 2$). (Note: Brownian motion requires a total of one state variable.) For a $1/f$ power spectral density extending from 10^{-50} Hz to 10^{50} Hz, only a hundred sections are required for a better than 5-percent fit. For a 1-percent fit, the number is probably less than 500. This means that the system would have only hundreds of state variables and would summarize the past with only hundreds of numbers.

Even if we demand a perfect fit, but only observe the process for a finite time, the number of state variables whose present values must be treated individually is not infinite. The effect of a finite observation time (T_{obs}) is to smooth the frequency response by averaging over a bandwidth (f_0) of approximately the reciprocal of that time.

$$f_0 \sim \frac{1}{T_{\text{obs}}}.$$

At most, we would need a pole and zero at every multiple of (f_0). Thus the maximum number of distinguishable state variables is the ratio of the high-frequency limit (f_h) to the reciprocal of the observation time.

$$\text{maximum} = \frac{f_h}{f_0}.$$

For purpose of comparison with the number for a 5-percent fit, choose the low-frequency limit to coincide with the lowest frequency allowed by the observation time. (The effect of all state variables whose time constants are much longer than the observation time can be grouped into one present value: the apparent steady value.) The result is a range for the number of state variables (N) given by

$$\log\left[\frac{f_h}{f_0}\right] < N < \frac{f_h}{f_0}.$$

The construction of Fig. 9 supports the view that $1/f$ noises with slightly different values of γ are fundamentally similar. They all have a uniform distribution of poles per decade of frequency and over the range: $1/2 < \gamma < 3/2$ have approximately the same number of poles per decade. The major difference in the system for different values of γ is in the relative weighting of the effect of each state variable on the present value of the output.

For the duration of this paper, the author will use the number of state variables required for a 5-percent fit as the minimum required for the process. Choosing a slightly better or worse fit would have only a small effect on the minimum number. Combining the result for the minimum number of state variables required with the characterization of the autocorrelation functions developed earlier, we arrive at the following description of the memory of $1/f$ noise for γ very near 1. The correlation of present events with those from a time τ long ago would result from the voltage on all capacitors whose discharge rate was too slow to allow for an appreciable change in the time τ. All capacitors whose poles were at low enough frequencies but above the lowest frequency f_0 would be included.

$$f_0 < |f| < \frac{1}{2\pi\tau}.$$

Since each capacitor has an equal influence on present behavior, the magnitude of the correlation for $\gamma = 1$ is just proportional to the total number of capacitors included.

$$R(\tau) \propto [1 \text{ per decade}] \times [\text{number of decades}]$$

$$\sim \log\left[\frac{1}{2\pi\tau} \div f_0\right]$$

$$\sim \text{constant} - \log\tau.$$

Of course, this is the same result as before.

V. Discussion

$1/f$ noise is an evolutionary random process. Its present behavior is strongly influenced by its entire history. Furthermore, its memory is dynamic; the influence of recent events is added to and gradually supersedes the influence of distant events. The influence of the distant past fades very slowly, either as a small power of elapsed time or logarithmically, very much slower than the exponential relaxation times associated with low-order differential equations commonly used for system modeling. These ideas will be illustrated first by considering $1/f$ noise in MOSFET's and then in informational systems.

Most physical systems are gradually approaching thermodynamic equilibrium and the influence of their initial conditions decreases with time. For example, a MOSFET consists of concentrations of material, which given enough time, would disperse. The usual description of its dynamics presumes that it can be divided into two subsystems, one that changes very slowly when compared with the time of observation and another that changes very fast. Its structure changes so slowly that for practical purposes, it is considered static. On the other hand, the distributions of the electrons and holes within the transistor have very short relaxation times and are considered to be in quasi-equilibrium.

The presence of $1/f$ noise in MOSFET's, down to the lowest frequency allowed by the limited observation time [9], suggests that the division into just two subsystems is inappropriate. Instead, the transistor must be composed of subsystems with relaxation times comparable to all time scales of interest. No matter how long it is observed, some processes will have reached equilibrium and the values of their initial conditions will be forgotten. A few processes will be demonstrably changing over the observation interval, while others will be changing too slowly for detection and will preserve their initial conditions almost completely. The length of time that one observes the system partitions the subsystems into three categories: fast, comparable, and slow. Changing the observation time merely alters the category into which a particular subsystem is placed; the process is fundamentally the same. To characterize the behavior of the transistor, one must know the present values of each of the state variables with times comparable to or faster than the observation time. In addition, one must know the aggregate effect of all the state variables that do not change and whose separate effects cannot be distinguished during the observation interval. All of these slowly changing variables are represented by one number: the apparent steady value of the process. Changing the observation interval alters which variables are grouped together and which must be treated individually.

Informational systems accumulate information. With passing time, they exhibit a general increase in structure and complexity, rather than a decrease as in the case of a transistor. Biological evolution, cultural development, the growth of governments, the development of economic systems, and personal growth and development are good examples. In addition works of art that develop central themes, such as novels and symphonies are also good examples. Voss and Clarke [8] have shown that when music (which is obviously not a random composition) is treated as if it were a random process, then the power spectral density of both its amplitudes and pitch (as functions of time) are $1/f$. The same may be true for many information accumulating, evolving systems; some parameters when treated as random processes may have power spectral densities that are $1/f$. $1/f$ noise combines the strong influence of past events with the influence of current events. The result is an overall context or pattern and somewhat predictable behavior, but with the possibility of new trends developing and of occasional surprises.

A $1/f$ process appears to extract trends and condense data into summaries that, in turn, determine the values of the system's state variables and influence present behavior. Focusing on the minimum number of state variables required, we have one with a time constant of say 0.1 units of time, one with 1 unit, 10 units, 100 units, and so on. The value of the state variable whose relaxation (or averaging) time is 1 s represents the average behavior or trend in the process over the past 1-s or so. The value of the 100-s state variable represents a summary of the most recent 100 s. Present behavior is influenced equally by each of these state variables. Each one represents a trend in the data, but over different time scales. Recent data might dominate the values of the shortest time state variables, but it will have a progressively diminishing impact on the longer ones. Persistent new trends will cause the process to adapt logarithmically with time, as more and more of the state variables reflect the new trends in the data rather than the old.

The models developed for $1/f$ noise might be useful for modeling some of the informational systems mentioned above. The idea that information describing the past is summarized and stored in the form of trends over different time scales is particularly appealing. It seems close to the way humans remember (and incorrectly remember) information, as parts of consistent patterns rather than as separate, unrelated pieces. The fact that music has $1/f$ noise statistics and that when notes are chosen at random, they sound most musical when their power spectral density is $1/f$, suggests a connection between the way humans perceive and remember, and the structure of $1/f$ noise. Because of this connection and the influence of human memory and behavior on the development of our institutions: the development of ourselves, our economic system, our government, and our culture may each have the statistics of a $1/f$ noise random process.

As a final note, consider the task of best predicting the future behavior of a $1/f$ process. Since present and future behavior are highly correlated with past behavior, one must understand the influence of the past to best predict the future. For a $1/f$ process, the past behavior can be summarized by the present values of the system's state variables. Approximately one per decade over the time span of interest is the minimum number required. Any form of regression analysis applied to a $1/f$ process would have to estimate not just the average value of a parameter over the entire extent of the data, but also its value averaged over 1 unit of time, 10 units, 100 units, 1000, 10 000, etc., extending from the shortest to the longest times of interest. Each of these averages would determine the value of one state variable, and each state variable will have an equal influence on the future behavior of the process. For best predictions, one would combine the present value of each of the state variables with an understanding of the mechanism by which each of them influences present behavior.

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References

[1] For an introduction to random process theory, including autocorrelation functions and power spectral densities, see: A. Papoulis, *Probability, Random Variables, and Stochastic Processes.* New York: McGraw-Hill, 1965.

[2] J. Bernamont, *Proc. Phys. Soc.*, 49 (extra part): 138, 1937.

[3] A. L. McWhorter, "$1/f$ noise and related surface effects in germanium," R.L.E. 295 and Lincoln Lab Tech Rep. 80, M.I.T., 1955.

[4] R. F. Voss, "$1/f$ (flicker) noise: A brief review," in *Proc. 33rd Annu. Symp. Frequency Contr.*, Atlantic City, NJ, 1979, pp. 40-46.

[5] F. N. Hooge, "$1/f$ Noise," *Physica*, vol. 83B, pp. 14-23, 1976.

[6] M. J. Campbell and B. W. Jones, "Cyclic changes in the insulin needs of an unstable diabetic," *Science*, vol. 177, pp. 889-891, Sept. 8, 1972.

[7] B. B. Mandelbrot and J. W. Van Ness, "Fractional brownian motions, fractional noises, and applications," *SIAM Rev.*, vol. 10, no. 4, pp. 422-437, 1968.

[8] R. F. Voss and J. Clarke, "$1/f$ noise in music: Music from $1/f$ noise," *J. Accoust. Soc. Amer.*, vol. 63, no. 1, pp. 258-263, 1978.

[9] M. A. Caloyannides, "Microcycle spectral estimates of $1/f$ noise in semiconductors," *J. Appl. Phys.*, vol. 45, no. 1, pp. 307-316, 1974.

[10] B. B. Mandelbrot and J. R. Wallis, "Some long-run properties of geophysical records," *Water Resources Res.*, vol. 5, no. 2, pp. 321-340, 1969.

[11] J. Clarke and T. Y. Hsiang, "Low frequency noise in tin and lead films at the superconducting transition," *Phys. Rev. B*, vol. 13, no. 11, pp. 4790-4800, 1976.

[12] A. A. Verveen and H. E. Derksen, "Fluctuation phenomena in nerve membrane," *Proc. IEEE*, vol. 56, no. 6, pp. 906-916, 1968.

[13] B. B. Mandelbrot, "Some noises with $1/f$ spectrum, a bridge between direct current and white noise," *IEEE Trans. Inform. Theory*, vol. IT-13, no. 2, pp. 289-298, 1967.

[14] F. N. Hooge, "Discussions of recent experiments on $1/f$ noise," *Physica*, vol. 60, pp. 130-144, 1972.

[15] M. S. Keshner, "Renewal process and diffusion models of $1/f$ noise," Sc.D. thesis submitted to the Department of Electrical Engineering and Computer Science of the Massachusetts Institute of Technology, June 1979.

[16] Erdelyi *et al.*, *Tables of Integral Transforms*, vol. 1. New York: McGraw-Hill, 1954.